Stochastic Processes and Stochastic Calculus - 10
(Short) Introduction to Interest Rate Models

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Overview

Remark on notation

Throughout all this lecture, Brownian motion will be indicated with

$$W_t \qquad \text{for “Wiener process”}$$

instead of B_t.

Zero-coupon bonds

Primary object

A zero-coupon bond $B(t,T)$ is (the price of a) contract (stipulated at time $t \leq T$) which guarantees the () 1 € to be paid at time T.

Facts:

- $T \mapsto B(t,T)$ is regular and $B(T,T) = 1$.
- $t \mapsto B(t,T)$ is highly irregular $\Rightarrow$ stochastic process

Principle: the amount 1 € at time t is worth

$$\frac{1}{B(t,T)} \quad \text{at time } T \geq t.$$

Starting from the principle: the amount 1 € at time t is worth

$$\frac{1}{B(t,T)} \quad \text{at time } T \geq t,$$

we can introduce interests in linear terms

$$1 + L(t,T)(T-t) = \frac{1}{B(t,T)}$$

$\Rightarrow$ LIBOR interest rate $L(t,T)$

$$L(t,T) = \frac{1}{(T-t)} \frac{1-B(t,T)}{B(t,T)} = \frac{\frac{1}{B(t,T)} - 1}{T-t}$$

or interests in continuously compounded terms

$$\exp\left(Y(t,T)(T-t)\right) = \frac{1}{B(t,T)}$$

$\Rightarrow$ Yield $Y(t,T)$

$$Y(t,T) = \frac{-\log B(t,T)}{(T-t)}$$

What happens in $L(t,T)$ if we let $T \to t$ (recall that $T \mapsto B(t,T)$ is "regular"):

Definition (instantaneous short rate)

$$r(t) := \lim_{h \to 0^+} L(t, t+h)$$

We obtain the usual numéraire: money market account:

$$B_t := \exp\left(\int_0^t r(s)ds\right)$$

(recall our change of notation for BM...)

Let us introduce the "usual" model hypothesis:

Hypothesis

1. (completeness) the filtration $\mathcal{F}_t$ is the natural filtration generated by a d-dimensional Wiener process (BM)

$$(W_t^1, W_t^2, \dots, W_t^d) \quad t \in [0, T],$$

but we will write W_t as if $d = 1$.

2. (no-arbitrage) there exists an equivalent probability measure P^* such that, under P^*, every "discounted" process

$$t \mapsto \frac{B(t,T)}{B_t} = B(t,T) \exp\left(- \int_0^t r(s) ds \right)$$

is a martingale.

A naive approach

Let us directly model $t \mapsto B(t, T)$ via an SDE:

$$d_t B(t, T) = B(t, T) \left(\alpha(t, T) dt + \sigma(t, T) dW_t \right)$$

Bad idea! How can we guarantee that $B(T, T) = 1$?

Why should we model interest rates?

Since it is not straightforward to model $B(t, T)$, let us see first motivations.

These come from derivatives whose underlying are interest rates:

caps, floors, swaps, ...

Caps

Cap $\Leftrightarrow$ sum of caplets. Caplet on $[S, T]$

$$(T - S)\,(L(S,T) - K)^{+} = (\text{some computations}\ldots) = K^{*}\left(\frac{1}{K^{*}} - B(S,T)\right)^{+}$$

where $K^{*} = 1 + (T - S)K$.

Hence a caplet is equivalent to a put option at time S on a bond of maturity T.

Similarly, a floorlet is equivalent to an option on a bond of future maturity.

For the swaps there is a theoretical formula (outside any model)

Swap rate

$$R = \frac{B(0, T_0) - B(0, T_n)}{\delta \sum_{i=1}^{n} B(0, T_i)}$$

where $0 < T_0 < T_1 < \ldots < T_n$ and $\delta = T_i - T_{i-1}$ (intervals of equal length).

Problem: LIBOR rates are known only up to 1 year but swaps could be e.g. over 15 years!

$\Rightarrow$ some model becomes necessary.

Models based on the short rate

1. we introduce a stochastic model for for the short rate $r(t)$

$$dr(t) = \alpha\left(t, r(t)\right) dt + \beta\left(t, r(t)\right) dW_t$$

   where W_t is a one-dimensional Brownian motion.

   Problem we only know $B_t = \exp\left(\int_0^t r(s)ds\right)$

2. model the equivalent martingale probability (actually, its density)

$$\frac{dP^*}{dP}$$

   Problem who chooses the martingale probability? Answer: the market!

First models

Vasicek	$dr(t) = (b - ar(t))\, dt + \sigma dW_t$
Cox-Ingersoll-Ross	$dr(t) = a\,(b - r(t))\, dt + \sigma \sqrt{r(t)} dW_t$

Where does the term $(b - ar(t))$ come from?

