Stochastic Processes and Stochastic Calculus - 7
Two Fundamental Results on Stochastic Integration

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Overview

Equivalent probabilities

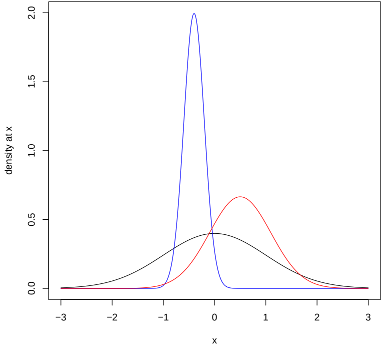
Definition

Two probabilities P^1 and P^2 on the same space $(\Omega, \mathcal{F})$ are said to be equivalent if they have the same null-sets, i.e.

$$P^1(A) = 0 \quad \Leftrightarrow \quad P^2(A) = 0$$

Examples

- Any two Gaussian probabilities $N(\mu_1, \sigma_1^2)$, $N(\mu_2, \sigma_2^2)$ are equivalent (see picture).
- If $B \in \mathcal{F}$ has $0 < P(B) < 1$, then $P^1 = P$ and $P^2 = P(\cdot|B)$ are not equivalent.



Radon-Nikodym theorem

If two probabilities P^1 and P^2 are equivalent, there is a density

$$L(\omega) = \frac{dP^2}{dP^1}(\omega) \quad \text{such that} \quad P^2(A) = \int_A L(\omega) dP^1(\omega)$$

and more generally, for every $Y : \Omega \to \mathbb{R}$

$$E^2[Y] = \int_\Omega Y dP^2 = \int_\Omega Y \cdot \frac{dP^2}{dP^1} dP^1 = \int_\Omega Y \cdot L dP^1 = E^1[Y \cdot L]$$

Some properties of L:

- L must be strictly positive
- $E^1[L] = E^2[1] = 1$.
- $\frac{1}{L}$ is the inverse density $\frac{dP^1}{dP^2}$.

Converse $\Rightarrow$ every r.v. $L > 0$ with $E^1[L] = 1$ yields an equivalent P^2 given by

$$E^2[Y] := E^1[Y \cdot L] = \int_\Omega Y \cdot L dP^1.$$

Example – translation of Gaussians

Let $P^1 = N(0,1)$ on $\Omega = R^1$,

$$P^1(dx) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

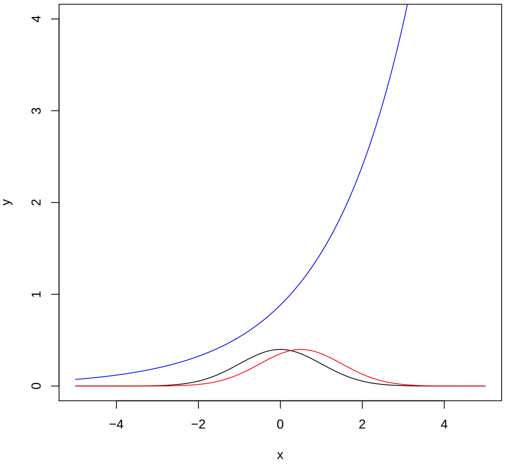
Fix $a \in \mathbb{R} \Rightarrow P^2 = N(a,1)$,

$$P^2(dx) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-a)^2}{2}\right).$$

The density $\frac{dP^2}{dP^1}$ is the quotient

$$\frac{dP^2}{dP^1}(x) = \frac{\exp\left(-\frac{(x-a)^2}{2}\right)}{\exp\left(-\frac{x^2}{2}\right)}$$

$$= \exp\left(ax - \frac{a^2}{2}\right).$$



Girsanov's theorem

Aim: characterize all equivalent probabilities on a a Brownian filtration

Let us start with a simplified version of Girsanov's theorem.

Fix $(\Omega, \mathcal{F}, P)$, where it is defined

- a Brownian motion $(B_t)_{t\in[0,T]}$
- its natural filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and such that $\mathcal{F}_T = \mathcal{F}$.

Theorem

Fix $a \in \mathbb{R}$ and define

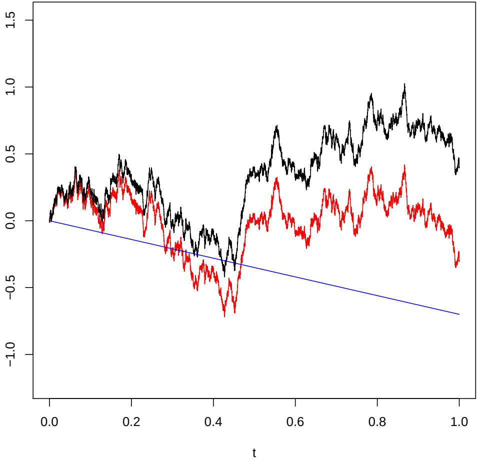
$$L_T(\omega) = \exp\left(aB_T(\omega) - \frac{a^2}{2}T\right).$$

Then

1. $L_T > 0$, a.e., $E[L_T] = 1$,
2. L_T is the density of an equivalent probability $P^\star$ such that
$$\frac{dP^\star}{dP} = L_T.$$
3. The process $(B_t - at)_{t\in[0,T]}$ is a Brownian motion w.r.t. $P^\star$.

