# Stochastic Processes and Stochastic Calculus - 8
# Stochastic Differential Equations
# and their link with Partial Differential Equations

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# Overview

1. Stochastic Differential Equations
   - Definitions
   - Existence and uniqueness
   - Examples
2. SDE's and Partial Differential Equations
3. Black-Scholes formulas
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# Ordinary differential equations

By ordinary differential equation we mean an equation whose unknown is a curve (function)

$$x : [t_0, T] \to \mathbb{R},$$

$$\begin{cases} \frac{dx}{dt}(t) = f(t, x(t)) & \text{for } t \in (t_0, T), \\ x(t_0) = x_0 \end{cases}$$

We can think of $f(t, y)$ as a prescribed velocity for the curve, at time $t$, if its position is $y$.

A solution is therefore a function such that

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \quad \text{for } t \in [t_0, T].$$

## Example

If $f(t, y) = ay$, then the equation is

$$\frac{dx}{dt}(t) = ax(t) \quad \Rightarrow \quad x(t) = x_0 e^{a(t-t_0)}.$$

# Stochastic Differential Equations - SDE's

In a similar way, we call Stochastic differential equation an equation whose unknown is a Itô process

$$X : \Omega \times [0, T] \to \mathbb{R},$$

in the form – using the "differential notation"

$$\begin{cases} dX_t = \alpha(t, X_t)dt + \beta(t, X_t)dB_t & \text{for } t \in (0, T), \\ X_0 = x_0 \end{cases}$$

In integral form, we mean that

$$X_t = x_0 + \int_0^t \alpha(s, X_s)ds + \int_0^t \beta(s, X_s)dB_s \quad \text{for } t \in [0, T].$$

We can think of

- $\alpha(t, y)$ as a prescribed velocity (drift) for the process, at time $t$, if its position is $y$.
- $\beta(t, y)$ as a prescribed intensity of "oscillations" (diffusion coefficient) for the process, at time $t$, if its position is $y$.

# Existence and uniqueness results

You may recall a classical result for ordinary differential equations (theorem of Cauchy-Lipschitz), stating that

$$\begin{cases} \frac{dx}{dt}(t) = f(t, x(t)) \quad \text{for } t \in (t_0, T), \\ x(t_0) = x_0 \end{cases}$$

admits a unique solution (for a small interval) around $t_0$, if

$f(t, y)$ is continuous and uniformly Lipschitz w.r.t. $y$, i.e. for some $K > 0$
$|f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2|$, for every $y_1, y_2 \in \mathbb{R}$.

## Theorem (Existence and uniqueness for SDE's)

Assume that drift $\alpha$ and diffusion $\beta$ are continuous and satisfy, for every $t \in [0, T]$, $y_1, y_2 \in \mathbb{R}$,

$$|\alpha(t, y_1) - \alpha(y_2)| \leq K|y_1 - y_2| \quad \text{and} \quad |\beta(t, y_1) - \alpha(y_2)| \leq K|y_1 - y_2|$$

$$|\alpha(t, y)| \leq C(1 + |y|) \quad \text{and} \quad |\beta(t, y)| \leq C(1 + |y|)$$

then the equation

$$\begin{cases} dX_t = \alpha(t, X_t)dt + \beta(t, X_t)dB_t & \text{for } t \in (0, T), \\ X_0 = x_0 \end{cases}$$

has a unique solution on $[0, T]$ (global, i.e. defined for all times).

## Example (geometric Brownian motion)

$$dS_t = S_t \mu dt + S_t \sigma dB_t$$

we have $\alpha(t, y) = y\mu$ and $\beta(t, y) = y\sigma$, $\Rightarrow$ the solution we found is unique:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t\right)$$

# Example (Langevin's equation)

We consider the SDE

$$\begin{cases} dX_t = rX_t dt + \sigma dB_t & \text{for } t \in (0, T), \\ X_0 = x_0 \end{cases}$$

Recall that the linear ordinary differential equation

$$\frac{dx}{dt}(t) = rx(t) + g(t)$$

admits an "explicit formula" for its solution,

$$x(t) = x(0)e^{rt} + e^{rt} \int_0^t e^{-rs} g(s) ds$$

Formally taking $g(t) = \sigma \frac{dB_t}{dt}$, we obtain

$$x(0)e^{rt} + e^{rt} \int_0^t e^{-rs} \sigma \frac{dB_s}{ds} ds = x(0)e^{rt} + e^{rt} \int_0^t e^{-rs} \sigma dB_s$$

where the right hand side in an Itô integral.

