Stochastic Processes and Stochastic Calculus - 3
Markov Chains

Dario Trevisan

Università degli Studi di Pisa

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Overview

A “lake model” of employment flows

We propose a simple model for

- flows between unemployment and employment
- employment and unemployment rates at “equilibrium”

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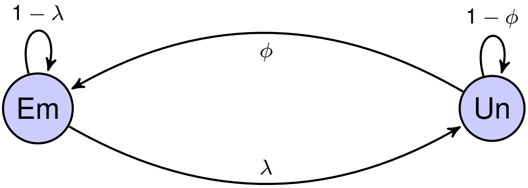

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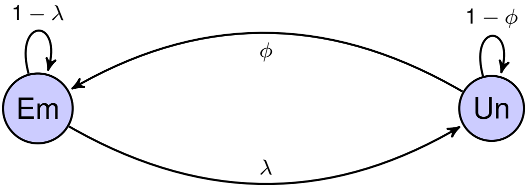

Fix a unit of time (e.g. one month) and introduce two parameters

$$\lambda := \text{probability that a worker loses his/her job in a month}$$

$$\phi := \text{probability that an unemployed one finds a job in a month}$$

We call $\lambda, \phi \in [0, 1]$ transition probabilities.

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NO. For example, we do not know the initial state X_0.

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But what about three or more? One year? At this stage we introduce the following assumption:

Markov property

At any time, regardless of the information about the past months, the next-month state of employment depends uniquely on the present one, i.e.

$$\mathbb{P}(X_{n+1} = E | X_n = U, X_{n-1} = \cdot, X_{n-2} = \cdot, \ldots, X_{n-k} = \cdot) = \mathbb{P}(X_{n+1} = E | X_n = U)$$

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- The model becomes very tractable analytically
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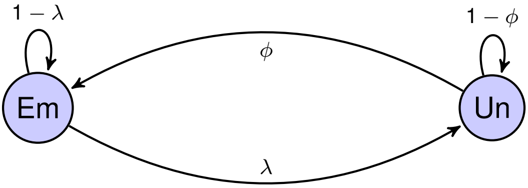
A more realistic assumption would be e.g.

$$\mathbb{P}(X_{n+1} = E | X_n = U, X_{n-1} = U, X_{n-2} = U, \ldots, X_{n-k} = U) \leq \mathbb{P}(X_{n+1} = E | X_n = U)$$

but how to make it quantitative?

Examples



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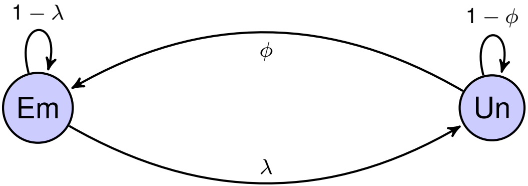

$$\begin{aligned}\mathbb{P}(X_2 = E, X_1 = U | X_0 = E) &= \mathbb{P}(X_2 = E | X_1 = U, X_0 = E)\mathbb{P}(X_1 = U | X_0 = E) \\ &= \mathbb{P}(X_2 = E | X_1 = U)\mathbb{P}(X_1 = U | X_0 = E) \\ &= \phi\lambda\end{aligned}$$

$$\mathbb{P}(X_3 = E, X_2 = U, X_1 = U | X_0 = E) = \phi(1 - \phi)\lambda$$

$$\mathbb{P}(X_3 = E | X_0 = E) = ?$$

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Definition (Markov process)

Given a filtration $(\mathcal{F}_t)_{t\in T}$, a process $X_t : \Omega \to E$ is Markov if

$$\text{for all } s \leq t \in T,\ A \subseteq E \quad \mathbb{P}(X_t \in A|\mathcal{F}_s) = \mathbb{P}(X_t \in A|X_s)$$

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- Usually one has $\mathcal{F}_t = \sigma(X_r : r \leq t)$, the natural filtration of X_t.

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The transition probabilities are $|\mathcal{S}|^2$ numbers that we write matrix notation:

$$\begin{pmatrix} 1-\lambda & \lambda \\ \phi & 1-\phi \end{pmatrix} \Rightarrow Q = (p_{ij})_{i,j \in \mathcal{S}} \quad p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i) \quad \text{for } i, j \in \mathcal{S}$$

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Remark (Q is a stochastic matrix)

We must have $p_{ij} \in [0, 1]$ for $i, j \in \mathcal{S}$ and

$$\sum_{j \in \mathcal{S}} p_{ij} = \sum_{j \in \mathcal{S}} \mathbb{P}(X_{n+1} = j | X_n = i) = \mathbb{P}(X_{n+1} \in \mathcal{S} | X_n = i) = 1$$

