Option pricing and numéraires

Dario Trevisan

Università degli Studi di Pisa

San Miniato - 15 September 2016

Overview

What is a numéraire?

A numéraire (or numeraire) is a chosen standard by which value is computed.

Example (currencies ar numeraires)

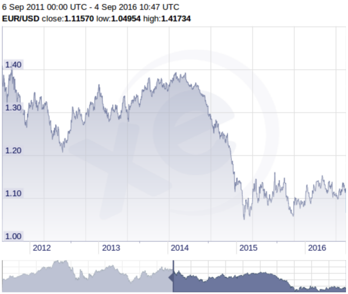
We may compute values w.r.t to USD 1$ or EUR 1 € or JPY (1 ¥).

Of course, others might prefer use commodities:

1 OZ of gold could be a numeraire.

Clearly, once we choose a numeraire e.g. 1 USD, we determine the value of other assets:

USD per 1 EUR



USD per 1 XAU

The change of numeraire problem

Problem

In theory, does it really matter which numeraire we choose?

Of course, in practice there are reasons to prefer gold to other commodities e.g. corn, live cattle, or one currency with respect to another (political reasons)...

But intuitively there should be no theoretical reason (at least at the scale of investors)

to measure value in gold or USD (there used to be also the “gold standard”)

$\Rightarrow$ Can we deduce/exploit this fact in our financial models?

The strong underlying principle will be always

absence of arbitrage opportunities.

Back to the Arrow-Debreu model

Recall (from Tuesday) the simple model with $N \geq 1$ securities (i.e. bonds, stocks or derivatives)

$$\vec{a} = (a_1, a_2, \ldots, a_N)$$

can be held long or short by any investor.

Two times: $t = 0$ and a fixed future $t = 1$.
At $t = 0$ we have the observed spot prices of the N securities

$$\vec{p} = (p_1, p_2, \ldots, p_N) = (p_i)_{i=1}^{N} \in \mathbb{R}^N$$

At $t = 1$, the market attains one state among M possible "scenarios"

$$s \in \{1, \ldots, M\}.$$

If s is attained $\Rightarrow$ "dividends" (prices) of the securities at $t = 1$

$$\vec{D}^s = \left(D_1^s, D_2^s, \ldots, D_N^s\right) \in \mathbb{R}^N$$

Theorem

No arbitrage $\Rightarrow$ existence of positive weights $(\pi_s)_{s=1,\dots,M}$ such that

$$\vec{p} = \sum_{s=1}^{M} \vec{D}^s \pi_s, \quad \text{i.e. } p_i = \sum_{s=1}^{M} D_i^s \pi_s \text{ for every } i \in \{1, \dots, N\}.$$

Next we assume that there is a risk-free security a_1 (e.g. a bond) such that in any scenario s we have $D_1^s = 1 \Leftrightarrow a_1$ is our numeraire

Define R and $\hat{\pi}$ (the interest rate and risk-neutral probability) by the relation

$$1 + R = \frac{1}{\sum_{s=1}^{M} \pi_s}, \quad \hat{\pi}_s = \frac{\pi_s}{\sum_{r=1}^{M} \pi_r}.$$

Then

$$p_1 = \frac{1}{1+R}, \quad p_i = E_{\hat{\pi}}\left[\frac{D_i}{1+R}\right] = \sum_{i=1}^{M} D_i^s \pi_s.$$

Change of numeraire

What happens if instead of a_1 we fix another asset, e.g. a_2 as numeraire?

Assume $D_2^s > 0$ for every state of the market.

$\Rightarrow$ the value of the asset a_i, measured in units of a_2, at $t = 1$ is

$$v_i^s := \frac{D_i^s}{D_2^s}, \qquad \text{if the market is in the state } s.$$

We rewrite the value at $t = 0$ of a_i as

$$p_i = \sum_{s=1}^{M} D_i^s \pi_s = \sum_{s=1}^{M} \frac{D_i^s}{D_2^s} \left(D_2^s \pi_s\right)$$

hence if we measure the value in units of a_2, since

$$p_2 = \sum_{r=1}^{M} \left(D_2^r \pi_r\right)$$

we have

$$\frac{p_i}{p_2} = \sum_{s=1}^{M} v^s \frac{\left(D_2^s \pi_s\right)}{\sum_{r=1}^{M} \left(D_2^r \pi_r\right)} = E_2\left[v^s\right],$$

where E_2 is a different probability measure than $E_{\hat{\pi}}$.

