Stochastic Processes and Stochastic Calculus - 5
Brownian Motion

Prof. Maurizio Pratelli

Università degli Studi di Pisa

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Overview

Brownian motion – some history

$\sim$ 1820 the botanist R. Brown reported for the first time the observation of the alert highly irregular random motion of minute particles ejected from the pollen grains suspendend in water.

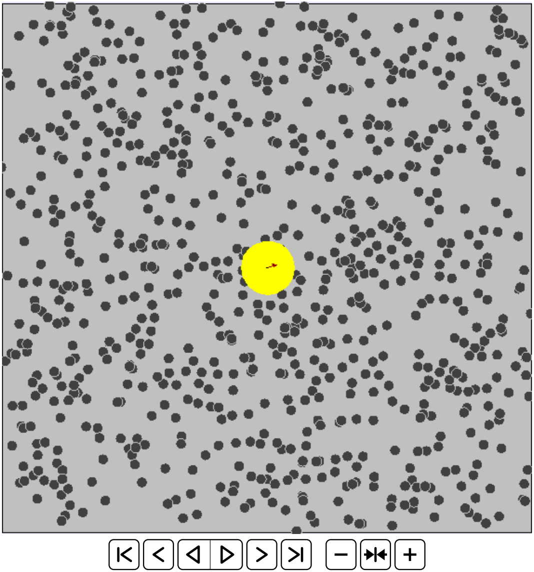
He then observed similar motion in inorganic minute particles (no biological phenomenon).

$\sim$ 1900 L. Bachelier published his thesis “Théorie de la Spéculation”: first attempt of mathematical formulation of Brownian motion and application to economics and finance

$\sim$ 1905 A. Einstein deduced from principles of statistical mechanics that Brownian motion was a result of thermal molecular motion.

He also stated the mathematical properties of Brownian motion.

Simulation of physical Brownian motion



Mathematical definitions

Fix $T > 0 \Rightarrow$ work with stochastic processes on "time" $[0, T]$.

Sometimes we may let $T = +\infty$, but condition $t \leq T$ becomes $t < T = +\infty$.

Definition

Given a stochastic process $(X_t)_{0 \leq t \leq T}$, the path at $\omega \in \Omega$ is the function

$$t \mapsto X_t(\omega).$$

Definition

A Brownian motion is a stochastic process $(B_t)_{0 \le t \le T}$ (possibly $T = +\infty$) such that

1. $B_0 = 0$
2. for any times $t_1 < t_2 < \ldots < t_n$, the (increments) r.v.'s

$$B_{t_1}, B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots, B_{t_n} - B_{t_{n-1}}$$

   are independent
3. for every $0 \le s < t$, $B_t - B_s$ has Gaussian law $\mathcal{N}(0, t - s)$
4. the paths of the process are continuous.

Such mathematical model of Brownian motion is also called Wiener process

$\sim$ 1923 N. Wiener provided the first mathematical construction i.e., showed that

mathematical Brownian motion (Wiener process) exists.

Wiener's construction of BM - Fourier analysis

N. Wiener was actively working in signal analysis (harmonic analysis).

$\sim$ 1800 Fourier $\Rightarrow$ every signal, i.e. a continuous function f on $[0, T]$ can be written as sum of wave-like signals:

$$f(t) = a_0\sqrt{\frac{1}{T}} + \sum_{n=1}^{\infty} a_n\sqrt{\frac{2}{T}} \cos\left(\frac{2\pi n}{T} t\right) + b_n\sqrt{\frac{2}{T}} \sin\left(\frac{2\pi n}{T} t\right).$$

Moreover,

$$a_0 = \int_0^T f(t)\sqrt{\frac{1}{T}} dt, \quad a_n = \int_0^T f(t)\sqrt{\frac{2}{T}} \cos\left(\frac{2\pi n}{T} t\right) dt, \quad n \geq 1$$

$$b_n = \int_0^T f(t)\sqrt{\frac{2}{T}} \sin\left(\frac{2\pi n}{T} t\right) dt, \quad n \geq 1$$

