Stochastic Processes and Stochastic Calculus - 6
Stochastic Integral and Itô's Formula

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Overview

Simple processes

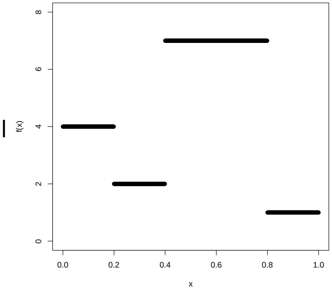
We want to define a stochastic integral

$$\int_0^t H_s dB_s, \quad \text{where } H_s \text{ is a stochastic process}$$

and $(B_s)_{s\in[0,T]}$ is a Brownian motion, with filtration $(\mathcal{F}_s)_{s\in[0,T]}$.

Definition

A simple process $H = (H_s)_{s\in[0,T]}$ is an adapted process with piecewise constant paths



More precisely, H is of the form

$$H_s(\omega) = \begin{cases} H_i(\omega) & \text{for } t_i < s \leq t_{i+1} \\ 0 & \text{for } t_n < s \end{cases}$$

where $0 \leq t_0 < t_1 < \ldots < t_n \leq T$ and each H_i is $\mathcal{F}_{t_i}$-measurable and bounded.

Integral of simple processes

Given a simple process

$$H_s(\omega) = \sum_{i=0}^{n-1} H_i(\omega) I_{\{t_i < s \le t_{i+1}\}}$$

we define its integral

$$\int_0^T H_s dB_s = \sum_{i=0}^{n-1} H_i(\omega) \left(B_{t_{i+1}} - B_{t_i}\right)$$

and for $t \in [0, T]$ such that $t_j < t \le t_{j+1}$, we define

$$\int_0^t H_s dB_s = \sum_{i=0}^{j-1} H_i(\omega) \left(B_{t_{i+1}} - B_{t_i}\right) + H_j \left(B_t - B_{t_j}\right).$$

Let us denote $\mathcal{I}_t(H) = \int_0^t H_s dB_s$ the stochastic process thus defined.

Properties of Itô stochastic integral

One verifies that

1. $t \mapsto \mathcal{I}_t(H)$ is a martingale (w.r.t. $\mathcal{F}$).
2. $E[\mathcal{I}_t(H)] = E\left[\int_0^t H_s dB_s\right] = 0$
3. $\mathrm{Var}(\mathcal{I}_t(H)) = E\left[\left(\int_0^t H_s dB_s\right)^2\right] = E\left[\int_0^t H_s^2 ds\right]$
4. The quadratic variation of the paths $t \mapsto \mathcal{I}_t(H)$ is

$$[\mathcal{I}(H)]_t = \int_0^t H_s^2 ds.$$

The mnemonic rule for the last identity reads

$$d\left(\int_0^t H_s dB_s\right) = H_t dB_t \quad \Rightarrow \left(d\left(\int_0^t H_s dB_s\right)\right)^2 = (H_t dB_t)^2 = H_t^2 (dB_t)^2 = H_t^2 dt.$$

Remark: The integral $\int_0^T H_s dB_s$ is not (necessarily) a Gaussian r.v.

Itô integral (of the first kind)

By an approximating procedure, using the continuity consequence of isometry property

$$E\left[\left(\int_0^t H_s dB_s\right)^2\right] = E\left[\int_0^t H_s^2 ds\right]$$

we can define the stochastic integral (of the first kind)

$$\int_0^t H_s dB_s \quad \text{provided that}$$

a) $t \mapsto H_t$ is adapted to $\mathcal{F}$ (non-anticipative)
b) $E\left[\int_0^T H_s^2 ds\right] < \infty$.

In this case, all the properties valid for simple processes hold true:

1. $t \mapsto \int_0^t H_s dB_s$ is a martingale (w.r.t. $\mathcal{F}$).
2. $E\left[\int_0^t H_s dB_s\right] = 0$
3. $E\left[\left(\int_0^t H_s dB_s\right)^2\right] = E\left[\int_0^t H_s^2 ds\right]$
4. The quadratic variation of the paths $t \mapsto \int_0^t H_s dB_s$ is $\left[\int_0^\cdot H_s dB_s\right]_t = \int_0^t H_s^2 ds$.

Itô integral (of the second kind)

By a stopping-time procedure, a stochastic integral of the second kind can be defined also if

a) $t \mapsto H_t$ is adapted to $\mathcal{F}$ (non-anticipative)
b) $\int_0^T H_s^2(\omega)ds < \infty$ for P-a.e. ω.

The second condition is weaker than $E\left[\int_0^T H_s^2 ds\right] < \infty$.

