Stochastic Processes and Stochastic Calculus
Value at Risk

Scuola Normale Superiore, Pisa, Italy

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Value at Risk (VaR)

- Bank regulators require banks to hold capital to cover the risk associated with their activities
- Capital requirements are specified in international agreements: Basel Accords
- VaR was first introduced by JP Morgan (1994)
- Basel II (2006): VaR approach for risk measurement
- Accurate measurement of risk is an essential step for risk management.

Value at Risk (VaR)

How bad can things get?

- VaR is a single number which summarizes the estimated worst loss over a time horizon T within a given confidence interval $(1-\alpha)$.
- If a bank has calculated that the 10-day 99% VaR is $10 million, it has been estimated that, with probability 99%, the maximum loss will not exceed $10 million in 10 days.

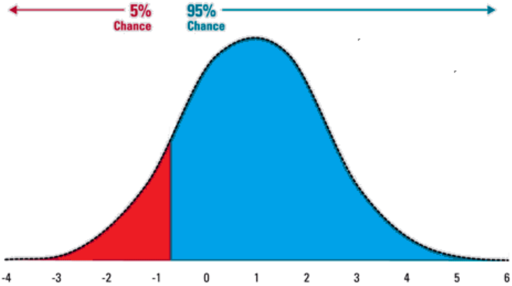
Value at Risk (VaR)

In probabilistic terms, VaR is a quantile.
Let X represent the daily returns of a portfolio. Choose $\alpha \in (0,1)$.

Definition

$$\mathrm{VaR}_\alpha(X) = -q_\alpha(X) = -\inf\{t : \mathbb{P}(X \le t) > \alpha\}$$

If X has a an invertible cdf F, then $q_\alpha = F^{-1}(\alpha)$

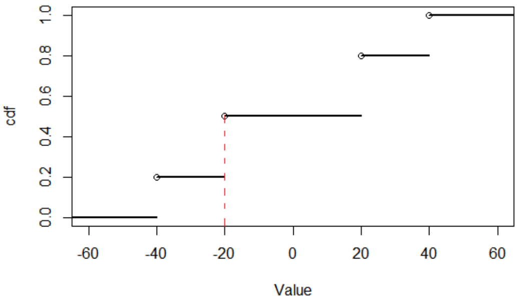


$$\mathrm{VaR}_{0.05} = 0.82$$

An example

Assume that the 1-year return of a portfolio is characterized by the following discrete distribution

Value (mln $)	-40	-20	20	40
Probability	0.2	0.3	0.3	0.2



$q_{0.2} = -20$ $\qquad$ $q_{0.4} = -20$

Properties

1. Monotonicity: $X \leq Y \Rightarrow \mathrm{VaR}(X) \geq \mathrm{VaR}(Y)$;
2. Translation invariance: $\mathrm{VaR}(X+k) = \mathrm{VaR}(X) - k$, $k \in \mathbb{R}$;
3. Homogeneity: $\mathrm{VaR}(\lambda X) = \lambda \, \mathrm{VaR}(X)$, $\lambda > 0$;

Disadvantages:

1. Does not satisfy subadditivity: $\mathrm{VaR}(X+Y)$ might be greater than $\mathrm{VaR}(X) + \mathrm{VaR}(Y)$;
2. Does not give any information on the losses that exceed VaR;

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Esempio 1. Assume that the returns of two independent projects have the following distributions:

Value (mln \$)	-100	-4	3
Probability	0.03	0.02	0.95

Hence $\mathrm{VaR}_{0.04}(A) = \mathrm{VaR}_{0.04}(B) = \$4$ mln.

When the projects are put in the same portfolio, the distribution of the returns is:

Value (mln \$)	-200	-104	-97	-8	-1	6
Probability	0.0009	0.0012	0.057	0.0004	0.038	0.9025

$\mathrm{VaR}_{0.04}(A + B) = \$97 \text{ mln} > \mathrm{VaR}_{0.04}(A) + \mathrm{VaR}_{0.04}(B)$.

Esempio 2. Consider one asset A which might result in a loss of \$ 100 mln, \$ 4 mln or a gain of \$ 3 mln with probability 0.02, 0.04 and 0.94 respectively.

$$\mathrm{VaR}_{0.05}(A) = \$4\text{mln}.$$

Let B be another asset which might result in a loss of \$ 800 mln, \$ 4 mln or a gain of \$ 3 mln with probability 0.02, 0.04 and 0.94 respectively.

$$\mathrm{VaR}_{0.05}(B) = \$4\text{mln}.$$

$$\mathrm{VaR}_{0.05}(A) = \mathrm{VaR}_{0.05}(B).$$

Calculation of VaR - Parametric method

Idea:

Assume that the daily returns X of a portfolio follow a normal distribution $N(\mu, \sigma^2)$. Then,

$$q_\alpha = \mu + \sigma z_\alpha$$

where z_α is the α-quantile of the standard normal distribution.