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Markov Chains - 2

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we say that a process $X_n : \Omega \to \mathcal{S}$ ($n \in \mathbb{N}$) is a markov chain if for all $n \in \mathbb{N}$,

$$\mathbb{P}(X_{n+1} = j|\mathcal{F}_n) = \mathbb{P}(X_{n+1} = j|X_n) \quad \forall j \in \mathcal{S} \quad \text{(Markov property)}$$

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The two conditions above yield

$$\mathbb{P}(X_{n+1} = j|X_n = i, X_{n-1} = \cdot, X_{n-2} = \cdot, \dots, X_{n-k} = \cdot) = p_{ij}$$

Graphical representation

We associate

- a node to each state $i \in \mathcal{S}$
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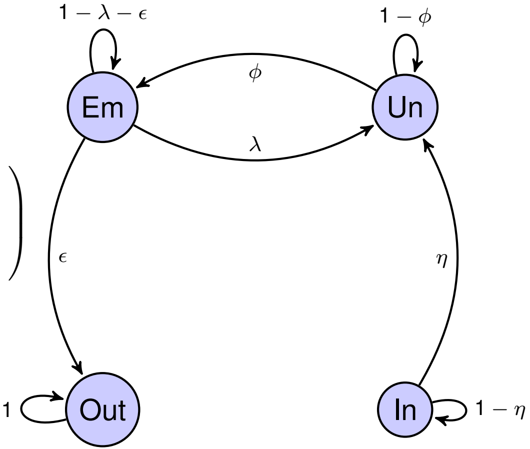
$$\mathcal{S} = (I, U, E, O)$$

$$Q = \begin{pmatrix} 1-\eta & \eta & 0 & 0 \\ 0 & \phi & 1-\phi & 0 \\ 0 & \lambda & 1-\lambda-\epsilon & \epsilon \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



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Theorem

Given a finite or countable $\mathcal{S}$, a stochastic matrix $Q = (p_{ij})_{i,j\in\mathcal{S}}$ and a probability measure μ_0 on $\mathcal{S}$.

∃) there exists a Markov chain $(X_n)_{n\in\mathbb{N}}$ (with respect to the natural filtration) with transition probability Q and law of X_0 equal to μ_0.

!) Such a chain X is unique in law, i.e. given any Markov chain Y with transition probability Q and law of X_0 equal to μ_0, one has

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The condition $X = Y$ in law means that for every $n \geq 1$, $A_0, \dots, A_n \subseteq \mathcal{S}$

$$\mathbb{P}(X_0 \in A_0, X_1 \in A_1, \dots, X_n \in A_n) = \mathbb{P}(Y_0 \in A_0, Y_1 \in A_1, \dots, Y_n \in A_n)$$

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The simplest case is that of 1-marginals, i.e. the law of X_n. For example

$$P(X_1 = j) = \sum_{i \in \mathcal{S}} P(X_1 = j | X_0 = i) P(X_0 = i) = \sum_{i \in \mathcal{S}} p_{ij} \mu_i^0$$

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Proposition (marginal laws)

$$P(X_n = \cdot) = \mu^0 (Q^n), \quad \text{for any } n \geq 0.$$

A geometric interpretation of Q^n

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By definition of power of a matrix

$$(Q^n)_{ij} = \sum_k p_{ik}(Q^{n-1})_{kj} = \sum_{k_1, k_2, \ldots k_{(n-1)}} p_{ik_1} p_{k_1 k_2} \cdot p_{k_{(n-1)} j}$$

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1. Each choice of $k_1, \ldots, k_{(n-1)}$ defines the path of ‘length” n

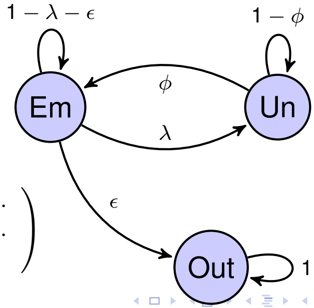
$$i \to k_1 \to k_2 \to \ldots \to k_{(n-1)} \to j.$$

2. Each path has total weight given by the product of the weights.
3. We sum the weights over all the paths of length n joining i to j.

$$\mathcal{S} = (U, E, O)$$

$$Q = \begin{pmatrix} 1-\phi & \phi & 0 \\ \lambda & 1-\lambda-\epsilon & \epsilon \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q^2 = \begin{pmatrix} \lambda\phi & \ldots & \ldots \\ \lambda(1-\phi)+\lambda(1-\lambda-\epsilon) & \ldots & \ldots \\ 0 & 0 & 1 \end{pmatrix}$$



Long time behavior

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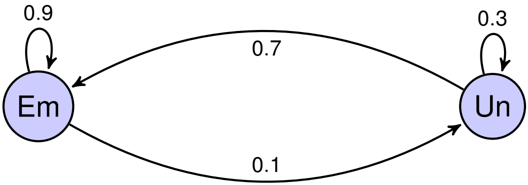
Consider the 2-states model, with $\lambda = 0.1$ and $\phi = 0.3$.