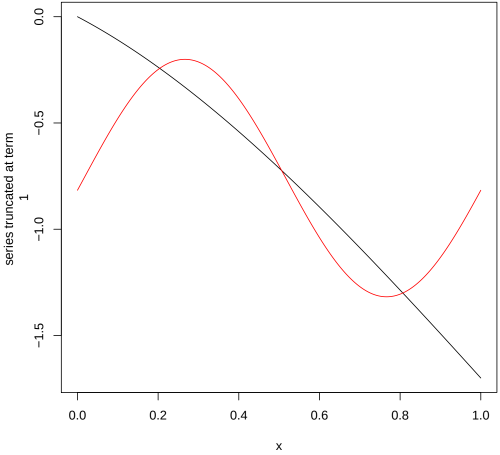
and the "Pythagorean" theorem holds:

$$\int_0^T f^2(t) dt = {a_0}^2 + \sum_{n=1}^{\infty} {a_n}^2 + \sum_{n=1}^{\infty} {b_n}^2 < \infty.$$

We reconstruct the signal by the Fourier coefficients $a_0, a_1, b_1, \dots, a_n, b_n, \dots$.

An example

Approximating $f(t) = 0.3t^3 - t^2 - t$, truncating the Fourier series at term N.



Wiener's idea: BM is an integrated version of pure (also called white) noise $\rightarrow$ the Fourier coefficients $a_0, a_1, b_1, \dots, a_n, b_n, \dots$ must be very decorrelated.

Consider $A_0, A_1, B_1, \dots, A_n, B_n, \dots$ independent Gaussian variables $\mathcal{N}(0,1)$ and formally define

$$\frac{d}{dt}B(t,\omega) := A_0(\omega)\sqrt{\frac{1}{T}}+\sum_{n=1}^{\infty} A_n(\omega)\sqrt{\frac{2}{T}}\cos\left(\frac{2\pi n}{T}t\right)+B_n(\omega)\sqrt{\frac{2}{T}}\sin\left(\frac{2\pi n}{T}t\right).$$

BUT the series does not converge. Its integral from 0 to t, defined

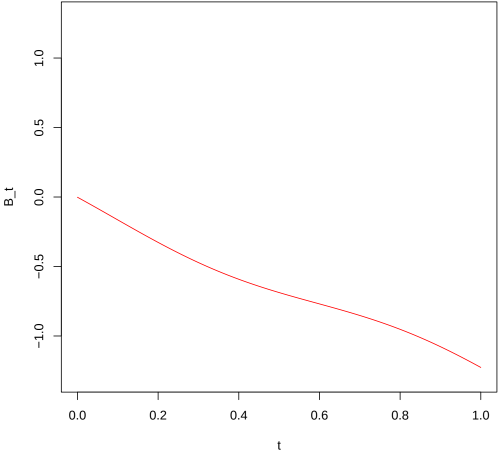
$$A_0(\omega)\int_0^t\sqrt{\frac{1}{T}}dt+\sum_{n=1}^{\infty} A_n(\omega)\int_0^t\sqrt{\frac{2}{T}}\cos\left(\frac{2\pi n}{T}s\right)ds+B_n(\omega)\int_0^t\sqrt{\frac{2}{T}}\sin\left(\frac{2\pi n}{T}s\right)ds.$$

converges to a function which gives a construction of Brownian motion:

$$B(t) := A_0\int_0^t\sqrt{\frac{1}{T}}dt+\sum_{n=1}^{\infty} A_n\int_0^t\sqrt{\frac{2}{T}}\cos\left(\frac{2\pi n}{T}s\right)ds+B_n\int_0^t\sqrt{\frac{2}{T}}\sin\left(\frac{2\pi n}{T}s\right)ds.$$

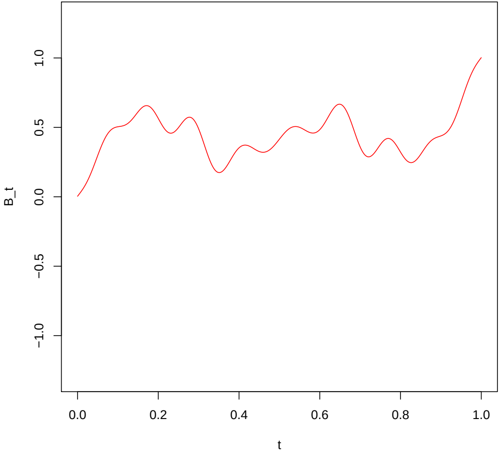
Simulations of Wiener process - 1

Approximating Wiener process by truncating the Fourier series at term $N = 1, \ldots, 100$.



