Stochastic processes and stochastic calculus
Exercises I

12.09.2016

1. Let X_1 and X_2 be discrete random variables with a Poisson distribution:

$$X_1 \sim \text{Poisson}(\lambda_1) \qquad \text{and} \qquad X_2 \sim \text{Poisson}(\lambda_2)$$

   Show that if X_1 and X_2 are independent, then $X_1 + X_2$ is a Poisson random variable with parameter $\lambda_1 + \lambda_2$.

2. Prove that, if X and Y are real-valued random variables with $\text{Var}(X) = \text{Var}(Y)$, then $X + Y$ and $X - Y$ are uncorrelated.

3. Let X be an exponential random variable with parameter λ:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0. \end{cases}$$

   a) Calculate $\mathbb{E}[X]$ and $\text{Var}(X)$.

   b) Show that the probability distribution of X is memoryless, i.e. if for any non-negative real numbers t and s, we have

$$\Pr(X > t + s \mid X > t) = \Pr(X > s).$$

   c) Let Y be another exponential random variable with parameter μ, independent of X. Show that

$$\mathbb{P}(X < Y) = \frac{\lambda}{\lambda + \mu}$$

d) Let Y be another exponential random variable with parameter μ, independent of X. Let $Z = \min(X, Y)$. Show that Z is an exponential random variable with parameter $\lambda + \mu$.

4. An economics consulting firm has created a model to predict recessions. The model predicts a recession with probability 80% when a recession is indeed coming and with probability 10% when no recession is coming. The unconditional probability of falling into a recession is 20%. If the model predicts a recession, what is the probability that a recession will indeed come?

5. Let $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ be two independent random variables. Calculate

$$\mathbb{E}\left[(X + Y + XY)^2 \mid X = 2\right].$$

6. Let X and Y be two random variables with joint density

$$f(x, y) = 2\,\mathbb{1}_{\{0 \le y \le x \le 1\}}.$$

Calculate $\mathbb{E}[X \mid Y = 1/2]$.

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Exercises II

13.09.2016

1. Let $Y_1, Y_2, \dots,$ be independent random variables uniformly distributed in $\{1, 2, 3, 4, 5\}$. Define

$$X_n = \max(Y_1, \dots, Y_n), \quad n \geq 1.$$

   a) Show that X_n is a Markov chain.

   b) Determine the transition probability matrix.

   c) Find the probability distribution function for the random variable X_3.

2. A Markov chain has transition probability matrix

$$P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

   Determine all stationary distributions of the chain.

3. Strikes in a factory occur according to a Poisson process with rate 2 per year. Find the probability that there is exactly one strike in the first 3 months and exactly 3 strikes in the subsequent 9 months.

4. Let $S_1, S_2, \dots,$ be i.i.d. random variable exponentially distributed with parameter λ. Define $T_n = S_1 + \cdots + S_n$ and $N_t = \sum_{n \geq 1} \mathbb{1}_{\{T_n \leq t\}}$. Show that N_t is a Poisson process.

5. Let $(X_n)_{n\geq 0}$ be i.i.d uniform on $[0,1]$ random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. For $n \geq 0$, let $F_n = \sigma(X_k,\, k \leq n)$, and consider the random variable $T = \inf\{n \geq 1 : X_n > X_0\}$. Show that T is a stopping time with respect to the filtration F_n.

6. Let X_i, $i \geq 0$ be integrable random variables, and $F_n = \sigma(X_0, ..., X_n)$. Assume that for $n \geq 1$,

$$\mathbb{E}[X_{n+1}|F_n] = aX_n + bX_{n-1},$$

where $a \in (0,1)$ and $a+b = 1$. For what value(s) of α, $S_n = \alpha X_n + X_{n-1}$ is a (F_n)-martingale?

7. Let $\xi_1, \xi_2, \ldots$ be i.i.d random variables with $\mathbb{E}[\xi_i^2] = \sigma^2 < \infty$ and $\mathbb{E}[\xi_i] = 0$. Define $X_n = \sum_{i=1}^n \xi_i$. Find the increasing predictable process A_n such that $X_n^2 - A_n$ is a martingale and $A_0 = 0$.

