Stochastic Processes and Stochastic Calculus
Cox, Ross & Rubinstein model

Scuola Normale Superiore, Pisa, Italy

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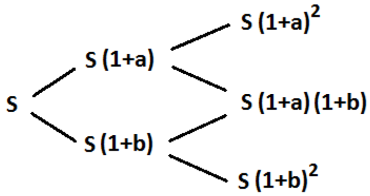
Cox, Ross & Rubinstein model

- It was first proposed by Cox, Ross and Rubinstein in 1979
- Binomial option pricing model
- Multi-period
- Discrete-time version of the Black-Scholes model

- One risky asset with price S_n, $0 \leq n \leq N$
- One riskless asset with return r: $S_n^0 = (1+r)^n$

Assumption: At any point in time, the relative price change for the risky asset is either a or b,

$$S_{n+1} = S_n(1+a) \qquad \text{or} \qquad S_{n+1} = S_n(1+b)$$



Formulation of the problem

Assume that the initial price S_0 is known and define

$$T_n = \frac{S_n}{S_{n-1}}, \qquad n = 1, \ldots, N.$$

Let

$$\Omega = \{1 + a, 1 + b\}^N, \qquad \mathcal{F} = \mathcal{P}(\Omega),$$

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \qquad \mathcal{F}_n = \sigma\{S_1, \ldots, S_n\} = \sigma\{T_1, \ldots, T_n\}.$$

$$\mathbb{P}\{(x_1, \ldots, x_N)\} = \mathbb{P}(T_1 = x_1, \ldots, T_N = x_N)$$

Discounted Price: $\tilde{S}_n = S_n / S_n^0$.

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3) If $r \leq a$ there exist arbitrage strategies.

- At time 0 borrow an amount S_0 and purchase 1 share of the risky asset.
- At time N pay the loan back and sell the asset.

Profit:

$$S_N - S_0(1+r)^N \geq S_0(1+r)^N - S_0(1+r)^N = 0$$

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$$\mathbb{P}(T = 1 + a) = p := \frac{b - r}{b - a}.$$

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Let C_n, P_n be the prices at time n of a European call/put on a share of stock with strike price K and maturity N.

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c(n,x) &= (1+r)^{-(N-n)}\mathbb{E}^*\left[\left(x\prod_{i=n+1}^{N}T_i-K\right)_+\right] \\
&= \frac{\sum_{j=0}^{N-n}\binom{N-n}{j}p^j(1-p)^{N-n-j}\left(x(1+a)^j(1+b)^{N-n-j}-K\right)_+}{(1+r)^{(N-n)}}
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6) The replicating strategy H_n is a function of n and S_{n-1}.

$$H_n^0(1+r)^n + H_n S_n = c(n, S_n)$$

$$\begin{cases} H_n^0(1+r)^n + H_n S_{n-1}(1+a) = c\left(n, S_{n-1}(1+a)\right) \\ H_n^0(1+r)^n + H_n S_{n-1}(1+b) = c\left(n, S_{n-1}(1+b)\right) \end{cases} \Rightarrow$$

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Y_N = \sum_{i=1}^{N} X_i, \qquad X_i = \log\frac{T_i}{1+r}.
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$$Var^*(X_i) = \left(\log \frac{1+a}{1+r} - \log \frac{1+b}{1+r}\right)^2 p(1-p) := \sigma_N^2.$$

CLT (for triangular arrays): $\dfrac{\sum_{i=1}^N X_i - N\mu_N}{\sqrt{N}\sigma_N} \xrightarrow{d} N(0,1)$

If $N\mu_N \to \mu$ and $\sqrt{N}\sigma_N \to \sigma$, then

$$Y_N = \sum_{i=1}^N X_i \xrightarrow{d} Y \sim N(\mu, \sigma^2).$$

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CLT (for triangular arrays): $\dfrac{\sum_{i=1}^N X_i - N\mu_N}{\sqrt{N}\sigma_N} \xrightarrow{d} N(0,1)$

If $N\mu_N \to \mu$ and $\sqrt{N}\sigma_N \to \sigma$, then

$$Y_N = \sum_{i=1}^N X_i \xrightarrow{d} Y \sim N(\mu, \sigma^2).$$

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Let R (instantaneous rate) be such that $r = RT/N$, i.e.

$$e^{RT} = \lim_{N\to\infty} (1 + r)^N.$$

Let $\psi(y) := (Ke^{-RT} - S_0 e^y)_+$.

$$\left| P_0^{(N)} - \mathbb{E}^* [\psi(Y_N)] \right| \leq K \left| \left(1 + \frac{RT}{N}\right)^{-N} - e^{-RT} \right| \xrightarrow{N\to\infty} 0.$$

Hence, since ψ is a continuous bounded function, it holds

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Assume

$$\log\frac{1+b}{1+r}=\frac{\sigma}{\sqrt{N}},\qquad \log\frac{1+a}{1+r}=-\frac{\sigma}{\sqrt{N}}$$

Then

$$N\mu_N\to-\frac{\sigma^2}{2},\qquad \sqrt{N}\sigma_N\to\sigma.$$

Hence,

$$Y_N\xrightarrow{d}Y\sim N\left(-\frac{\sigma^2}{2},\sigma^2\right).$$

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We just need to calculate $\mathbb{E}\left[\left(Ke^{-RT} - S_0 e^{-\sigma^2/2+\sigma Y}\right)_+\right]$ for $Y \sim N(0,1)$.

Define

$$d_1 = \frac{1}{\sigma}\left[\log\frac{S_0}{K} + RT + \frac{\sigma^2}{2}\right], \qquad d_2 = d_1 - \sigma.$$

Then

$$Ke^{-RT} \geq S_0 \exp\left(-\frac{\sigma^2}{2} + \sigma Y\right) \Leftrightarrow Y \leq -d_2.$$

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\begin{aligned}
&\mathbb{E}\left[\left(Ke^{-RT} - S_0 e^{-\sigma^2/2+\sigma Y}\right)_+\right] \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d2} \left[Ke^{-RT} - S_0 e^{-\sigma^2/2+\sigma y}\right] e^{-\frac{y^2}{2}} \,\mathrm{d}y
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where F is the cdf of the standard normal distribution.

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Call/Put parity equation:

$$C_0 = S_0 F(d_1) - Ke^{-RT} F(d_2).$$

Summary

Limitations of the model

- In the real-world markets, the prices can take any positive value;
- Trading takes place almost continuously;

Advantages

- Relatively simple;
- Gives results comparable to the Black-Scholes model;