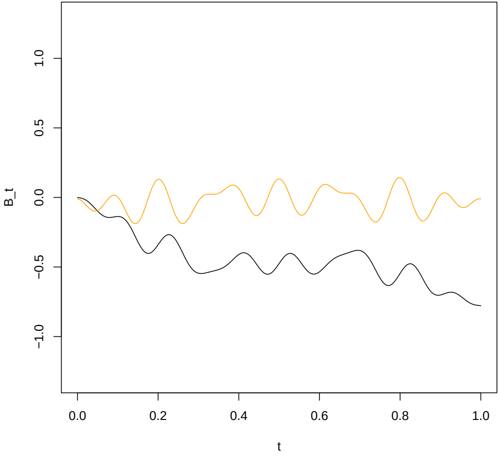
Simulations of Wiener process - 2

Another realization of WBM by truncating the series at term $N = 1, 10, 20, \ldots, 500$.



Simulations of Wiener process - 3

We plot in orange the rescaled "velocity" $\frac{dB_t}{dt} \cdot \sqrt{dt} \Rightarrow dB_t \sim \sqrt{dt}$



Continuity of paths

Wiener's construction $\Rightarrow$ properties 1 2 and 3 of BM are true.
What about continuity? paths of BM are irregular but continuous (no jumps).

Definition

Let $\lambda \in (0,1)$. A function $t \mapsto f(t) \in \mathbb{R}$ is called λ-Hölder continuous if there is some constant $C > 0$ such that

$$|f(t) - f(s)| \leq C|t-s|^{\lambda}, \quad \text{for } s, t \in [0, T].$$

Since $B_t - B_s$ is $\mathcal{N}(0, t-s)$, i.e. $B_t - B_s = \sqrt{t-s}Z$ where Z is $\mathcal{N}(0,1)$,

$$E\left[|B_t - B_s|\right] = E\left[|Z|\right]\sqrt{t-s} = \sqrt{\frac{2}{\pi}}\sqrt{t-s},$$

we could say that BM is "approximatively" $\frac{1}{2}$-Hölder continuous.

Theorem

There is a modification $\tilde{B}$ of BM whose paths are λ-Hölder, for every $\lambda < \frac{1}{2}$.

Modification means $\tilde{B}$ satisfies $\tilde{B}_s = B_s$ with probability 1, for all $s \in [0, T]$.

Continuity of BM follows from the following general result.

Theorem (Kolmogorov criterion)

Let $(X_t)_{t\in[0,T]}$ be a process such that for some $\alpha, \beta > 0$,

$$E\left[|X_t - X_s|^{\alpha}\right] \leq C_{\alpha,\beta}|t-s|^{1+\beta}, \quad \text{for every } s, t \in [0,T].$$

Then there is a modification $\tilde{X}$ of X whose paths are

$$\lambda\text{-Hölder, for any } \lambda < \frac{\beta}{\alpha}.$$

For BM, we choose $1 \leq \alpha < \infty$ and obtain

$$E\left[|B_t - B_s|^{\alpha}\right] = E\left[|Z|^{\alpha}\right]|t-s|^{\frac{\alpha}{2}} = C_{\alpha}\,|t-s|^{\frac{\alpha}{2}}$$

hence

$$\beta = \frac{\alpha}{2} - 1 = \frac{\alpha-2}{2} \quad \Rightarrow \lambda < \frac{1}{2}\frac{\alpha-2}{\alpha}$$

letting $\alpha \to \infty$,

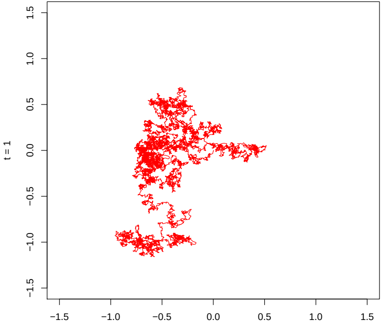
$$\lim_{\alpha\to\infty} \frac{1}{2}\frac{\alpha-2}{\alpha} = \frac{1}{2}.$$

Brownian motion in the plane, in space . . .

We can obtain Brownian motion in the plane (a model of one observed by Brown) by taking

$$t \mapsto (B_t^1, B_t^2)$$

where the two one-dimensional BM's B^1, B^2 are independent



Similarly, we obtain BM in space (B_t^1, B_t^2, B_t^3) and so on. . .

