796AA aspetti matematici computazione quantistica
A.A. 2022/23 - Test 10/01/2023

The test is one hour long. Only justified answers will be accepted.

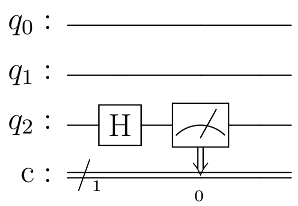
Problem 1

On a three-qubit quantum system $(\mathbb{C}^2)^{\otimes 3}$, consider the the pure state[1] given by

$$|\Psi\rangle = c\,(|000\rangle + |111\rangle)\,,$$

where $c > 0$ is a suitable constant.

1. Determine c. If one drops the requirement $c > 0$, is c unique?
2. Assume that the system is in the state $|\Psi\rangle$ and the third (rightmost) qubit is measured (in the computational basis). What are the possible outcomes, and what are the associated probabilities? What is the state on the remaining two qubits after the measurement if the outcome is 0? Is it separable or entangled?
3. Assume instead that the system is initially in the state $|\Psi\rangle$, one acts with the Hadamard gate on the third qubit and then measures the third qubit (circuit below). What are the possible outcomes, and what are the associated probabilities? What is the state on the remaining two qubits after the measurement if the outcome is 0? Is it separable or entangled?



4. Describe a circuit (in terms of elementary gates) that generates the state $|\Psi\rangle$ if initialized in the state $|000\rangle$.

A solution:

1. Since $|000\rangle$ and $|111\rangle$ are orthonormal, we have

$$\||\Psi\rangle\|^2 = c^2 2,$$

hence $c = 1/\sqrt{2}$. If one drops the positivity assumption, any complex number of the form $e^{i\theta}/\sqrt{2}$ would define a state, hence c is not unique.

2. Measuring the third qubit in the computational basis yields either 0 or 1, with probabilities respectively

$$P(0) = \frac{1}{2}\,\||000\rangle\|^2 = \frac{1}{2}, \quad P(1) = \frac{1}{2}\,\||111\rangle\|^2 = \frac{1}{2}.$$

[1] also called GHZ state after Daniel Greenberger, Michael Horne and Anton Zeilinger

The state on the three qubits, after observing 0 is given by the (normalized) orthogonal projection on the subspace spanned by vectors of the form $|\varphi\rangle \otimes |0\rangle$, hence it is simply $|000\rangle$ since $|111\rangle$ is orthogonal to such subspace. To obtain the state on the two qubits, we simply discard the third one and obtain $|\varphi\rangle = |00\rangle$. The state is separable (it is a product state).

3. Acting with the Hadamard gate on the third qubit yields

$$
\begin{aligned}
|\Psi_1\rangle &= \frac{1}{\sqrt{2}} \left(|00\rangle \otimes H|0\rangle + |11\rangle \otimes H|1\rangle \right) \\
&= \frac{1}{\sqrt{2}} \Big(|00\rangle \otimes (|0\rangle + |1\rangle) \sqrt{2} + |11\rangle \otimes (|0\rangle - |1\rangle) \sqrt{2} \Big) \\
&= \frac{1}{2} \left(|000\rangle + |001\rangle + |110\rangle - |111\rangle \right)
\end{aligned}
$$

Measuring the third qubit gives again outcomes 0 or 1 with uniform probabilities

$$P(0) = \frac{1}{4} \left\| |000\rangle + |110\rangle \right\|^2 = \frac{1}{2}, \quad P(1) = \frac{1}{2}.$$

The state on the three qubits, after observing 0 is given by the (normalized) orthogonal projection on the subspace spanned by vectors of the form $|\varphi\rangle \otimes |0\rangle$, which in this case reads

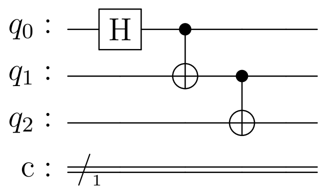
$$\frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right) \otimes |0\rangle .$$

Discarding the third qubit we obtain the state

$$\frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right),$$

which we recognize as a Bell state, a fundamental example of entangled state.

4. A circuit implementing $|\Psi\rangle$ is given by composing a Hadamard gate followed by two CNOT gates as shown in the following diagram:



Let us analytically check that this is indeed the case. After applying the Hadamard gate on the first qubit we obtain the state

$$\frac{1}{\sqrt{2}} \left(|000\rangle + |100\rangle \right).$$

Once we apply the first CNOT gate, we obtain

$$\frac{1}{\sqrt{2}} \left(|000\rangle + |110\rangle \right),$$

since CNOT is not activated on $|000\rangle$. The second CNOT finally gives the target state $|\Psi\rangle$, since it leaves unchanged $|000\rangle$ and maps $|110\rangle$ in $|111\rangle$.