Remark: for integrals of the second kind

1. $t \mapsto \int_0^t H_s dB_s$ is a NOT (necessarily) a martingale
2. the paths are continuous, with quadratic variation until time t still given by

$$\left[\int_0^{\cdot} H_s dB_s\right]_t = \int_0^t H_s^2 ds.$$

Itô's processes

We introduce a class of processes for which some stochastic calculus holds.

In some sense we are going to take derivatives and integrate these processes.

Definition

We call Itô's process any stochastic process $(X_t)_{t\in[0,T]}$ in the form

$$X_t = X_0 + \int_0^t H_s dB_s + \int_0^t K_s ds, \quad \text{for } t \in [0, T].$$

where $(H_s)_{s\in[0,T]}$, $(K_s)_{s\in[0,T]}$ are adapted and

$$\int_0^T H_s^2(\omega) ds < \infty, \int_0^T |K_s(\omega)|\, ds < \infty, \quad \text{for } P\text{-a.e. } \omega.$$

For brevity we use the mnemonic "differential" notations

$$X_t = X_0 + \int_0^t H_s dB_s + \int_0^t K_s ds \quad \leftrightarrow \quad dX_t = H_t dB_t + K_t dt$$

Given an Itô process

$$X_t = X_0 + \int_0^t H_s dB_s + \int_0^t K_s ds \quad \leftrightarrow \quad dX_t = H_t dB_t + K_t dt$$

one can prove that H_s and K_s are uniquely determined by X

In order to calculate the quadratic variation of X, we use the mnemonic rules

$$(dt)^2 = 0, \quad dtdB_t = 0, \quad (dB_t)^2 = dt.$$

According to the rules

$$\begin{aligned} d[X]_t &= (dX_t)^2 = (H_t dB_t + K_t dt)^2 = H_t^2 (dB_t)^2 + 2H_t K_t dB_t dt + K_t^2 (dt)^2 \\ &= H_t^2 dt \end{aligned}$$

Theorem

The quadratic variation of $X = X_0 + \int_0^t H_s dB_s + \int_0^t K_s ds$ is

$$[X]_t = \int_0^t H_s^2 ds.$$

Given an Ito's process

$$X_t = X_0 + \int_0^t H_s dB_s + \int_0^t K_s ds$$

and any adapted process $(L_s)_{s\in[0,T]}$, one can define

$$\int_0^t L_s dX_s := \int_0^t L_s H_s dB_s + \int_0^t L_s K_s ds$$

provided that

$$\int_0^t (L_s H_s)^2 \, ds < \infty \quad \text{and} \quad \int_0^t |L_s K_s| \, ds < \infty.$$

"differentiation" of Itô processes - Itô's formula

A fundamental result in stochastic calculus is Itô formula, which extends the chain rule for derivatives.

Recall (chain rule)

If

- $(x_t)_{t \in [0,T]}$ is differentiable, with derivative $\frac{dx}{dt}$,
- $f : \mathbb{R} \to \mathbb{R}$ is differentiable, with derivative $f' : \mathbb{R} \to \mathbb{R}$

then the composition $f(x_t)$ is differentiable and

$$\frac{d}{dt} f(x_t) = f'(x_t) \frac{dx}{dt}, \quad \text{or} \quad f(x_t) = f(x_0) + \int_0^t f'(x_s) \frac{dx}{ds}(s) ds.$$

In some sense, Itô formula is an extension to Itô processes $x_t = X_t$, which are

less regular than differentiable.

The trade-off is that $f : \mathbb{R} \to \mathbb{R}$ must be twice differentiable.

Theorem (Itô's formula)

Let $(X_t)_{t\in[0,T]}$ be an Itô process with "differentials"

$$dX_t = H_t dB_t + K_t dt$$

and $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Then $f(X_t)$ is an Itô process with

$$d\left(f(X_t)\right) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\left[X\right]_t,$$

i.e.

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\left[X\right]_s \\ &= f(X_0) + \int_0^t f'(X_s)H_s dB_s + \int_0^t f'(X_s)K_s ds + \frac{1}{2}\int_0^t f''(X_s)H_s^2 ds \end{aligned}$$

Idea of proof

Recall that if f is differentiable, we can write a tangent line at any $y \in \mathbb{R}$

$$f(z) = f(y) + f'(y)(z - y) + \text{smaller errors}$$

If f is twice differentiable we can write a “second order” Taylor expansion

$$f(z) = f(y) + f'(y)(z - y) + \frac{1}{2} f''(y)(z - y)^2 + \text{smaller errors}$$

i.e. we approximate the graph of f with a tangent parabola

If we choose $z = X_s$ and $y = X_s$, we obtain

$$f(X_s) = f(X_t) + f'(X_t)(X_s - X_t) + \frac{1}{2} f''(X_t)(X_t - X_s)^2 + \text{smaller errors}$$

Summing over a partition and recalling the quadratic variation, i.e.

$$(X_t - X_s)^2 \approx [X]_t - [X]_s \approx d[X]_t,$$

we obtain the result.