Brownian motion and martingales

Brownian paths are very irregular and oscillating, e.g. they are nowhere differentiable.

Oscillating nature of BM is also related to the following results.

Theorem

Let $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the natural filtration of the Wiener process $(B_t)_{t\in[0,T]}$. Then

1. $t \mapsto B_t$ is a martingale w.r.t. $\mathcal{F}$
2. $t \mapsto (B_t)^2 - t$ is a martingale w.r.t. $\mathcal{F}$

These properties will be fundamental in the construction of the Itô's stochastic integral with respect to BM.

Quadratic variation of BM

We have seen that $t \mapsto B_t$ is "approximatively" $\frac{1}{2}$-Hölder continuous.

A similar but important way of measuring regularity of paths of BM is via their quadratic variation.

Precisely: with probability 1,

$$\lim_{n\to\infty} \sum_{i=1}^{2^n} \left(B_{t\frac{i}{2^n}} - B_{t\frac{i-1}{2^n}} \right)^2 = t.$$

Notation/Definition

For process $(X_t)_{t\in[0,T]}$ we let $([X]_t)_{t\in[0,T]}$ be its quadratic variation process (if it exists)

$$[X]_t := \lim_{n\to\infty} \sum_{i=1}^{2^n} \left(X_{t\frac{i}{2^n}} - X_{t\frac{i-1}{2^n}} \right)^2.$$

In case of BM, we have $[B]_t = t$. The property can be expressed by the mnemonic rule of "differentials"

$$(dB_t)^2 = d[B]_t = dt, \quad \text{in general} \quad (dX_t)^2 = d[X]_t.$$

Wiener-Itô stochastic integrals

N. Wiener already noticed that it is not possible to define integrals w.r.t. BM,

$$\int_0^t h_s \frac{dB_s}{ds} ds$$

because $\frac{dB_s}{ds}$ does not exist (here h_s is a deterministic function).

Wiener's solution: define directly a new mathematical quantity

$$\int_0^t h_s dB_s$$

K. Itô extended Wiener's approach to define an integral

$$\int_0^t H_s dB_s$$

where H_s is another stochastic process (belonging to some suitable class) which is in some sense limit of "Riemann sums"

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} H_{s_i} \left(B_{s_{i+1}} - B_{s_i}\right).$$

Let us consider first the case of a deterministic function $h = h(s)$ which is constant on some intervals

$$h(s) = \sum_{i=0}^{n-1} a_i I_{]s_i, s_{i+1}]}(s)$$

It is natural to define

$$\int_0^T h(s) dB_s = \sum_{i=0}^{n-1} a_i \int_{s_i}^{s_{i+1}} dB_s = \sum_{i=0}^{n-1} a_i \left(B_{s_{i+1}} - B_{s_i}\right)$$

Since the increments $\left(B_{s_{i+1}} - B_{s_i}\right)$ are Gaussian and independent, the random variable

$$\omega \mapsto \left(\int_0^T h(s) dB_s\right)(\omega)$$

is Gaussian centred with variance

$$\text{Var}\left(\int_0^T h(s) dB_s\right) = \text{Var}\left(\sum_{i=0}^{n-1} a_i \left(B_{s_{i+1}} - B_{s_i}\right)\right) = \sum_{i=0}^{n-1} a_i^2 (s_{i+1} - s_i) = \int_0^T h^2(s) ds.$$

Wiener integral

By continuity, this definition extends to all functions $h \in L^2(0, T)$, i.e. with

$$\int_0^T h^2(s)ds < \infty.$$

Theorem

Let $h \in L^2(0, T)$. The Wiener integral

$$\int_0^T h(s)dB_s$$

is well-defined and is a Gaussian random variable:

$$\int_0^T h(s)dB_s \sim \mathcal{N}\left(0, \int_0^T h^2(s)ds\right).$$

Towards Itô integral

In the next lecture, we show how to extend this definition to stochastic integrands H_s.

The Itô integral $\int_0^T H_s dB_s$ is defined provided that

1. the process H_s is adapted or non-anticipative, i.e. H_s uses only information from the past history of B up to time s;
2. the square-integrability condition holds

$$E\left[\int_0^T H_s^2 ds\right] = \int_\Omega \int_{[0,T]} H_s^2(\omega) ds dP(\omega) < \infty.$$