796AA aspetti matematici computazione quantistica
A.A. 2022/23 - Test 14/02/2023

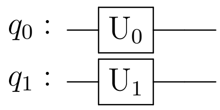
The test is one hour long. Only justified answers will be accepted.

Problem 1

Assume that a 2-qubit system $(\mathbb{C}^2)^{\otimes 2}$ is initially in the Bell state

$$\left|\Psi^+\right\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right).$$

Prove that it is impossible to find a circuit that uses only single-qubit gates (as depicted below) which transforms $|\Psi^+\rangle$ into the state $|00\rangle$.



A solution:

If there were a unitary $U = U_1 \otimes U_2$ transforming $|\Psi^+\rangle$ into $|00\rangle$, using its inverse $U^* = U_1^* \otimes U_2^*$, one transforms $|00\rangle$ into $|\Psi^+\rangle$, which would then be separable, since

$$|\Psi^+\rangle = U^*|00\rangle = (U_1^*|0\rangle) \otimes (U_2^*|0\rangle).$$

Problem 2

Consider a three-qubit quantum system $(\mathbb{C}^2)^{\otimes 3}$, and the pure state[1] given by

$$|\Psi\rangle = c\left(|100\rangle + |010\rangle + |001\rangle\right),$$

where $c > 0$ is a suitable constant.

1. Determine c.
2. Assume that the system is in the state $|\Psi\rangle$ and the third (rightmost) qubit is measured (in the computational basis). What are the possible outcomes, and what are the associated probabilities? What is the state on the remaining two qubits after the measurement if the outcome is 0? Is it separable or entangled?
3. Assume that the system is initially in the state $|\Psi\rangle$, i.e., with density operator $\rho = |\Psi\rangle\langle\Psi|$. Compute the reduced density operator σ associated with the subsystem consisting of the first two qubits (i.e., σ is the partial trace with respect to the third qubit).
4. Is the state σ obtained in the previous point separable or entangled? (*Hint: write the density matrix as* 2×2 *block matrix, take the transpose in each block, and check if it remains positive semi-definite. Why this should reveal entanglement?*)

[1] also called W state after Wolfgang Dür

A solution:

1. Since the state vectors in the computational basis $|100\rangle$, $|010\rangle$, $|001\rangle$ are orthogonal, we have

$$\|\Psi\|^2 = c^2(1+1+1) = 3c^2,$$

hence $c = 1/\sqrt{3}$.

2. When measuring the third most qubit in the computational basis, the outcomes are 0 or 1 (or correspondingly -1, 1 if we measure the observable σ_z). The probabilities are as usual computing the squared norms of the projection on the corresponding eigenspaces. We find

$$P_{|Psi\rangle}(\text{“third qubit is 0”}) = \|\frac{1}{\sqrt{3}}(|100\rangle + |010\rangle)\|^2 = \frac{2}{3},$$

$$P(\text{“third qubit is 1”}) = \|\frac{1}{\sqrt{3}}|001\rangle\|^2 = \frac{1}{3}.$$

After measuring and observing 0, the state collapses to

$$\frac{1}{\sqrt{2}}\left(|10\rangle + |01\rangle\right),$$

which is the Bell state from the previous exercise (it is entangled).

3. Consider the composite system $(\mathbb{C}^2)^{\otimes 2} \otimes \mathbb{C}^2$, and write

$$\begin{aligned}\rho = |\Psi\rangle\langle\Psi| &= \frac{1}{3}\left(|100\rangle + |010\rangle + |001\rangle\right)\left(\langle 100| + \langle 010| + \langle 001|\right) \\ &= \frac{1}{3}\left(|100\rangle + |010\rangle\right)\left(\langle 100| + \langle 010|\right) + \frac{1}{3}|001\rangle\langle 001| \\ &+ \frac{1}{3}|001\rangle\left(\langle 100| + \langle 010|\right) + \frac{1}{3}\left(|100\rangle + |010\rangle\right)\langle 001|.\end{aligned}$$

After the definition of partial trace, we operationally need to “contract” the summation upon the pairs where the third qubits have the same value. This has the effect that the last line above can be entirely dropped, hence

$$\sigma = \frac{1}{3}\left(|10\rangle + |01\rangle\right)\left(\langle 10| + \langle 01|\right) + \frac{1}{3}|00\rangle\langle 00|.$$

4. The state σ although not pure, is entangled. To see this, let us follow the hint: write the density matrix in the canonical basis in block form

$$\sigma = \frac{1}{3}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Consider the transpose in each block

$$\tilde{\sigma} = \frac{1}{3}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right).$$

We compute $\det(\tilde{\sigma}) = -1$, hence the matrix is not positive semidefinite. Why this implies that σ is entangled? if it were separable, then one could represent the density matrix as

$$\sigma = \sum_{i \in I} p_i \rho_i \otimes \sigma_i,$$

with mixed states ρ_i, σ_i on $\mathbb{C}^2$ and a probability distribution $(p_i)_{i \in I}$. Then,

$$\tilde{\sigma} = \sum_{i \in I} p_i \rho_i^t \otimes \sigma_i,$$

where σ_i^t denotes the transposed matrix, which has the same eigenvalues as σ_i (hence it is positive). Hence, we would deduce that $\tilde{\sigma}$ is a sum of positive matrices, hence it should be positive as well.