It is called mean reversion.

Consider the ordinary differential equation

$$df(t) = a(m - f)dt \quad \Rightarrow \quad f(t) = m + (c - m)e^{-at}$$

We have $(a > 0)$

$$\lim_{t \to +\infty} f(t) = m$$

Interpretation: "mean reversion" $\Rightarrow$ convergence to equilibrium

but we add some uncertainty (noise)

Explicit solution to Vasicek model

It is not difficult to prove that

$$dr(t) = (b - ar(t))\,dt + \sigma dW_t$$

has the solution

$$r(t) = \frac{b}{a} + \frac{b}{a}\left(r^*(0) - \frac{b}{a}\right)e^{-at} + \sigma e^{-at}\int_0^t e^{as}dW_s$$

Stochastic integral is a Wiener integral (deterministic integrand) $\Rightarrow r(t)$ is a Gaussian random variable.

For the Cox-Ingersoll-Ross (C.I.R.) model

$$dr(t) = a(b - r(t))\,dt + \sigma\sqrt{r(t)}dW_t$$

we must provide conditions so that

$$r(t) \geq 0,$$

otherwise $\sqrt{r(t)}$ has no meaning...

Theorem

If $ab > \frac{\sigma^2}{2}$, then $r(t,\omega) > 0$ a.e.

A new (real) phenomenon: short interest rates can become negative!

Model based on short interest rates – conclusion

We saw just two simple models – Vasicek and CIR.

General fact: in short rate models, it can be proved that

$$B(t, T) = F^T(t, r(t))$$

where F^T is built as a solution to some partial differential equation.

Instantaneous forward rate

Recall the definition of Yield

$$Y(t,T) = \frac{-\log B(t,T)}{(T-t)}$$

We introduce a new quantity:

$$f(t,T) = -\frac{\partial}{\partial T}\frac{\log B(t,T)}{(T-t)} = \frac{\partial Y}{\partial T}(t,T)$$

Principle: model

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t$$

in such a way that there exists P^* equivalent (to P) martingale probability.

Theorem (Heat-Jarrow-Morton)

$\left(\frac{B(t,T)}{B_t}\right)_{t\in[0,T]}$ is a martingale if and only if

$$\alpha(t,T) = \sigma(t,T)\int_0^t \sigma(t,s)ds, \quad \text{for every } t \in [0,T].$$

Heat-Jarrow-Morton is very interesting from a mathematical viewpoint but less tractable numerically.

Moreover, where does the definition

$$f(t,T) = -\frac{\partial \log B(t,T)}{\partial T}$$

come from?

Consider forward interest rates in $[S, T]$ implicit at time $t < S < T$.

The amount 1 at time t is worth

$$\frac{1}{B(t,S)} \quad \text{at time } S$$

and

$$\frac{1}{B(t,T)} \quad \text{at time } T.$$

If we assume linear interests:

$$\frac{1}{B(t,S)}\left(1 + L(t,S,T)(T-S)\right) = \frac{1}{B(t,T)}$$

$$\Rightarrow L(t,S,T) = \left(\frac{B(t,S)}{B(t,T)} - 1\right) / (T-S)$$

If we assume continuously compounded interests:

$$\frac{1}{B(t,S)} \exp F(t,S,T)(T-S) = \frac{1}{B(t,T)}$$

$$\Rightarrow F(t,S,T) = \log\left(\frac{B(t,S)}{B(t,T)}\right) / (T-S) = \frac{\log B(t,S) - \log B(t,T)}{T-S}$$

We obtain

$$f(t,T) = \lim_{h \downarrow 0} F(t,T,T+h).$$

Change of numéraire

An important tool in interest rate modes is the

principle of change of numéraire

Take as numeraire $B(t,T)$ and consider

$$\frac{dP^T}{dP^*} = \frac{B(t,T)}{B_t\,B(0,T)} \qquad \Leftarrow \text{T-forward measure.}$$

An example (at the blackboard)

“Modern” point of view (market models)

A more recent approach is the following (market models):

Fix a finite number of maturities

$$0 < T_0 < T_1 < \ldots < T_n$$

and directly model n processes, for $i = 0, 1, \ldots, n-1$

$$L_i(t) := L(t, T_i, T_{i+1})$$

i.e. the LIBOR interest on the interval $[T_i, T_{i+1}]$.

References – suggested readings

- T. Björk, Arbitrage Theory in Continuous Time, OUP Oxford, 2009.
- T. Mikosch, Elementary Stochastic Calculus, with Finance in View, Advanced Series on Statistical Science & Applied Probabilit. World Scientific, 1998.
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