A visualization

We “shift” a path (black) of BM by the blue line $-0.7t \Rightarrow$ the red path.



$(B_t - 0.7t)$ is a Browninan motion if we change “weights” the original probability according to L_T.

Sketch of proof

Define

$$L_t(\omega) = \exp\left(aB_t(\omega) - \frac{a^2}{2}t\right)$$

then L_t is a solution to (recall previous lecture)

$$dL_t = aL_t dB_t, \quad L_0 = 1.$$

The property $E[L_T] = 1$ is a simple verification.

Let us see that $B_t - at$ is $N(0,t)$ under P^*. Take any function $\varphi : \mathbb{R} \to \mathbb{R}$

$$\begin{aligned} E^*[\varphi(B_t - at)] &= E[\varphi(B_t - at)L_t] = E\left[\varphi(B_t - at)\exp\left(aB_t - \frac{a^2}{2}t\right)\right] \\ &= \int_{\mathbb{R}} \varphi(\sqrt{t}z - at)\exp\left(a\sqrt{t}z - \frac{a^2}{2}t\right)\frac{\exp\left(-\frac{z^2}{2}\right)}{\sqrt{2\pi}}dz \\ &= \int_{\mathbb{R}} \varphi\left(\sqrt{t}(z - a\sqrt{t})\right)\frac{\exp\left(-\frac{(z-a\sqrt{t})^2}{2}\right)}{\sqrt{2\pi}}dz \\ &= E[\varphi(B_t)] \end{aligned}$$

General Girsanov's theorem

Girsanov's theorem holds for more general "shifts" than $t \mapsto at$.

Let $(K_s)_{s\in[0,T]}$ be an adapted process such that

$$\int_0^T K_s^2 ds < \infty, \qquad P\text{-a.e.}$$

and define

$$L_T = \exp\left(\int_0^T K_s dB_s - \frac{1}{2}\int_0^T K_s^2 ds\right).$$

Notice that in the case $K_s = a$ constant, we recover

$$L_T = \exp\left(aB_T - \frac{a^2}{2}T\right)$$

In general, $L_T > 0$ and always

$$E[L_T] \leq 1 \text{but it can happen } E[L_T] < 1.$$

Theorem (Girsanov)

Define

$$L_T = \exp\left(\int_0^T K_s dB_s - \frac{1}{2}\int_0^T K_s^2 ds\right).$$

If $E[L_T] = 1$, then under the probability P^* defined by the density L_T,

$$\left(B_t - \int_0^t K_s ds\right)_{t\in[0,T]} \quad \text{is a Brownian motion.}$$

The condition $E[L_T] = 1$ is the difficult obstruction to apply the theorem

We have criterion that is sufficient in many practical cases.

Theorem (Novikov's condition)

If $E\left[\exp\left(\frac{1}{2}\int_0^T K_s^2 ds\right)\right] < \infty$, then $E[L_T] = 1$ and Girsanov theorem applies.

Itô integral and martingales

Recall (lecture 5) that every Itô integral of the first kind is a martingale:

$$E\left[\int_0^T H_s^2 ds\right] < \infty \quad \Rightarrow \quad t \mapsto \int_0^t H_s dB_s \text{ is a martingale.}$$

There are many ways to build martingales (lecture 2), e.g.

$$\text{let } X \text{ be a r.v. with } E\left[X^2\right] < \infty \text{ and define } M_t := E\left[X|\mathcal{F}_t\right]$$

where $\mathcal{F}_t$ is the natural filtration of the BM $(B_t)_{t\in[0,T]}$.

Problem: Is every martingale $(M_t)_{t\in[0,T]}$ an Itô integral (of first kind)?

Since Itô integrals of the first kind satisfy

$$E\left[\left(\int_0^t H_s dB_s\right)^2\right] = E\left[\int_0^T H_s^2 ds\right] < \infty, \quad \text{for } t \in [0,T].$$

we restrict the problem to square-integrable martingales, i.e.

$$E\left[M_t^2\right] < \infty, \text{for } t \in [0,T].$$

The answer to the problem is YES if $(\mathcal{F}_t)_{t\in[0,T]}$ is the natural filtration of BM.

Theorem (Martingale representation theorem)

Every martingale $(M_t)_{t\in[0,T]}$ such that

$$E\left[M_t^2\right] < \infty, \text{for } t \in [0, T]$$

can be written as an Itô integral of the first kind

$$M_t = M_0 + \int_0^t H_s dB_s$$

for some adapted process $(H_s)_{s\in[0,T]}$ with $E\left[\int_0^T H_s^2 ds\right] < \infty$.