We just need to estimate the parameters μ, σ based on historical data.

What if X depends on many risk factors (prices, interest rates, spreads...)?

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Calculation of VaR: delta-normal approach

- Assume that the vector of risk factors $R = (R_1, \dots, R_N)$ follows a multivariate normal distribution $N(m, \Sigma)$;
- Assume that the portfolio value P is a linear function of $R_1, \dots, R_N$:

$$P = \sum_{i=1}^{N} w_i R_i;$$

- It follows that $P \sim N(\mu_P, \sigma_P^2)$ with

$$\mu_P = \sum_{i=1}^{N} w_i m_i, \qquad \sigma_P^2 = w \Sigma w^T.$$

Note that if Γ is the correlation matrix and $\sigma = (\sigma_1, \dots, \sigma_N)$ is the vector of volatilities, then

$$\Sigma = \sigma \Gamma \sigma^T.$$

An example

Consider a portfolio which considers in a long position \$ 100 mln worth of British pounds sterling (GBPs) and a short position \$100 mln worth of Euros.

Assume that returns are normally distributed with mean zero and a daily standard deviation of $\sigma_1 = 80$ basis points for the Euro and $\sigma_2 = 70$ basis points for the GBP.

Assume a correlation of $\rho = 0.8$ between the GBP and the Euro rates.

- VaR for the long position:

$$\mathrm{VaR}^{l}_{0.05} = -100 * \sigma_2 * z_{0.05} = 100 * 0.007 * 1.645 = \$1.151\text{mln}$$

- VaR for the short position:

$$\mathrm{VaR}^{s}_{0.05} = -100 * \sigma_1 * z_{0.05} = 100 * 0.008 * 1.645 = \$1.316\text{mln}$$

An example

The return of the portfolio is normally distributed $N(0, \sigma_P^2)$ with

$$\sigma_P^2 = 100^2\sigma_1^2 + 100^2\sigma_2^2 - 2 * 100 * 100 * \rho * \sigma_1 * \sigma_2 = 0.234$$

VaR of the entire portfolio:

$$\text{VaR}_{0.05}^P = -\sigma_P z_{0.05} = \$0.795\text{mln}$$

$$\text{VaR}_{0.05}^P < \text{VaR}_{0.05}^l + \text{VaR}_{0.05}^s$$

With a high correlation between the two risk factors (the \$/Euro rate and the \$/GBP rate), it is more likely that gains on one part of the trade compensate losses on the other.

Calculation of VaR - Parametric method

- Choice of the time horizon

$$T - day\text{VaR} \approx 1 - day\text{VaR} * \sqrt{T}$$

This is the exact formula if the changes in successive days are independent and $N(0, \sigma^2)$.

- Back-testing: Examines how well VaR has performed in the past calculating the percentage of losses exceeding VaR.

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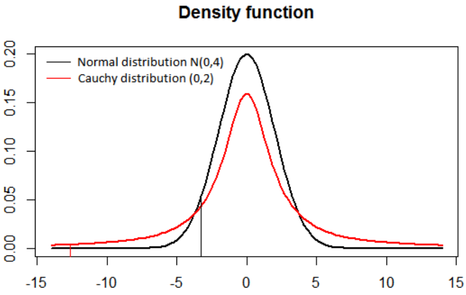
Misspecification

If we assume normality but the distribution is heavy-tailed, VaR is being underestimated at high confidence levels and overestimated otherwise.



$N(0,4) : q_{0.05} = -3.29$ $\qquad$ $\text{Cauchy}(0,2) : q_{0.05} = -12.63$

Calculation of VaR - Full valuation methods

Estimate future values of the portfolio and calculate empirical quantiles

1. Historical method: use actual daily returns from the past period assuming that the future will be like the past;
    - No artificial assumption of normal distribution;
    - Limited by the data set available;
    - Sensitive to the non-stationarity assumption
2. Monte Carlo simulation: random simulation of future data assuming a particular stochastic model;
    - Need to assume a stochastic model;
    - Computationally more expansive;

Summary

- Easy calculation and interpretation
- It can be used for any type of portfolio

Even if it is the most widely used risk measure, it is not the best method!

"An airbag that works all the time except when you have a car accident..."
D. Einhoorn "Global Association of Risk Professionals Review" (2008)