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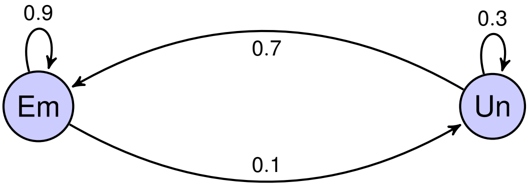

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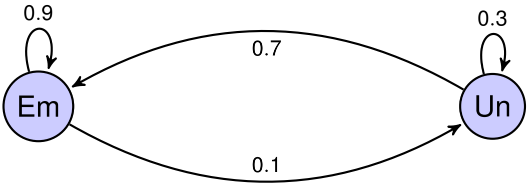

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$\mathbb{P}(X_n = E)$ will be close to 1. How to compute it?

$$Q = \begin{pmatrix} 0.9 & 0.1 \\ 0.7 & 0.3 \end{pmatrix} \quad Q^2 = \begin{pmatrix} 0.88 & 0.12 \\ 0.84 & 0.16 \end{pmatrix} \quad Q^4 \sim \begin{pmatrix} 0.87 & 0.13 \\ 0.87 & 0.13 \end{pmatrix}$$

Invariant distributions

Do the laws of $(X_n)_{n \geq 1}$ converges as $n \to \infty$ towards some limit law $\bar{\mu}$ on $\mathcal{S}$?

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Since the law of X_n is $\mu^0(Q^n)$, we should find

$$\bar{\mu} = \lim_{n\to\infty} \mu^0(Q^n)$$

but also

$$\bar{\mu}Q = \lim_{n\to\infty} \mu^0(Q^n)Q = \lim_{n\to\infty} \mu^0(Q^{n+1}) = \bar{\mu}$$

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In linear algebra notation, $\bar{\mu}$ is a row eigenvector of Q with eigenvalue 1.

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Choose any μ^0 and consider the averages

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These are “almost invariant”:

$$\bar{\mu}^n - \bar{\mu}^n Q = \frac{1}{n}\left(\mu^0 - Q^n \mu^0\right) \to 0$$

Existence of Invariant distributions

Theorem (Perron-Frobenius)

If the number of states $\mathcal{S}$ is finite, there exists at least one invariant distribution:

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Notice that the proof gives a "way" to find $\bar{\mu}$, essentially by taking powers Q^n and averaging.

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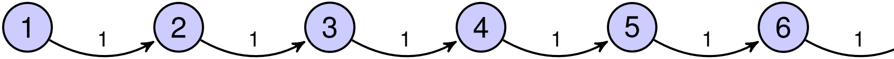

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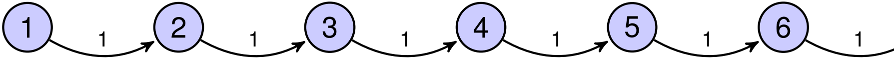

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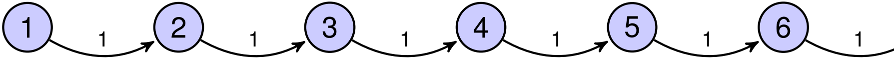

$$P(X_{n+1} = i + 1 | X_n = i) = 1$$

- We have $P(X_{n+1} = i) = 0$, for every $i \leq n$
- Any invariant probability satisifes $\bar{\mu}_i = 0$ for every i

Convergence of the laws

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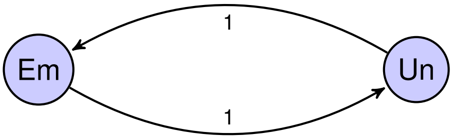
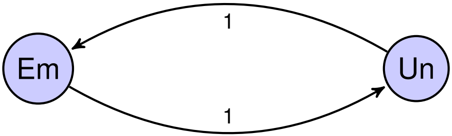
In general the answer is NO. Let us consider one example.



Why? If $P(X_0 = E) = 1$, we have $P(X_n = E) = 1$ or 0 alternatively...



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$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Q^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad Q^3 = Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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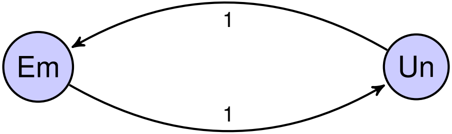

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Question: what are the invariant distributions for Q?