796AA aspetti matematici computazione quantistica
A.A. 2022/23 - Test 08/06/2023

The test is one hour long. Only justified answers will be accepted.

Problem 1

Consider the two-qubit gate known as the Controlled-NOT (CNOT) gate. It is a fundamental gate in quantum computing and it operates on two qubits, one control qubit and one target qubit.

1. Apply the CNOT gate to the two-qubit Bell state
$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle.$$
with the first qubit as the control qubit and the second qubit as the target qubit. Write the new state $|\psi\rangle$ as a linear combination of the computational basis states $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$.
2. Compute the density matrix for the state $|\psi\rangle$ obtained in Part 1.
3. Assume that a quantum system is in the state $|\psi\rangle$ defined in Part 1. By measuring the system in the Bell basis, what is the probability of observing the system in the state $|\Phi^+\rangle$?
4. Assume that a quantum system is in the state $|\psi\rangle$ defined in Part 1. Compute the expectation value of the Pauli X operator for the first qubit.

796AA aspetti matematici computazione quantistica
A.A. 2022/23 - Test 22/06/2023

The test is one hour long. Only justified answers will be accepted.

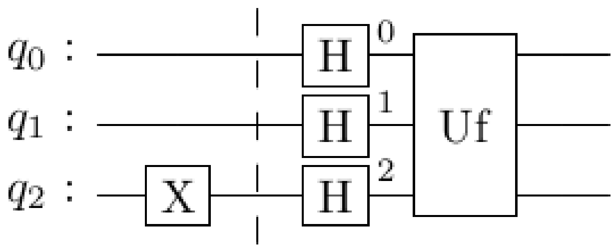
Problem 1

Let $f : \{0,1\}^2 \to \{0,1\}$ be a Boolean function that attains the value 1 at exactly one point in its domain, i.e., there exists a unique pair $(x_1, x_2) \in \{0,1\}^2$ such that $f(x_1, x_2) = 1$.

1. How many possibilities are there for such a function? Write down their explicit expression.
2. Suppose we are given a black-box function $f : \{0,1\}^2 \to \{0,1\}$ with the property mentioned above. Our goal is to find out which function this is, among all the possible ones. How many classical queries are needed for this task?
3. Let us now look at the problem from a quantum perspective. Suppose we are given a black-box quantum gate U_f computing such a function, that is, acting as:

$$|x_1, x_2, y\rangle \to |x_1, x_2, y \oplus f(x_1, x_2)\rangle.$$

   Determine the output of the following quantum circuit for each of the possible functions f:



4. Show that the possible outputs obtained above are pairwise orthogonal.
5. What can you deduce about the quantum complexity of our problem? (As usual, complexity is understood here as number of queries).

796AA aspetti matematici computazione quantistica
A.A. 2022/23 - Test 06/07/2023

The test is one hour long. Only justified answers will be accepted.

Problem 1

Let $s \in \{0,1\}^n$ be an (unknown) binary string of length n which is not identically zero and let $f : \{0,1\}^n \to \{0,1\}^n$ be a function such that $f(x) = f(y)$ if and only if

$$x = y \quad \text{or} \quad x \oplus y = s,$$

where $\oplus$ denotes componentwise sum modulo 2.

As usual, suppose we have an oracle that computes f. The goal of this exercise is to develop an algorithm that determines s using $O(n)$ queries to the oracle.