Change of numeraire ⇔ change of probability

We found an instance of the general "mechanisms":

Passing from the numeraire a to b corresponds to a change of probability, from

$$\pi_s^a := \frac{(D_a^s \pi_s)}{\sum_{r=1}^M (D_a^r \pi_r)}$$

to

$$\pi_s^b = \frac{(D_b^s \pi_s)}{\sum_{r=1}^M (D_b^r \pi_r)}.$$

The value of the asset a_i (w.r.t. the numeraire b) at $t = 0$ is given by the expectation w.r.t. π^b of the values at time $t = 1$

$$\frac{p_i}{p_b} = E^b \left[\frac{D_i^s}{D_b^s} \right] = \sum_{s=1}^M \frac{D_i^s}{D_b^s} \pi_s^b.$$

Continuous-time (Itô) models

Let us model the market with

- a probability space $(\Omega, \mathscr{A}, P)$,
- a filtration $(\mathcal{F}_t)_{t\in[0,T]}$
- Itô processes $\vec{S}_t = (S^i_t)_{i=1,\dots,N}$

A portfolio $\vec{H}_t = \left(H^1_t, \dots, H^N_t\right)$ has value

$$V_t := \vec{H}_t \cdot \vec{S}_t = \sum_{i=1}^{N} H^i_t S^i_t$$

A numeraire is a strictly positive Itô process D_t.

The prices actualized with respect to D become

$$\frac{S^i_t}{D_t}.$$

Change of numeraire and self-financing strategies

Proposition

The self-financing condition is invariant with respect to any chosen numeraire, i.e.

$$dV_t = \sum_{i=1}^{N} H_t^i dS_t^i, \quad t \in (0, T)$$

if and only if

$$d\left(\frac{V_t}{D_t}\right) = \sum_{i=1}^{N} H_t^i d\left(\frac{S_t^i}{D_t}\right), \quad t \in (0, T)$$

We use Itô formula for product

$$d(VG) = GdV + VdG + dGdV$$

with $G = 1/D$.
Since

$$d\left(\vec{H}\cdot\vec{S}\right) = \vec{H}d\left(\vec{S}\right)$$

we have

$$\begin{aligned} d(VG) &= G\vec{H}d\vec{S} + \left(\vec{H}\cdot\vec{S}\right)dG + dG\,d\left(\vec{H}\cdot\vec{S}\right) \\ &= G\vec{H}d\vec{S} + \left(\vec{H}\cdot\vec{S}\right)dG + dG\vec{H}d\vec{S} \\ &= \vec{H}\cdot\left(G\vec{S} + \vec{S}dG + dGd\vec{S}\right) \\ &= \vec{H}\cdot d\left(\vec{S}G\right) \end{aligned}$$

Change of probability

Theorem

Let P^0 be a probability (equivalent to P) such that every

$$\frac{S_t^i}{S_t^0}, \quad i \in \{1, \ldots, N\}$$

is a martingale, and also

$$\frac{D_t}{S_t^0}.$$

Consider the new probability

$$P^D = \frac{1}{D_0} \cdot \frac{D_T}{S_T^0} P^0.$$

Then each

$$\frac{S_t^i}{D_t} \quad i \in \{1, \ldots, N\}$$

is a P^D-martingale (as also $\frac{D_t}{D_t} = 1$).