One can verify that formal substitution gives the the unique solution

$$X_t = x_0 e^{rt} + \sigma e^{rt} \int_0^t e^{-rs} dB_s$$

called Ornstein-Uhlenbeck process.

# SDE's and Partial Differential Equations (PDE's)

We show that SDE's and certain linear Partial Differential Equations are linked (a simplified version of the Feynmann-Kac formula). The fundamental tool we use is Itô formula.

Assume that $(X_t)_{t\in[0,T]}$ is a solution to the equation

$$\begin{cases} dX_t = rX_t dt + \sigma(t, X_t) dB_t & \text{for } t \in (0, T), \\ X_0 = x_0 \end{cases}$$

Let us consider the linear differential operator

$$A_t = rx \frac{\partial}{\partial x} + \frac{\sigma^2(t,x)}{2} \frac{\partial^2}{(\partial x)^2},$$

acting on functions $u(t,x)$ that are differentiable (at least)

once w.r.t. $t \in (0, T)$ and twice w.r.t. $x \in \mathbb{R}$:

we write

$$(A_t u)(t,x) = rx \frac{\partial u}{\partial x}(t,x) + \frac{\sigma^2(t,x)}{2} \frac{\partial^2 u}{(\partial x)^2}(t,x)$$

We show that the operator $A_t$ appears when we apply Itô formula. Recall that

$$dX_t = rX_t dt + \sigma(t, X_t) dB_t \Rightarrow (dX_t)^2 = (\sigma(t, X_t))^2 \, dt.$$

Let $u(t,x)$ be differentiable once w.r.t. $t \in (0, T)$ and twice w.r.t. $x \in \mathbb{R}$. Then

$$\begin{aligned} d\left(u(t, X_t)\right) &= \frac{\partial u}{\partial t}(t, X_t) dt + \frac{\partial u}{\partial x}(t, X_t) dX_t + \frac{1}{2} \frac{\partial^2 u}{(\partial x)^2}(t, X_t) \left(dX_t\right)^2 \\ &= \left( \frac{\partial u}{\partial t}(t, X_t) + rX_t \frac{\partial u}{\partial x}(t, X_t) + \frac{1}{2} \left(\sigma(t, X_t)\right)^2 \right) dt + \frac{\partial u}{\partial x}(t, X_t) \sigma(t, X_t) dB_t \\ &= \left( \frac{\partial u}{\partial t}(t, X_t) + (A_t u)(t, X_t) \right) dt + \frac{\partial u}{\partial x}(t, X_t) \sigma(t, X_t) dB_t \end{aligned}$$

If we apply Itô formula to $e^{-rt} u(t, X_t)$ we obtain instead

$$d\left(e^{-rt} u(t, X_t)\right) = e^{-rt} \left\{ \left( \frac{\partial u}{\partial t}(t, X_t) + (A_t u)(t, X_t) - ru(t, X_t) \right) dt + \frac{\partial u}{\partial x} \sigma(t, X_t) dB_t \right\}$$

We found (all the functions are evaluated at $(t, X_t)$)

$$d\left(e^{-rt}u(t,X_t)\right) = e^{-rt}\left\{\left(\frac{\partial u}{\partial t} + (A_t u) - ru\right)dt + \frac{\partial u}{\partial x}\sigma dB_t\right\},$$

Assume that $u(t,x)$ is a (classical, differentiable) solution to the PDE

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + (A_t u)(t,x) - ru(t,x) = 0, & \text{for } (t,x) \in (0,T)\times\mathbb{R} \\ u(T,x) = f(x) & \text{for } x \in \mathbb{R} \end{cases}$$

and assume that

$$\int_0^t e^{-rs}\frac{\partial u}{\partial x}(s,X_s)\,\sigma(s,X_s)dB_s$$

is a martingale (e.g. a stochastic integral of the first kind).

$\Rightarrow e^{-rt}u(t,X_t)$ is a a martingale an in particular

$$e^{-rt}u(t,X_t) = E\left[e^{-rT}u(T,X_T)\,|\mathcal{F}_t\right] = E\left[e^{-rT}f(X_T)\,|\mathcal{F}_t\right].$$

# Two different approaches to Black-Scholes equation

Recall that in the Samuelson-Black-Sholes model

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) + \sigma B_t\right), \quad \text{and} \quad S_t^0 = e^{-rt}.$$

We illustrate two approaches to prove

## Thoerem (Black-Scholes formulas)

The price of a call option $(S_T - K)^+$ at time $t \in [0, T]$ is given by

$$C_t(\omega) = C(t, S_t(\omega))$$

where

$$C_t(x) = xN(d_+) + ke^{-r(T-t)}N(d_-)$$

$$N(d) = \int_{-\infty}^{d} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz \quad \text{(c.d.f. of a } \mathcal{N}(0,1))$$

$$d_\pm = \frac{\log\left(\frac{x}{k}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Similar formula for put option $(K - S_t)^+$.

# First approach to BS formulas

We consider a general option $X$ whose value at time $T$ is $f(S_T)$.

We use partial differential equations (original approach by Black-Scholes).

W.r.t. the general case studied before, we have $\sigma(t,x) = \sigma \cdot x \Rightarrow$ the PDE is

$$\begin{cases} \frac{\partial F}{\partial t}(t,x) + rx\frac{\partial F}{\partial x}(t,x) + \frac{\sigma^2 x^2}{2}\frac{\partial^2 F}{(\partial x)^2}(t,x) - rF(t,x) = 0, & \text{for } (t,x) \in (0,T) \times \mathbb{R}^+ \\ F(T,x) = f(x) & \text{for } x > 0 \end{cases}$$

**Theorem**

The no-arbitrage price of the option $X$ at time $t$ is

$$F(t, S_t(\omega)).$$

# Ideas of proof

We use the general result from the previous section, with $\sigma(t,x) = \sigma \cdot x \Rightarrow$

$$e^{-rt}F(t,S_t) = E^*\left[e^{-rT}F(T,S_T)|\mathcal{F}_t\right] = E^*\left[e^{-rT}f(S_T)|\mathcal{F}_t\right]$$

where $E^*$ means that we use the equivalent martingale probability.

To obtain Black-Scholes formulas, we solve explicitly the PDE.

# Second approach to BS formulas

We use stochastic calculus:

1. Every (square integrable) option can be replicated by a self-financing portfolio (martingale representation theorem)
2. Under an equivalent probability $P^*$ (Girsanov theorem) the asset's price is given by
$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma B_t^*\right)$$
where $B_t^*$ is a $P^*$-Brownian motion.
3. The value at time $t \in [0, T]$ of the replicating portfolio is given by
$$V_t = e^{-r(T-t)} E^* \left[ f(S_T) | \mathcal{F}_t \right].$$

$\Rightarrow$ we have to find an expression for $E^* \left[ f(S_T) | \mathcal{F}_t \right]$.

We have to find an expression for $E^* \left[ f(S_T) | \mathcal{F}_t \right]$.

$$S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma B_t^*\right)$$

gives

$$S_T = S_t \exp\left(\left(r - \frac{\sigma^2}{2}\right)(T - t) + \sigma \left(B_T^* - B_t^*\right)\right)$$

By the properties conditional expectation

$$\begin{aligned} E^* \left[ f(S_T) | \mathcal{F}_t \right] &= E^* \left[ f\left( S_t \exp\left(\left(r - \frac{\sigma^2}{2}\right)(T - t) + \sigma \left(B_T^* - B_t^*\right)\right)\right) | \mathcal{F}_t \right] \\ &= G(t, S_t), \end{aligned}$$

where, by independence of increments of $B^*$,

$$G(t, x) = E^* \left[ f\left( x \exp\left(\left(r - \frac{\sigma^2}{2}\right)(T - t) + \sigma \left(B_T^* - B_t^*\right)\right)\right)\right]$$

To compute

$$G(t,x) = E^*\left[f\left(x\exp\left(\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma\left(B_T^*-B_t^*\right)\right)\right)\right]$$

we use the fact that $B_T^* - B_t^* = \sqrt{T-t}Z$, where $Z = \mathcal{N}(0,1)$.

$$G(t,x) = \int_{-\infty}^{+\infty} f\left(x\exp\left(\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma\sqrt{T-t}z\right)\right)\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz.$$

To obtain BS-formula for call, write $f(x) = (x-k)^+$ (we skip some computations on integrals...)

# Comments on the two approaches

1. The first method (PDE's) is "classic" (analytical, does not require stochastic integration) but it works only for particular models (Markovian models)
2. The second method requires "advanced" tools form stochastic integration, but it is open to more general models (possibly non Markovian).