A consequence (which is in fact equivalent) is that

Corollary

Every square-integrable r.v. X (i.e. $E\left[X^2\right] < \infty$) can be written in the form

$$X = E[X] + \int_0^T H_s dB_s.$$

In reality, one proves first the corollary, i.e. the representation

$$X = E[X] + \int_0^T H_s dB_s.$$

The idea of the proof is that the subspace of random variables $X \in L^2(\Omega, \mathcal{F}, P)$ which are

orthogonal to all stochastic integrals $\int_0^T H_s dB_s$

is reduced to constant random variables.

The process $(H_s)_{s\in[0,T]}$ can be intepreted as a "derivative" of X with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, hence the representation formula is a kind of "fundamental theorem of calculus".

In general, it might be a task to find explicit formulas for $(H_s)_{s\in[0,T]}$.

Combined with Girsanov theorem, the martingale representation theorem gives the following:

Theorem

If the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is the natural filtration of the Brownian motion, all equivalent probabilities can be obtained by the Girsanov theorem, i.e. any density L is of the form

$$L = \exp\left(\int_0^T K_s dB_s - \frac{1}{2}\int_0^T K_s^2 ds\right).$$

Now we consider the consequences of the previous results to the Samuelson-Black-Scholes model.

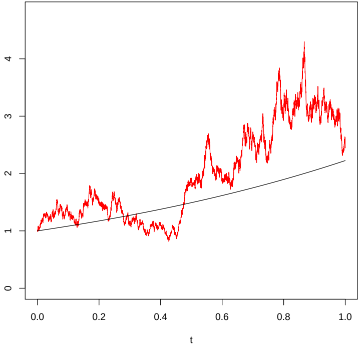
Applications: the Samuelson-Black-Scholes model

We saw (in lecture 6) $\Rightarrow$ Samuelson's equation for financial asset

$$dS_t = S_t\left(\mu dt + \sigma dB_t\right) \Rightarrow S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right)$$

while for a risk-less asset (e.g. bond) is similar (but zero volatility)

$$dS_t^0 = S_t^0 r dt \quad \Rightarrow \quad S_t^0 = \exp\left(rt\right)$$



In the picture:
(red) a path of the risky asset S_t, with $\mu = 1$, $\sigma = 1$
(black) a path of the risk-less asset S_t^0, with $r = 0.8$.

The discounted value of S_t is

$$\tilde{S}_t := \frac{S_t}{S_t^0} = S_0 \left(\left(\mu - r - \frac{\sigma^2}{2} \right) t + B_t \right)$$

in Itô differential notation

$$d\tilde{S}_t = \tilde{S}_t \left((\mu - r)\, dt + \sigma dB_t \right)$$
$$= \sigma \tilde{S}_t d \left(B_t + \left(\frac{\mu - r}{\sigma} \right) t \right)$$

We look for an equivalent probability P^* such that $B_t + \left(\frac{\mu - r}{\sigma}\right) t$ becomes a BM.
⇒ Girsanov theorem gives that

$$\frac{dP*}{dP} = \exp \left(- \left(\frac{\mu - r}{\sigma} \right) B_T - \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 T \right)$$

⇒ under P^* the process $B_t^* = B_t + \left(\frac{\mu - r}{\sigma}\right) t$ is a Brownian motion hence

$$d\tilde{S}_t = \sigma \tilde{S}_t dB_t^* \quad \Leftrightarrow \tilde{S}_t = \tilde{S}_0 + \int_0^t \sigma \tilde{S}_r dB_r^*$$

is a martingale. Hence P^* is a risk-neutral measure.

Moreover, completeness of the market means that every contingent claim X can be written as

$$\tilde{X} := \frac{X}{S_T^0} = c + \int_0^T H_r d\tilde{S}_r,$$

i.e. the final value at time T of a self-financing portfolio.

By the martingale representation theorem, if $\tilde{X}$ is square integrable w.r.t. P^* then

$$\tilde{X} = E^*\left[\tilde{X}\right] + \int_0^T K_r dB_r^*,$$

We can rewrite the stochastic integral in terms of $d\tilde{S}_r$, since

$$d\tilde{S}_r = \sigma \tilde{S}_r dB_r^* \quad \Rightarrow \quad dB_r^* = \frac{1}{\sigma \tilde{S}_r} d\tilde{S}_r,$$

and therefore

$$\tilde{X} = E^*\left[\tilde{X}\right] + \int_0^T \frac{K_r}{\sigma \tilde{S}_r} d\tilde{S}_r.$$

In summary:

- Girsanov theorem $\Rightarrow$ P^* risk-neutral measure (no-arbitrage)
- Martingale representation $\Rightarrow$ self-financing portfolio (market complete)