8. Let $X_1, \ldots, X_N$ be independent random variables with $\mathbb{E}[X_i] = 0$, $i = 1, \ldots, N$. For $n \in \{1, \ldots, N\}$, define $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and $S_n = X_1 + \cdots + X_n$. For $m < n$, calculate $\mathbb{E}[S_n \mid \mathcal{F}_m]$ and $\mathbb{E}[S_n \mid S_m]$.

Stochastic processes and stochastic calculus
Exercises III

14.09.2016

1. Prove that $B_t^2 - t$ and $B_t^3 - 3tB_t$ are martingales.

2. Let Y be a random variable such that $\mathbb{E}[|Y|] < \infty$. Define

$$M_t = \mathbb{E}[Y \mid \mathcal{F}_t].$$

   Show that M_t is an $\mathcal{F}_t$-martingale.

3. Prove that the following stochastic processes are martingales:

   a) $e^{t/2} \cos B_t$

   b) $(B_t + t)\exp(-B_t - t/2)$.

4. Show that

$$\int_0^1 t \, \mathrm{d}B_t$$

   is a Gaussian random variable. Calculate the expectation and the variance.

5. Compute the expectation and the variance of

$$\frac{1}{T}\int_0^T B_t \, \mathrm{d}t$$

and

$$\frac{1}{T}\int_0^T B_t \,\mathrm{d}B_t.$$

Are these variables normally distributed?

6. Let $x > 0$ be a constant and define

$$X_t = \left(x^{1/3} + \frac{1}{3}B_t\right)^3, \qquad t \geq 0.$$

Show that

$$\mathrm{d}X_t = \frac{1}{3}X_t^{1/3}\mathrm{d}t + X_t^{2/3}\mathrm{d}B_t, \qquad B_0 = x.$$

7. a) For c, α constants, $B_t \in \mathbb{R}$ define

$$X_t = \exp\left(ct + \alpha B_t\right).$$

Prove that

$$\mathrm{d}X_t = \left(c + \frac{1}{2}\alpha^2\right) X_t \mathrm{d}t + \alpha X_t \mathrm{d}B_t.$$

b) For c, $\alpha_1 \ldots, \alpha_n$ constants, $B_t = (B_t^1, \ldots, B_t^s) \in \mathbb{R}^n$ define

$$X_t = \exp\left(ct + \sum_{j=1}^n \alpha_j B_t^j\right).$$

Prove that

$$\mathrm{d}X_t = \left(c + \frac{1}{2}\sum_{j=1}^n \alpha_j^2\right) X_t \mathrm{d}t + X_t \left(\sum_{j=1}^n \alpha_j \mathrm{d}B_t^j\right).$$

Stochastic processes and stochastic calculus
Exercises IV

15.09.2016

1. Let W_t^1, W_t^2 be two independent Wiener processes and let a, b be constants. Show that

$$W_t := \frac{aW_t^1 + bW_t^2}{\sqrt{a^2 + b^2}}$$

is also a Wiener process.

2. In each of the cases below find the process H_t such that

$$F = \mathbb{E}[F] + \int_0^T H_t \, \mathrm{d}B_t$$

a) $F = B_T^2$

b) $F = \exp(B_T)$

3. Let B_t be a Brownian motion and define $Y_t = B_t + t$. Find $\mathbb{Q}_T \sim \mathbb{P}$ such that $(Y_t)_{t \leq T}$ becomes a $\mathbb{Q}_T$-Brownian motion.

4. Let $\mathbb{Q}$ and $\mathbb{P}$ be two probability measures on $(\Omega, \mathcal{F})$. Assume that Q is absolutely continuous with respect to $\mathbb{P}$ such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = L.$$

Let $\mathcal{G} \subset \mathcal{F}$ and show that for every random variable X

$$\mathbb{E}_{\mathbb{Q}}\left[X \mid \mathcal{G}\right] = \frac{\mathbb{E}_{\mathbb{P}}\left[X\, L \mid \mathcal{G}\right]}{\mathbb{E}_{\mathbb{P}}\left[L \mid \mathcal{G}\right]}.$$

5. Let H_t be an adapted bounded process and let Z_t be the solution of $dZ_t = -H_t Z_t dB_t$ such that $Z_0 = 1$. Define the probability measure $d\mathbb{Q} = Z_T\, d\mathbb{P}$ and prove that

$$\mathbb{E}_{\mathbb{P}}\left[Z_T \log Z_T\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{2}\int_0^T H_s^2\, ds\right].$$