Communicating states

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Definition (Communicating states)

Given states $i, j \in \mathcal{S}$, we say that $i \rightsquigarrow j$ if there is $n \in \mathbb{N}$ such that

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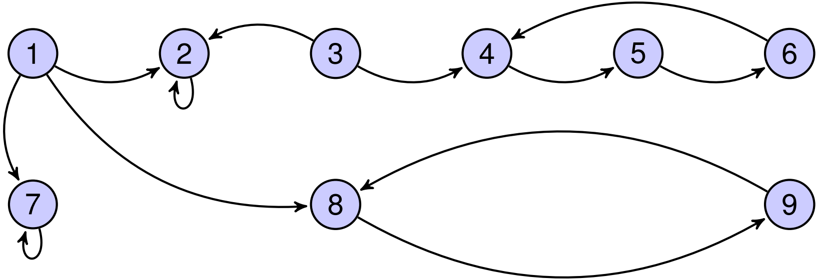

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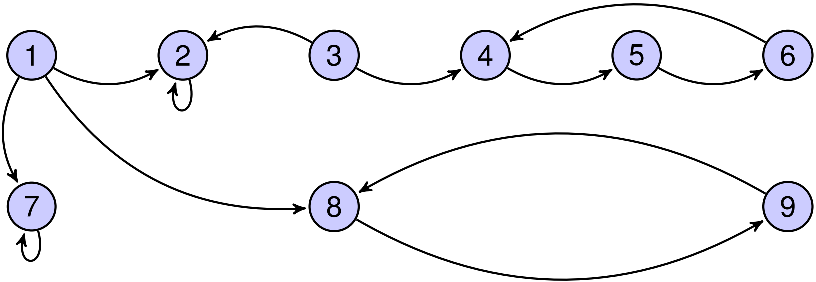

Notice that $1 \rightsquigarrow 2$ but not $2 \rightsquigarrow 1$.

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- in terms of the transition matrix, this means that for every i, j there exists $m \in \mathbb{N}$ such that $(Q^m)_{ij} > 0$.
- The name "irreducible" comes from the fact that any Markov chain can be decomposed into smaller chains which are irreducible, plus some "remainder" (called transitory states).

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Notice that if Q^m has strictly positive entries, then Q^{m+1} also is strictly positive.

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because

$$\sum_k P(X_1 = k | X_0 = i) = 1.$$

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Theorem (Markov)

If Q is a regular transition probability (on a finite state space $\mathcal{S}$) then there is a unique invariant distribution $\bar{\mu}$ and

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In terms of Markov chain, this answer positively to the problem

$$P(X_n = j) \to \bar{\mu}(j), \quad \text{as } n \to \infty,$$

for every $j \in \mathcal{S}$, whatever the initial law of X_0 was.

Ergodicity

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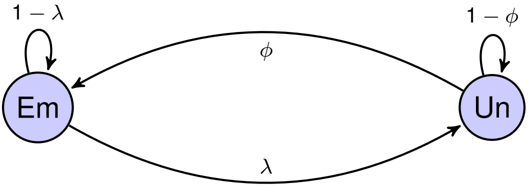
Going back to our model of employment flows,



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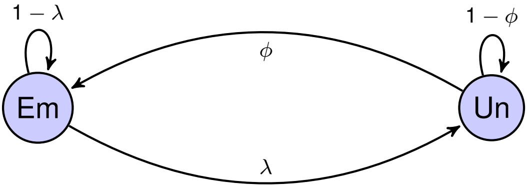

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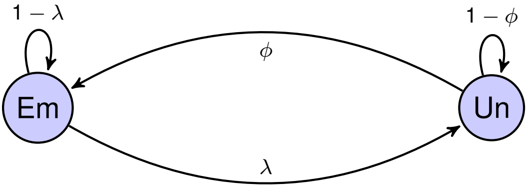

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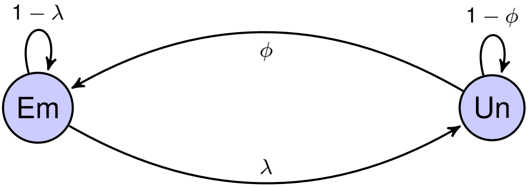

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Beware: this holds for systems at equilibrium!