Proof

$$E^0\left[\cdot|\mathcal{F}_t\right] \Rightarrow \text{the conditional expectation w.r.t. } P^0 \text{ and}$$
$$E^D\left[\cdot|\mathcal{F}_t\right] \Rightarrow \text{the conditional expectation w.r.t. } P^D.$$

$\frac{S_t^i}{S_t^0}$ is a P^0 martingale $\Rightarrow \frac{S_t^i}{S_t^0} = E^0\left[\frac{S_T^i}{S_T^0}|\mathcal{F}_t\right]$.

We have a formula for conditional expectation w.r.t. different probabilities:

$$E^D\left[X|A\right] = \frac{E^0\left[XfI_A\right]}{E^0\left[fI_A\right]} = \frac{E^0\left[Xf|A\right]}{E^0\left[f|A\right]} \quad \text{with } f = \frac{1}{D_0}\cdot\frac{D_T}{S_T^0}.$$

Theorem

For any (P^D integrable) random variable X,

$$E^D\left[X|\mathcal{F}_t\right] = \frac{E^0\left[Xf|\mathcal{F}_t\right]}{E^0\left[f|\mathcal{F}_t\right]}, \quad \text{with } f = \frac{dP^D}{P^0} = \frac{1}{D_0}\cdot\frac{D_T}{S_T^0}.$$

$$\Rightarrow \quad E^D\left[\frac{S_T^i}{D_T}|\mathcal{F}_t\right] = \frac{E^0\left[\frac{S_T^i D_T}{D_T D_0 S_T^0}|\mathcal{F}_t\right]}{E^0\left[\frac{1}{D_0}\cdot\frac{D_T}{S_T^0}|\mathcal{F}_t\right]} = \frac{D_0}{D_0}\,\frac{E^0\left[\frac{S_T^i}{S_T^0}|\mathcal{F}_t\right]}{E^0\left[\frac{D_T}{S_T^0}|\mathcal{F}_t\right]} = \frac{\frac{S_t^i}{S_t^0}}{\frac{D_t}{S_t^0}} = \frac{S_t^i}{D_t}.$$

Pricing via change of numeraire

A consequence of the previous theorem is the possibility to compute prices w.r.t. P^D instead of P^0.

Corollary

The value at time $t \in [0, T]$ of a asset X can be computed as

$$V_t = S_t^0 E^0 \left[\frac{X}{S_T^0} | \mathcal{F}_t \right] = D_t E^D \left[\frac{X}{D_T} | \mathcal{F}_t \right].$$

Also the self-financing hedging strategy can be computed w.r.t. D_t.

If $D_t = S_t^i$ for some i, this has the advantage of reducing the number of parameters by one.

Let us consider some examples of applications.

A remark

Assume that an Itô process $S_t > 0$ is in the form

$$(dS)_t = S_t(\mu_t dt + \sigma_t dW_t),$$

hence the quadratic variation is

$$d[S]_t = \text{``}dSdS\text{''} = S_t^2\sigma_t^2 dt.$$

The process S_t^{-1} is an Itô process, with and by Itô formula with

$$f(x) = \frac{1}{x}, \quad f'(x) = -\frac{1}{x^2}, \quad f''(x) = +2\frac{1}{x^3},$$

we have

$$\begin{aligned} dS^{-1} &= f'(S)dS + \frac{1}{2}f''(S)d\,[S] \\ &= -\frac{1}{S^2}dS + \frac{1}{S^3}S^2\sigma^2 dt \\ &= -\frac{1}{S_t^2}\left(S_t(\mu_t dt + \sigma_t dW_t)\right) + S_t^{-1}\sigma_t^2 dt \\ &= S^{-1}\left(\left(-\mu_t + \sigma_t^2\right)dt - \sigma_t dW_t\right) \end{aligned}$$

Siegel paradox on exchange rates

Assume that we have

- two different currencies, 1 and 2,
- and corresponding (determistic) bonds

$$B_t^1 = e^{r^1 t}, \quad B_t^2 = e^{r^2 t}$$

  with interest rates r^1, r^2.
- B^i is expressed in currency $i \in \{1, 2\}$
- a stochastic exchange rate R_t

$$dR_t = R_t(\mu dt + \sigma dW_t)$$

  so that a unit of currency 2 equals R_t units of currency 1

In currency 1, the market is given by two assets:

$$(B_t^1, B_t^2 R_t).$$

In currency 2, the market is given by two assets:

$$(B_t^1 R_t^{-1}, B_t^2).$$

In the equivalent martingale measure P^0 (risk neutral measure) we have

$$\frac{B_t^2 R_t}{B_t^1} = e^{(r^2 - r^1)t} R_t$$

must be a martingale. By Ito formula (with respect to the measure P^0)

$$d\left(B_t^2 R_t / B_t^1\right) = e^{(r^2 - r^1)t} R_t \left((r^2 - r^1)dt + dR\right) \Rightarrow dR = R\left((r^1 - r^2)dt + \sigma dW_t^0\right).$$

Also with respect to P^0, the equation for the inverse exchange rate is

$$d\left(R^{-1}\right) = R^{-1}\left(\left(-(r^1 - r^2) + \sigma^2\right) dt - \sigma dW_t^0\right)$$

there is an extra term σ^2 which gives no symmetry.

$\Rightarrow$ The choice of one numeraire “transfers” all the risk to the others assets.

Choose currency 2 as a numeraire, i.e. $D_t = B_t^2 R_t$. Then in the probability P^D, we have that

$$\frac{B_t^1}{B_t^2 R_t} \quad \text{is a } P^D\text{-martingale.}$$

this leads to the “natural” equation

$$d\left(R^{-1}\right)_t = R_t^{-1}\left(-(r^1 - r^2)dt - \sigma dW_t^D\right)$$

Swap options

Consider a market with three assets S^0, S^1, S^2,

$$dS_t^0 = S^0 r dt, \quad dS_t^1 = S^1 \left(\mu_1 dt + \sigma_1 dW_t^1\right), \quad dS_t^2 = S_t^2 \left(\mu_2 dt + \sigma_2 dW_t^2\right)$$

with W_t^1, W_t^2 independent.

We want to price the swap option that gives the possibility to the holder to exchange S_T^2 with S_T^1 without additional costs. Its value at time T is

$$(S_T^1 - S_T^2)^+ = \left(\frac{S_T^1}{S_T^2} - 1\right)^+ S_T^2$$

Idea: swap option is a call option with strike price 1, if numeraire is S^2.

$$
\begin{aligned}
d\left(\frac{S^1}{S^2}\right) &= \frac{S^1}{S^2}\left((\dots)\,dt - \sigma_2 dW_t^2 + \sigma_1 dW_t^1\right) \\
&= \frac{S^1}{S^2}\left((\dots)\,dt + \sqrt{\sigma_1^2+\sigma_2^2}\,\frac{\sigma_1 dW_t^1 - \sigma_2 dW_t^2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right) \\
&= \frac{S^1}{S^2}\left((\dots)\,dt + \sqrt{\sigma_1^2+\sigma_2^2}\,dW^*\right)
\end{aligned}
$$

Where W^* is a Brownian motion w.r.t. P.
If we use the risk-neutral probability P^2, corresponding to the numeraire S^2,

$$
d\left(\frac{S^1}{S^2}\right) = \frac{S^1}{S^2}\left(\sqrt{\sigma_1^d 2 + \sigma_2^2}\,dB^*\right)
$$

where B^* is a P^2-Brownian motion and

$$
\begin{aligned}
V_t &= S_t^2 E^2\left[\frac{(S_T^1 - S_T^2)^+}{S_T^2}\,|\mathcal{F}_t\right] = S_t^2 E^2\left[\left(\frac{S_T^1}{S_T^2} - 1\right)^+ |\mathcal{F}_t\right] \\
&= S_t^2\,\mathcal{C}\left(t, T, \frac{S_t^1}{S_t^2}, 1, 0, \sqrt{\sigma_1^2+\sigma_2^2}\right)
\end{aligned}
$$

where $\mathcal{C}(t, T, x, K, r, \sigma)$ Black-Scholes formula for price of call option.