Arbitrage and Pricing Theory

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Overview

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Why derivatives?

- insuring against price movements (hedging)
- increasing exposure to price movements
- speculation
- getting access to hard-to-trade assets or markets

Common types of derivatives – Futures

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[1] 2 Sep 2016

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Example: the today[1] (spot) price of 1 OZ of gold is $1,325.36$ USD, but the price for a future contract GCG17 (Feb ’17) is $1,329.9$ USD.

Such a contract allows e.g. an investor who needs gold in Feb ’17 to protect against uprising of the price (hedge risk).

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Such a contract allows e.g. an investor who needs gold in Feb '17 to protect against uprising of the price (hedge risk).

Of course it also allows for speculation. Suppose we buy now the contract for 1 OZ gold and at the delivery month (Feb) the spot price of gold has become $1,339.9$ USD. Then we could sell our 1 OZ of gold and obtain a net gain of

$$1,339.9 - 1,329.9 = 10\text{USD}$$

If the price is lower than $1,329.9$ USD, our gains will become losses.

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European option: If the holder can exercise only on the expiration date

American option: If the holder can exercise at any time before the expiration date.

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- gold is worth $X > 1,330 + 2$ USD per OZ. Then, we exercise our right, buy gold and then sell it immediately on the market at X USD, thus getting $X - 1,330$ USD. In total, we gain $X - 1,330$ USD, but we have to subtract the initial 2 USD, hence we made profit (we are in the money).

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- gold is worth less than $X = 1,330$ USD per OZ. There is no reason to exercise our right, so we end up with 2 USD losses (out of the money).

Financial leverage

Options allows us to amplify gains (but losses as well): in the previous example, with only 2 USD I could enter the gold market and (partially) act as if I had 1 OZ of gold (which would cost $1,320$ USD today).

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- (almost) instantaneously: e.g. the same good is sold at different prices (mispricing) on different financial markets and no execution risks (transportation costs)
- in future date: e.g. we make sure profit by buying something today which is worth more tomorrow (and no storage costs. . .).

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Common assumption is that arbitrage opportunities do NOT exist in reality.

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For simplicity, we consider only the present $t = 0$ and a fixed future $t = 1$.

At $t = 0$ we have the observed spot prices of the N securities

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Example

$$a_1 = \text{"gold"},\ a_2 = \text{"future contract on gold"},\ a_3 = \text{"call option on gold"}$$

$$t = 0 \ \rightarrow \text{today} \qquad t = 0 \ \rightarrow \text{Feb '17}$$

$$s \in \{\text{possible prices of gold on Feb '17}\}$$

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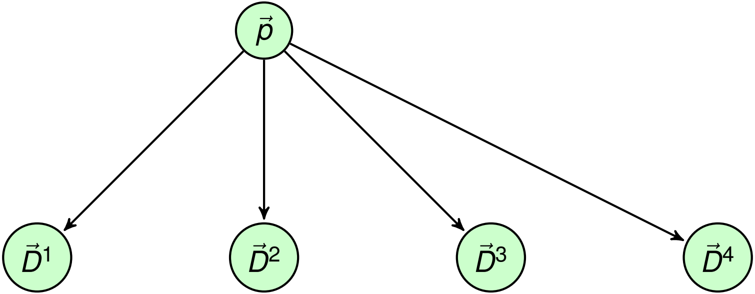

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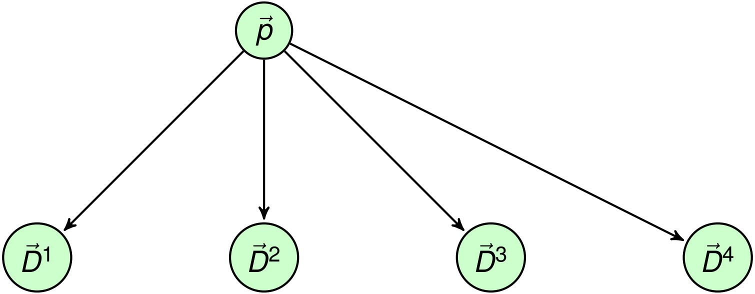

This resembles a Markov chain...but we have no transition probabilities $(\hat{\pi}_1, \ldots, \hat{\pi}_M)$.

Portfolio

An investor position $\vec{\theta}$ on the market represented by the amounts of securities

$$\theta^i \in \mathbb{R} \quad (\theta^i > 0 \text{ if he is short, } \theta^i < 0 \text{ if he is long on } a_i)$$

for $i \in \{1, \ldots, N\}$.

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This distinction usually does not matter too much if

the free-lunch today can be “safely” invested to get free-lunch tomorrow.

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Conversely, if there is such π, then there cannot be arbitrage opportunities.

Prices are discounted expected values w.r.t. risk-neutral probability

Define R and $\hat{\pi}$ by the relation

$$1 + R = \frac{1}{\sum_{s=1}^{M} \pi_s}, \quad \hat{\pi}_s = \frac{\pi_s}{\sum_{r=1}^{M} \pi_r}.$$

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Corollary (prices are discounted expectations)

For every $i \in \{1, \ldots, N\}$

$$p_i = \frac{1}{1+R} E[D_i] = \frac{1}{1+R} \sum_{s=1}^{M} D_i^s \hat{\pi}_s.$$

No-arbitrage assumption $\Rightarrow$ existence of some “transition probabilities”

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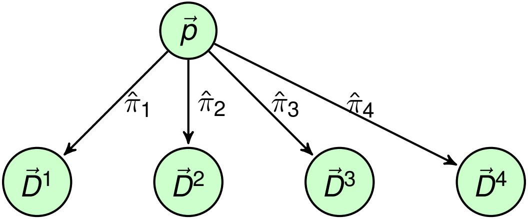

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and the linear subspace of $\mathbb{R}^{M+1}$:

$$L = \left\{ (-\vec{\theta} \cdot \vec{p}, \vec{\theta} \cdot D^1, \vec{\theta} \cdot D^2, \ldots, \vec{\theta} \cdot D^M) \, : \, \vec{\theta} \in \mathbb{R}^M \right\}.$$

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$$L = \left\{(-\vec{\theta} \cdot \vec{p}, \vec{\theta} \cdot D^1, \vec{\theta} \cdot D^2, \ldots, \vec{\theta} \cdot D^M) \,:\, \vec{\theta} \in \mathbb{R}^M\right\}.$$

If $R_+^{M+1} \cap L \neq \{0\}$, we have an arbitrage opportunity taking $\vec{\theta}$ such that

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and not all equal to 0.

Assume now that no arbitrage opportunities exist. The argument is geometrical.

Consider the cone in the $M+1$ dimensional space

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Therefore the sets $R_+^{M+1} \setminus \{0\}$ and L are disjoint.

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which separates $R_+^{M+1} \setminus \{0\}$ and L, i.e.

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Hence, we define $\pi_s := \lambda^s / \lambda_0$ and obtain

$$\vec{p} = \sum_{s=1}^{M} \vec{D}^s \pi_s.$$

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may be true, but actual market investors act on the basis of more information!