Example

Samuelson's model for the evolution of the value of a financial asset S_t:

$$dS_t = S_t(\mu dt + \sigma dB_t) \quad \leftrightarrow \quad S_t = S_0 + \int_0^t S_r \sigma dB_r + \int_0^t S_r \mu dr \quad r \in [0, T]$$

S_t appears both in the left and right terms $\Rightarrow$ equation for S.

Example

If the volatility σ is equal to 0, the equation becomes

$$dS_t = S_t \mu dt \Rightarrow \frac{dS_t}{dt} = S_t \mu \Rightarrow S_t = S_0 e^{\mu t}.$$

The solution in the general case is given by

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t\right), \quad t \in [0, T]$$

Definition

The process $\exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t\right)$ is called geometric Brownian motion.

Suppose we know that S_t is strictly positive and let

$$Y_t = \log(S_t), \quad \text{i.e. } Y_t = f(S_t), \text{ with } f(x) = \log(x).$$

Since $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, we have

$$\begin{aligned} dY_t &= \frac{1}{S_t} dS_t - \frac{1}{2S_t^2} d\,[S]_t \\ &= \frac{1}{S_t} S_t(\mu dt + \sigma dB_t) - \frac{1}{2S_t^2} S_t^2 \sigma^2 dt \\ &= \mu dt + \sigma dB_t - \frac{1}{2}\sigma^2 dt \\ &= (\mu - \frac{\sigma^2}{2})dt + \sigma dB_t. \end{aligned}$$

Hence,

$$\log(S_t) = \log(S_0) + (\mu - \frac{\sigma^2}{2})t + \sigma B_t, \quad \Rightarrow \quad S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t\right).$$

General positive Itô processes

The same proof works if the equation is

$$dS_t = S_t \left(H_t dB_t + K_t dt \right)$$

where H_t, K_t are adapted processes.

The solution is

$$dS_t = S_0 \exp \left(\int_0^t \left(K_s - \frac{H_s^2}{2} \right) ds + \int_0^t H_s dB_s \right).$$

Remark

It can be proved that strictly positive every Itô process X_t satisfies an equation of the form

$$dX_t = X_t \left(H_t dB_t + K_t dt \right).$$

Itô formula – multidimensional case

We extend Itô formula to the vector-valued case (more than one process).

If $(B_t^1, B_t^2, \dots, B_t^d)$ is a d-dimensional Brownian motion, i.e. d independent BM's, the mnemonic rule is

$$dB_t^i dB_t^j = 0, \quad \text{for } i \neq j.$$

The idea behind the rule above is that, even if each dB_t^i has size $\sqrt{dt}$, the independent oscillation cancel the term.

Recall that a twice differentiable function $F : \mathbb{R}^d \to \mathbb{R}$ admits at every point $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ a vector gradient

$$\left(\frac{\partial F}{\partial x^i}(x) \right)_{i=1,\dots,d}$$

and a matrix Hessian

$$\left(\frac{\partial^2 F}{\partial x^i \partial x^j}(x) \right)_{i,j=1,\dots,d}$$

Theorem (Itô formula – vector case)

Let $X^1, X^2, \dots, X^d$ be Itô processes and $F : \mathbb{R}^d \to \mathbb{R}$ be twice differentiable. Then,

$$F(X_t^1, X_t^2, \dots, X_t^d) \text{ is a Itô process}$$

and

$$d\left(F(X_t^1, X_t^2, \dots, X_t^d)\right) = \sum_{i=1}^{d} \frac{\partial F}{\partial x^i}(X_t^1, \dots, X_t^d) dX_t^i + \\ + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 F}{\partial x^i \partial x^j}(X_t^1, \dots, X_t^d) d[X^i, X^j]_t$$

where $d[X^i, X^j]_t$ is computed accordingly to the mnemonic rules and

$$d[X^i, X^j]_t = dX_t^i dX_t^j.$$

An application

One of the classical formulas of differential calculus is Leibniz rule

$$d(x_t y_t) = x_t(dy_t) + y_t(dx_t).$$

For Itô processes, we have the following rule obtained choosing $F(x, y) = xy$:

$$\frac{\partial F}{\partial x} = y, \quad \frac{\partial F}{\partial y} = x,$$

$$\frac{\partial^2 F}{\partial^2 x} = 0, \quad \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = 1, \quad \frac{\partial^2 F}{\partial^2 y} = 0.$$

Itô formula gives

$$X_t Y_t = F(X_t, Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t$$

In integrated form this gives an integration by parts formula

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - [X, Y]_t.$$