1. We will work on a quantum system formed by $2n$ qubits, split into two registers of n qubits each. As customary, all qubits are initialized to zero; therefore we denote the initial state as
$$|\psi_0\rangle = |0^n\rangle\,|0^n\rangle\,.$$
Let us now apply a Hadamard gate to each qubit in the first register. Write the expression of the state $|\psi_1\rangle$ obtained after this step.
2. In our setting, a query to the oracle takes the form
$$|x, 0^n\rangle \to |x, f(x)\rangle\,.$$
Write the expression of the state $|\psi_2\rangle$ obtained after applying a query to state $|\psi_1\rangle$.
3. Starting from the state $|\psi_2\rangle$, let us measure the second register in the computational basis. Suppose the result of the measurement is some $y \in \{0,1\}^n$. Write the expression of the state $|\psi_3\rangle$ of the system after this measurement.
4. Let us now discard the second register for the state $|\psi_3\rangle$ and focus on the first one. Apply a Hadamard gate to each qubit in the first register and write the expression of the state $|\phi_4\rangle$ of the first register obtained after this step. Which strings $z \in \{0,1\}^n$ are such that $|z\rangle$ has nonzero amplitude when writing $|\phi_4\rangle$ in the computational basis? What results do we to obtain if we apply a measurement to the first register?
5. By repeating $O(n)$ times the steps in procedure outlined above, how would you build a linear system with solution s?

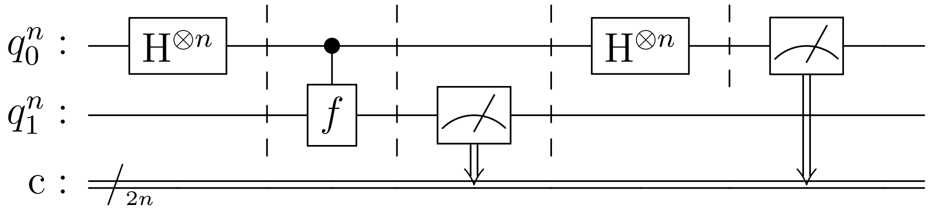


Figure 1: Circuit diagram for steps 1-4 above.

A solution:

1. The application of Hadamard gates puts the first register in a state of uniform superposition, therefore we have
$$|\psi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle\, |0^n\rangle\,.$$

2. After a query we obtain
$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle\, |f(x)\rangle\,.$$

3. From the hypotheses it follows that y has exactly two preimages w.r.t. f, which differ by s and can therefore be denoted as $|t\rangle$ and $|t \oplus s\rangle$. After measuring the second register, we have
$$|\psi_3\rangle = \frac{1}{\sqrt{2}} \left(|t\rangle + |t \oplus s\rangle\right) |y\rangle\,.$$

4. Applying Hadamard gates to the first register yields
$$\begin{aligned} |\phi_4\rangle &= \frac{1}{\sqrt{2^{n+1}}} \left(\sum_{x\in\{0,1\}^n} (-1)^{t\cdot x} |x\rangle + \sum_{x\in\{0,1\}^n} (-1)^{(t\oplus s)\cdot x} |x\rangle \right) \\ &= \frac{1}{\sqrt{2^{n+1}}} \left(\sum_{x\in\{0,1\}^n} (-1)^{t\cdot x} |x\rangle \left(1 + (-1)^{s\cdot x}\right) |x\rangle \right). \end{aligned}$$

   From the expression of $|\phi_4\rangle$ it follows that $|z\rangle$ has nonzero amplitude iff $s \cdot z = 0 \mod 2$. Therefore if we apply a measurement to the first register we obtain a state associated with a string orthogonal to s.

5. Repeating this procedure $O(n)$ times we expect to be able to find $n-1$ independent linear equations forming a linear system whose solution is s. The system can then be solved classically. It can be proved that at each iteration we have probability $\geq \frac{1}{2}$ of finding a new equation that is linearly independent w.r.t. the ones found previously.

796AA aspetti matematici computazione quantistica
A.A. 2022/23 - Test 20/09/2023

The test is one hour long. Only justified answers will be accepted.

Problem 1

Consider a four-qubit system initially in the state

$$|\psi\rangle = c(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle),$$

with $c \in \mathbb{C}$.

1. Determine all the possible values of c.
2. With the usual convention that binary strings correspond to binary expansions of natural numbers, compute the average outcome of measuring the system in the computational basis.
3. Discuss whether equality holds in Heisenberg's uncertainty inequality for the pair of observables $X^{\otimes 4}$ and $Z^{\otimes 4}$ on the state $|\psi\rangle$.
4. Suppose we perform a measurement on the first qubit (from the right) of the four-qubit system in the computational basis. What is the probability of obtaining the outcome 0?
5. Assuming that the measurement in the previous point is performed with outcome 0, write the resulting density operator.
6. Describe a quantum circuit that prepares the state $|\psi\rangle$ starting from the initial state $|0000\rangle$.

Answers (without justification):

1. $c = e^{i\theta}/2$, $\theta \in [0, 2\pi)$
2. the mean is $(0 + 5 + 10 + 15)/4$
3. Equality holds since e.g. $|\psi\rangle$ is eigenvector for $X^{\otimes 4}$, hence it is sharp.
4. The probability is $1/2$.
5. The state vector is $(|0000\rangle + |0100\rangle)/\sqrt{2}$.
6. The following circuit does the job: