Stochastic Processes and Stochastic Calculus - 9
Complete and Incomplete Market Models

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Overview

Self-financing portfolio

Consider a market consisting in $d+1$ assets with prices

$$(S_t^0, S_t^1, \dots S_t^d)_{t\geq 0}.$$

A portfolio is self-financing if its value changes only because the asset prices change.
No money is withdrawn or inserted after the initial forming of the portfolio.

- A portfolio strategy $(H_t^0, H_t)_{t\geq 0}$ is an $(d+1)$-dim adapted process
- The corresponding value process is

$$V_t = \sum_{i=0}^{d} H_t^i S_t^i = H_t^0 S_t^0 + H_t \cdot S_t$$

- A portfolio is self-financing if

$$\Delta V_n = H_n^0 \Delta S_n^0 + H_n \cdot \Delta S_n \quad \text{(discrete time)}$$

$$\mathrm{d}V_t = H_t^0 \mathrm{d}S_t^0 + H_t \cdot \mathrm{d}S_t \quad \text{(continuous time)}$$

Self-financing portfolio

In terms of discounted prices:

$$\mathrm{d}\tilde{V}_t = H_t \cdot \mathrm{d}\tilde{S}_t$$

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Proposition

For any adapted process $H_t = (H_t^1, \dots, H_t^d)_{t \geq 0}$ and any initial value $V_0 = x$, there exists a unique adapted process $(H_t^0)_{t \geq 0}$ such that the strategy $(H_t^0, H_t)_{t \geq 0}$ is self-financing.

Proof.

$$\tilde{V}_t = x + \int_0^t H_s \cdot \mathrm{d}\tilde{S}_s = H_t \cdot \tilde{S}_t + H_t^0$$

$$H_t^0 = x + \int_0^t H_s \cdot \mathrm{d}\tilde{S}_s - H_t \cdot \tilde{S}_t.$$

□

Complete markets

Definition

An $\mathcal{F}_T$-measurable random variable X is an attainable claim if there exists a self-financing portfolio worth X at time T.

Definition

A market is complete if every contingent claim is attainable.

Theorem

Assume that the market is arbitrage-free. Then, the following two statements are equivalent:

- *the market is complete*
- *the martingale probability is unique.*

Complete market models

- It is theoretically possible to perfectly hedge contingent claims.
- Gives a unique no-arbitrage price.
- Allows us to derive a simple theory of pricing and hedging.
- Is a rather restrictive assumption.

Black-Scholes model

Assuming a constant volatility B-S model gives a unique no-arbitrage price of an option

$$\mathrm{d}S_t = S_t(\mu \mathrm{d}t + \sigma \mathrm{d}B_t)$$

The pricing formula depends only on one non-observable parameter: σ

$$C(t, S_t) = xN(d_1) - Ke^{-r(T-t)}N(d_2),$$

where

$$d_{1,2} = \frac{\log(S_t/K) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

In practice two methods are used to evaluate σ.

Black-Scholes model

1. **The historical method**: Since

$$S_T = S_0 \exp\left[\sigma B_T - \left(\mu - \frac{\sigma^2}{2}\right) T\right]$$

the random variables

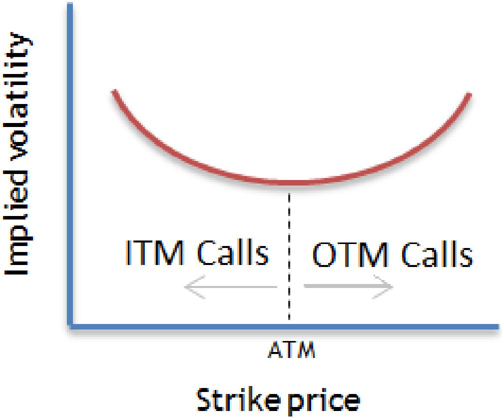
$$\log\left(\frac{S_T}{S_0}\right), \log\left(\frac{S_{2T}}{S_T}\right), \ldots, \log\left(\frac{S_{NT}}{S_{(N-1)T}}\right),$$

are independent Gaussian distributed with variance $\sigma^2 T$.
Estimate σ using asset prices observed in the past.

2. **The implied method**: we recover σ by inversion of the Black-Scholes formula using quoted options.
No explicit formulas! Numerical methods need to be used.

Volatility smile

Options based on the same underlying but with different strike and expiration time yield different implied volatilities.



Time-dependent volatility models

$$\mathrm{d}S_t = S_t \left(\mu(t)\mathrm{d}t + \sigma(t)\mathrm{d}B_t \right)$$

Similar formulas as in the Black-Scholes model replacing

$$\sigma^2(T-t) \rightsquigarrow \int_t^T \sigma^2(s)\mathrm{d}s.$$

$$S_t = S_0 \exp\left(\int_0^t \left(\mu(s) - \frac{\sigma^2(s)}{2} \right) \mathrm{d}s + \int_0^t \sigma(s)\mathrm{d}B_s \right)$$

Does not avoid the volatility smile!

Local volatility models

The volatility depends on the time and the stock price:

$$\mathrm{d}S_t = S_t\left(\mu(t, S_t)\mathrm{d}t + \sigma(t, S_t)\mathrm{d}B_t\right)$$

Note that $\mathcal{F}_t^S = \mathcal{F}_t^{B^*}$. The market is still complete.

For each $X \in L^2(\mathcal{F}_T^B, \mathbb{P}^*)$, there exists a replicating portfolio

$$V_t = e^{-r(T-t)}\mathbb{E}^*[X|\mathcal{F}_t] = F(t, S_t), \qquad H_t = \frac{\partial F}{\partial x}(t, S_t).$$

Need for more realistic models...

- In local volatility models, σ is perfectly correlated with the stock price.
- Empirical studies reveal that the previous models can not capture heavy tails and asymmetries present in log-returns in practice.
- The real market is incomplete.

Stochastic volatility models

Model volatility as a random process driven by its own source of randomness.

It is consistent with the highly variable and unpredictable nature of volatility.

Let B_t^1, B_t^2 be two independent Brownian motions.

$$\begin{cases} \mathrm{d}S_t = S_t \left(\mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t^1 \right) \\ \mathrm{d}\sigma_t = \alpha(t, \sigma_t)\mathrm{d}t + \beta(t, \sigma_t)\mathrm{d}B_t^2 \end{cases}$$

Stochastic volatility models

Let $B_t := (B_t^1, B_t^2)$ and $\mathcal{F}_t = \mathcal{F}_t^B$.

Girsanov theorem: $\left(B_t - \int_0^t H_s \, \mathrm{d}s\right)_t$ is a 2-dim $\mathbb{P}^*$-Brownian motion

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(\int_0^T H_s \, \mathrm{d}B_s - \frac{1}{2}\int_0^T \|H_s\|_2^2 \, \mathrm{d}s\right)$$

i.e.

$$\widehat{B}_t^1 := B_t^1 - \int_0^t H_s^1 \, \mathrm{d}s \qquad \text{and} \qquad \widehat{B}_t^2 := B_t^2 - \int_0^t H_s^2 \, \mathrm{d}s$$

are two independent Brownian motions w.r.t. $\mathbb{P}^*$.

Stochastic volatility models

If

$$H_t^1 = -\frac{\mu_t - r}{\sigma_t},$$

then

$$\mathrm{d}S_t = S_t \left(r\mathrm{d}t + \sigma_t \mathrm{d}\widehat{B}_t^1 \right)$$

which means that the discounted price is a $\mathbb{P}^*$-martingale

$$\mathrm{d}\tilde{S}_t = \tilde{S}_t \, \sigma_t \, \mathrm{d}\widehat{B}_t^1.$$

There is no restriction on the process H_t^2.

Consequently, there are many probability measures under which the traded asset is a martingale.

Stochastic volatility models

Note that $\mathcal{F}_t^S \supsetneq \mathcal{F}_t^{B^1}$. The model is not complete!

Let $X \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}^*)$. By martingale representation theorem:

$$\tilde{X} = X_0 + \int_0^T K_s^1 \,\mathrm{d}\widehat{B}_s^1 + \int_0^T K_s^2 \,\mathrm{d}\widehat{B}_s^2$$

for some processes K_t^1, K_t^2. Hence

$$\tilde{X} = X_0 + \int_0^T \frac{K_s^1}{\sigma_s \tilde{S}_s} \,\mathrm{d}\tilde{S}_s + \int_0^T K_s^2 \,\mathrm{d}\widehat{B}_s^2$$

But the second integral can not be written as an integral w.r.t. $\mathrm{d}\tilde{S}_s$.

Incomplete market models

Under a stochastic volatility model, the market is incomplete.

- No unique price.
- More random sources than traded assets.
- It is not always possible to hedge a generic contingent claim.
- Captures more empirical characteristics.

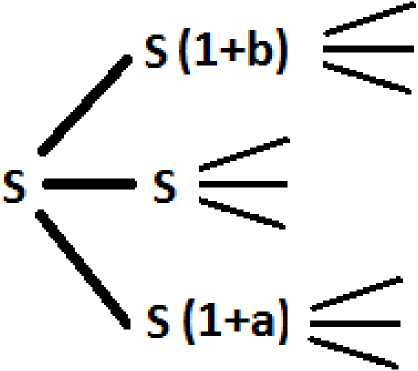
Limitations

- Analytically less tractable.
- No closed form solutions for option prices. Option prices can only be calculated by simulation
- The practical applications of stochastic volatility models are limited.

Trinomial model

An attempt to improve the Binomial Model (CRR)...

We add a third possible state at which the stock price will not change.



Trinomial Model

Absence of arbitrage $\Rightarrow \quad a < R < b.$

Indeed, absence of arbitrage implies the existence of a probability $\mathbb{P}^*$ such that discounted prices are $\mathbb{P}^*$-martingales.

Let $\mathbb{P}^*(S_1 = 1 + a) = p_1$ and $\mathbb{P}^*(S_1 = 1 + b) = p_2$.
Then,

$$S_0 = \mathbb{E}^* \left[\frac{S_1}{1+R} \right] = \frac{S_0(1+a)p_1 + S_0(1+b)p_2 + S_0(1-p_1-p_2)}{1+R},$$

or equivalently

$$1 + R = (1+a)p_1 + (1+b)p_2 + (1-p_1-p_2).$$

Hence, necessarily $a < R < b$.

Pricing in the Trinomial Model

For the one-step trinomial model, the discounted price is a $\mathbb{P}^*$-martingale if and only if

$$1 + R = (1 + a)p_1 + (1 + b)p_2 + (1 - p_1 - p_2),$$

where

$$\mathbb{P}^*(S_1 = 1 + a) = p_1 \qquad \text{and} \qquad \mathbb{P}^*(S_1 = 1 + b) = p_2.$$

We have to solve one equation with two unknown quantities.

No unique risk-neutral price!

Hedging in the Trinomial Model

Consider a financial derivative on the asset S with value

$$X_t = f(S_t).$$

At time 0, we want to construct a hedging strategy for X_1

$$H_1^0 S_1^0 + H_1 S_1 = f(S_1).$$

Hence, (H_1^0, H_1) must satisfy

$$\begin{cases} H_1^0 S_1^0 + H_1 S_0(1+a) = f\left(S_0(1+a)\right) \\ H_1^0 S_1^0 + H_1 S_0 = f\left(S_0\right) \\ H_1^0 S_1^0 + H_1 S_0(1+b) = f\left(S_0(1+b)\right) \end{cases} \tag{1}$$

We have to solve a system of three equations with two unknown quantities.

We are unable to replicate the portfolio!