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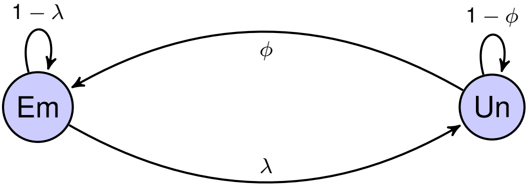

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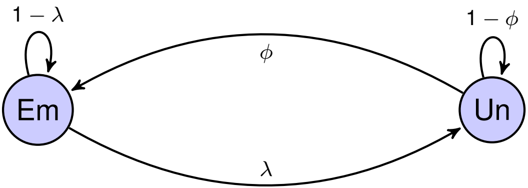

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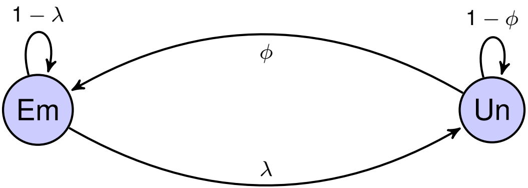

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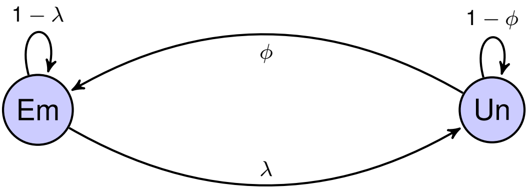

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Then the ergodic principle reads

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Of course, this theorem applies also to regular Markov chains.

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and build a process $Y_0, Y_{1/4}, Y_{2/4}, Y_{3/4}, \dots$.
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Continuous time Markov chains

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What happens if we split times into infinitely many small intervals, i.e. continous times?
We have $Q^{1/\infty} = Q^0 = Id$, but imagine, for large n,

$$Q^{1/n} \sim Id + \frac{1}{n} R + \text{smaller terms.}$$

We describe the chain by means of the matrix R (called transition rate matrix).

Recalling the formal expansion

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Assuming $Q^{1/n}$ to be a stochastic matrix, then we must have

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We interpret R_{ij} as the rate at which we jump from i to j.

Actually, we can describe the continuous Markov chain as follows:

1. When the particle is on the state $i \in \mathcal{S}$, take independent "alarm clocks" A_j, one for every $j \in \mathcal{S}$ with exponential laws

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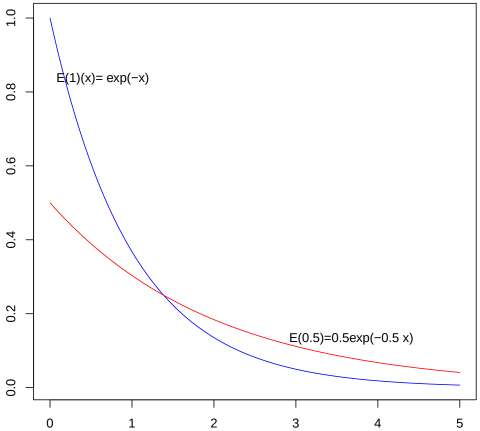

Poisson process

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An important example of such continuous time jump process has state space

$$\mathcal{S} = \mathbb{N} = \{0, 1, \ldots\}$$

and transition rates

$$R_{ij} = \begin{cases} 1 & \text{if } j = i + 1 \\ -1 & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

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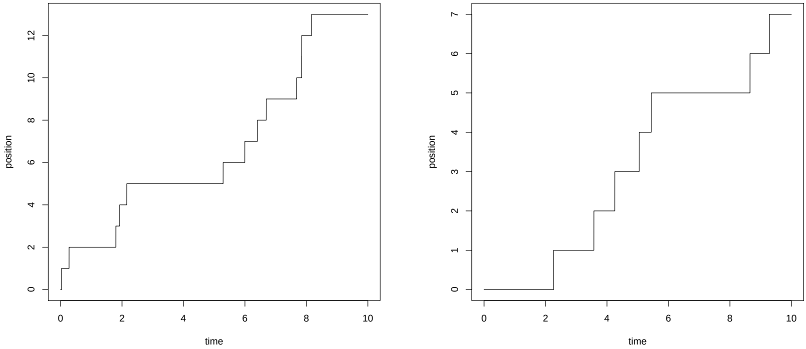
Two realizations of a Poisson process:



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It is possible to prove that

- N is a Markov process (obvious)
- For every $t \in [0, \infty)$, N_t has Poisson law

$$P(N_t = k) = \frac{e^{-k}}{k!}, \quad k \in \mathbb{N}.$$

  In particular,

$$\mathbb{E}\left[N_t\right] = t.$$

- the increments are independent, e.g. for $s < t < u < v$,

  $N_t - N_s$ and $N_v - N_u$ are independent random variables.

One can also change the intensity of jumps (not the size), by changing the transition rates: for $\lambda > 0$,

$$R_{ij}^{\lambda} = \lambda R_{ij}$$