796AA aspetti matematici computazione quantistica
A.A. 2022/23 - Raccolta di Esercizi [1]

Problem 1

Recall that any single qubit mixed state $\rho \in \mathbb{D}(\mathbb{C}^2)$ can be represented as

$$\rho = \frac{1}{2}\left(1 + b_x\sigma_x + b_y\sigma_y + b_z\sigma_z\right)$$

for a Bloch vector $b = (b_x, b_y, b_z)$ with $b \in \mathbb{R}^3$ and $|b|^2 = b_x^2 + b_y^2 + b_z^2 \leq 1$ and Pauli matrices σ_x, σ_y, $\sigma_z \in \mathbb{C}^{2\times 2}$.

1. prove that ρ is pure if and only if b belongs to the Bloch sphere, i.e. $|b| = 1$,
2. compute the Bloch vectors b for the states

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

   Are these states pure?

A solution:

1. This was proved in the lectures. 2. For the first state, we have $b = (0, 0, 1)$, for the second state $b = (1, 0, 0)$, for the third one $b = (-1, 0, 0)$. In all the cases the states are pure (using the first question).

Problem 2

Consider a pure state $|\psi\rangle \in \mathbb{C}^2$ for a single qubit system and write explicitly (e.g. in terms of the Bloch sphere parametrization as in the previous problem) Heisenberg's uncertainty inequality for the pair of observables σ_x, σ_z. Investigate whether equality may occur.

A solution:

Write $\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\left(1 + b_x\sigma_x + b_y\sigma_y + b_z\sigma_z\right)$ for some b in the Bloch sphere. We compute

$$\langle\sigma_x\rangle_\rho = \mathrm{Tr}(\sigma_x\rho) = \frac{1}{2}\mathrm{Tr}(\sigma_x\left(1 + b_x\sigma_x + b_y\sigma_y + b_z\sigma_z\right)) = b_x,$$

and similarly

$$\langle\sigma_z\rangle_\rho = \mathrm{Tr}(\sigma_z\rho) = \frac{1}{2}\mathrm{Tr}(\sigma_z\left(1 + b_x\sigma_x + b_y\sigma_y + b_z\sigma_z\right)) = b_z.$$

[1] Aggiornato il 14 settembre 2023.

To compute the uncertainty, notice that, by developing the square:

$$\Delta(A)^2_\rho = \langle (A - \langle A\rangle_\rho 1)^2\rangle_\rho = \mathrm{Tr}(A^2\rho) - \mathrm{Tr}(A\rho)^2.$$

Therefore,

$$\Delta(\sigma_x)^2 = \mathrm{Tr}(\sigma_x^2\rho) - b_x^2 = \mathrm{Tr}(\rho) - b_x^2 = 1 - b_x^2,$$
$$\Delta(\sigma_z)^2 = \mathrm{Tr}(\sigma_z^2\rho) - b_z^2 = \mathrm{Tr}(\rho) - b_z^2 = 1 - b_z^2,$$

Finally, we have by the commutation relations that

$$[\sigma_x, \sigma_z] = -2i\sigma_y,$$

so that $\langle i[\sigma_x, \sigma_z]\rangle_\rho = -2b_y$ and Heisenberg uncertainty inequality reads

$$|b_y| \leq \sqrt{1-b_x^2}\sqrt{1-b_z^2}.$$

Since b belongs to the Bloch sphere, we have $|b_y| = \sqrt{1-b_x^2-b_z^2}$, so that the inequality becomes

$$\sqrt{1-b_x^2-b_z^2} \leq \sqrt{1-b_x^2}\sqrt{1-b_z^2}.$$

To check for equality, we square both sides and simplify some terms, so that the inequality is seen to be equivalent to

$$0 \leq b_x^2 b_z^2,$$

and we see that equality occurs if and only if $b_x = 0$ or $b_z = 0$.

Problem 3

Suppose that two pure state vectors $|\phi\rangle$, $|\psi\rangle \in \mathbb{C}^2$ are orthogonal $\langle\phi|\psi\rangle = 0$. Show that there exists a unitary $U : \mathbb{C}^2 \otimes \mathbb{C}^2$ that "clones" both $|\phi\rangle$ and $|\psi\rangle$, i.e.,

$$U|\phi\rangle \otimes |0\rangle = |\phi\rangle \otimes |\phi\rangle, \quad U|\psi\rangle \otimes |0\rangle = |\psi\rangle \otimes |\psi\rangle.$$

Why this does not contradict the no-cloning theorem?

A solution:

The vectors $\{|\phi\rangle \otimes |0\rangle, |\psi\rangle \otimes |0\rangle\}$ are orthonormal. Similarly, the vectors $\{|\phi\rangle \otimes |\phi\rangle, |\psi\rangle \otimes |\psi\rangle\}$ are orthonormal. Therefore, to define U as required it is sufficient to complete both sets of vectors to orthonormal basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$ and impose that the added (two) vectors are mapped to the added vectors. This does not contradict the no-cloning theorem because anyway U does not "clone" all the states, e.g. we have

$$U\frac{1}{\sqrt{2}}\left(|\phi\rangle + |\psi\rangle\right) \otimes |0\rangle = \frac{1}{\sqrt{2}}\left(|\phi\rangle \otimes |\phi\rangle + |\psi\rangle \otimes |\psi\rangle\right),$$

which does not coincide with the state

$$\frac{1}{2}\left(|\phi\rangle + |\psi\rangle\right) \otimes \left(|\phi\rangle + |\psi\rangle\right),$$

as seen by computing the scalar product between the two, that yields $\frac{1}{\sqrt{2}}$.

Problem 4

On $\mathbb{H} = \mathbb{C}^2$ define the states $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. Write $|+-\rangle = |+\rangle \otimes |-\rangle$, $|-+\rangle = |-\rangle \otimes |+\rangle$, and express the state

$$\frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$$

in the computational basis on $\mathbb{H}^{\otimes 2}$.

A solution:

We have

$$|-+\rangle = \frac{1}{2}\left(|0\rangle - |1\rangle\right) \otimes \left(|0\rangle + |1\rangle\right) = \frac{1}{2}\left(|00\rangle + |01\rangle - |10\rangle - |11\rangle\right)$$

and similarly

$$|+-\rangle = \frac{1}{2}\left(|0\rangle + |1\rangle\right) \otimes \left(|0\rangle - |1\rangle\right) = \frac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right).$$

Summation yields

$$\frac{|+-\rangle - |-+\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right).$$

Problem 5

Given self-adjoint operators M_A, M_B respectively on Hilbert spaces $\mathbb{H}^A$, $\mathbb{H}^B$, prove that $M_A \otimes M_B$ is self-adjoint on $\mathbb{H}^A \otimes \mathbb{H}^B$. Can you describe its spectrum in terms of the spectra of M_A and M_B?

A solution:

We already proved during the lectures. The spectrum is given by the products

$$\sigma(M_A \otimes M_B) = \{\lambda_A \lambda_B \,:\, \lambda_A \in \sigma(M_A), \lambda_B \in \sigma(M_B)\},$$

since one can diagonalize $M_A \otimes M_B$ using the orthonormal basis of eigenvectors $(e_i \otimes f_j)_{i,j}$ where $(e_i)_i$, respectively (f_j), are the orthonormal basis of eigenvectors of M_A, respectively M_B.

Problem 6

Consider on a joint system $\mathbb{H}^A \otimes \mathbb{H}^B$ an operator $M = M^A \otimes M^B$. Prove that $\mathrm{Tr}^B(M^A \otimes M^B) = M^A \,\mathrm{Tr}(M^B)$ and $\mathrm{Tr}^A(M^A \otimes M^B) = \mathrm{Tr}(M^A)M^B$. In particular, if $\rho = \rho_A \otimes \rho_B$ for state operators ρ_A, ρ_B, then the reduced state from ρ on the system $\mathbb{H}^A$ is $\mathrm{Tr}^B(\rho) = \rho_A$ and similarly $\rho_B = \mathrm{Tr}^A(\rho)$.

A solution:

Consider orthonormal basis $(e_i)_i \subseteq \mathbb{H}^A$, $(f_j)_j \subseteq \mathbb{H}^B$ so that

$$M = \sum_{i,j,k,\ell} M_{ij,kl} |e_i, f_j\rangle\langle e_k f_\ell|,$$

with $M_{ij,k\ell} = \langle e_i \otimes f_j | M e_k f_\ell\rangle$, which in this case yields

$$M_{ij,k\ell} = M^A_{ik} M^B_{j\ell}.$$

The partial trace $\mathrm{Tr}^B(M)$ is given by

$$\begin{aligned}\mathrm{Tr}^B(M) &= \sum_{i,k} |e_i\rangle\langle e_k| \left(\sum_j M_{ij,kj}\right) = \sum_{i,k} M^A_{ij} |e_i\rangle\langle e_k| \left(\sum_j M^B_{jj}\right) \\ &= \mathrm{Tr}(M^B) \sum_{i,k} M^A_{ij} |e_i\rangle\langle e_k| = \mathrm{Tr}(M^B) M^A.\end{aligned}$$

The argument for $\mathrm{Tr}^A(M) = \mathrm{Tr}(M^A)M^B$ is similar.

Problem 7

Write the matrix corresponding to the operator $H \otimes H$, in the computational basis, where H is the Hadamard operator.

A solution:

Recall that in the computational basis we represent H as

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Since the matrix representing $M_A \otimes M_B$ in the computational basis is the Kronecker product of the two matrices representing M_A, M_B, we have in this case that $H \otimes H$ is represented by

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Problem 8

Consider the Pauli operators $X = \sigma_x = |1\rangle\langle 0| + |0\rangle\langle 1|$, $Y = \sigma_y = i|1\rangle\langle 0| - i|0\rangle\langle 1|$.

1. Find the matrix representation (with respect to the computational basis in $\mathbb{H}^{\otimes 2}$) of
$$A = \sigma_x \otimes \sigma_y, \quad \text{and} \quad B = \sigma_y \otimes \sigma_x.$$
2. Prove that A, B are self-adjoint operators and compute their spectra.
3. Compute $[A, B]$.
4. Assume that the system is prepared in the Bell state $(|00\rangle + |11\rangle)/\sqrt{2}$. What is the probability of observing 1 if we measure A?

A solution:

1. Since
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
we obtain the Kronecker product representations
$$A = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$
and
$$B = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$
We see that both A, B are self-adjoint. To compute their spectra, we may notice that
$$\frac{1}{\sqrt{2}}\left(|00\rangle + i|11\rangle\right), \quad \text{to be added...}$$
provide an orthonormal basis of eigenvectors, or use the fact that the spectra of σ_x, σ_y are $\{-1, 1\}$, hence the spectrum of A and B is also $\{-1, 1\}$.

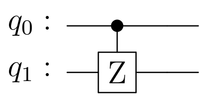
2. We see that $[A, B] = 0$, since
$$AB = (\sigma_x \otimes \sigma_y)(\sigma_y \otimes \sigma_x) = (\sigma_x\sigma_y) \otimes (\sigma_y\sigma_x) = -i^2\sigma_z \otimes \sigma_z$$
and
$$BA = (\sigma_y \otimes \sigma_x)(\sigma_x \otimes \sigma_y) = (\sigma_y\sigma_x) \otimes (\sigma_x\sigma_y) = -i^2\sigma_z \otimes \sigma_z.$$

3. to be added

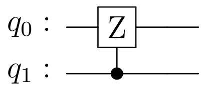
Problem 9

Let $Z = \sigma_z$ be the Pauli operator and consider the two controlled Z gates denoted respectively as $\Lambda^1(Z)$ and $\Lambda_1(Z)$ in Scherer's book on a 2-qubit system (represented below). Write explicitly their matrix representations and argue that $\Lambda^1(Z) = \Lambda_1(Z)$.

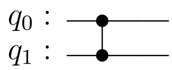
$\Lambda^1(Z)$:



$\Lambda_1(Z)$:



In view of this identity, it is usually simply represented as follows:



A solution:

Recall that $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$. We have the identity

$$\begin{aligned}\Lambda^1(Z) &= Z \otimes |1\rangle\,(1) + 1 \otimes |0\rangle\langle 0| \\ &= |01\rangle\langle 01| - |11\rangle\langle 11| + |00\rangle\langle 00| + |10\rangle\langle 10|.\end{aligned}$$

We obtain

$$\Lambda^1(Z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

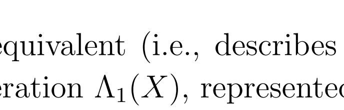
On the other side,

$$\begin{aligned}\Lambda^0(Z) &= |1\rangle\,(1) \otimes Z + |0\rangle\langle 0| \otimes 1 \\ &= |10\rangle\langle 10| - |11\rangle\langle 11| + |00\rangle\langle 00| + |01\rangle\langle 01|,\end{aligned}$$

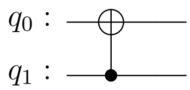
so that the matrix coincides with the one above and we obtain that $\Lambda^1(Z) = \Lambda_1(Z)$.

Problem 10

Show that the following quantum circuit on a register of 2 qubits



is equivalent (i.e., describes the same unitary transformation) to the controlled X operation $\Lambda_1(X)$, represented as



A solution:

We recall the matrix representations for the Hadamard operator

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

the CNOT $\Lambda^1(X)$ operator (the control qubit is the right one)

$$\Lambda^1(X) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and the $\Lambda_1(X)$ operator (the control qubit is the left one)

$$\Lambda_1(X) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix},$$

written in block notation. The operator $H \otimes H$ is represented in block form as

$$H \otimes H = \frac{1}{\sqrt{2}} \begin{pmatrix} H & H \\ H & -H \end{pmatrix},$$

We multiply the block matrices

$$H \otimes H \Lambda_1(X) = \frac{1}{\sqrt{2}} \begin{pmatrix} H & HX \\ H & -HX \end{pmatrix}$$

and then

$$(H \otimes H \Lambda_1(X))\, H \otimes H = \frac{1}{2} \begin{pmatrix} H^2 + HXH & H^2 - HXH \\ H^2 - HXH & H^2 + HXH \end{pmatrix}.$$

We have $H^2 = 1$, and

$$HXH = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

so that

$$H^2 + HXH = 2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad H^2 - HXH = 2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and we recognize the matrix representing $\Lambda^1(X)$.

Problem 11

Given a state vector on n qubits, $|\psi\rangle \in \mathbb{H}^{\otimes n}$, represent it in terms of the computational basis as

$$|\psi\rangle = \sum_{s \in \{0,1\}^n} \alpha_s |s\rangle.$$

Consider the observable $\sigma_z \otimes 1_{\mathbb{C}^2}^{\otimes n-1}$.

1. What is the probability that the measurement outcome is 1, in terms of the α coefficients?
2. Assume that the outcome of the measurement is 1. What is the state of the system after such a measurement?

A solution:

1. The eigenspace V associated to the eigenvalue 1 for $\sigma_z \otimes 1_{\mathbb{C}^2}^{\otimes n-1}$ consists of the span of vectors in the computational basis $|s\rangle$ with $s = (s_0, s_1, \ldots, s_{n-1})$ such that $s_0 = 0$. Therefore, the orthogonal projection onto this space is

$$P = \sum_{s \in \{0,1\}^{n-1}} |0s\rangle\langle 0s|.$$

We obtain

$$P|\psi\rangle = \sum_{s \in \{0,1\}^{n-1}} \alpha_{0s} |0s\rangle,$$

hence the probability of measuring 1 is

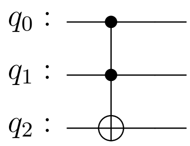
$$\|P|\psi\rangle\|^2 = \sum_{s \in \{0,1\}^{n-1}} |\alpha_{0s}|^2$$

and after observing one the state vector $|\psi\rangle$ collapses into

$$P|\psi\rangle / \|P|\psi\rangle\| = \frac{\sum_{s \in \{0,1\}^{n-1}} \alpha_{0s}|0s\rangle}{\sqrt{\sum_{s \in \{0,1\}^{n-1}} |\alpha_{0s}|^2}}.$$

Problem 12

Write the Toffoli gate using kets, bras and tensor products, and in matrix form with respect to the canonical basis on three qubits. Recall that Toffoli gate is a doubly-controlled not operation, represented as follows:



A solution:

Writing $1 = |0\rangle\langle 0| + |1\rangle\langle 1|$ for the identity operator and $X = |1\rangle\langle 0| + |0\rangle\langle 1|$,e we have

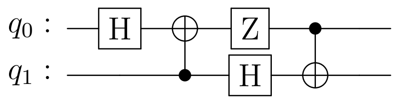
$$T = |11\rangle\langle 11| \otimes X + (1^{\otimes 3} - |11\rangle\langle 11|) \otimes 1 = |11\rangle\langle 11| \otimes (X - 1) + 1^{\otimes 3}.$$

In matrix notation

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Problem 13

Consider the following quantum circuit U:



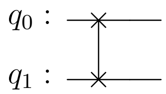
1. Compute the matrix representing the circuit transformation U in the computational basis.
2. Write a circuit representing the inverse transformation U^{-1}
3. If the initial state of the system is $|00\rangle$, what is the probability distribution of the readout in the computational basis?

A solution:

1.

Problem 14

The SWAP gate for 2 qubits is defined as $S|x\rangle \otimes |y\rangle = |y\rangle \otimes |x\rangle$ for $x, y \in \{0, 1\}$, and in circuit notation denoted as



1. Write the matrix representation of S in the computational basis.
2. Is S self-adjoint? Compute its spectrum.
3. Express S as a composition of controlled not gates.

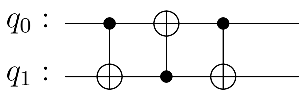
A solution:

1. We have $S|00\rangle = |00\rangle$, $S|11\rangle = |11\rangle$, $S|01\rangle = |10\rangle$ and $S|10\rangle = |01\rangle$, so that the matrix reads

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2. S is self-adjoint (it is real and symmetric). To compute its spectrum, notice that $S^2 = 1$, so $\sigma(S) \subseteq \{-1, 1\}$, but $S \neq 1$, so necessarily $\sigma(S) = \{-1, 1\}$.

3. It is straightforward to check that $S = \Lambda^1(X)\Lambda_1(X)\Lambda^1(X)$ (also represented as a circuit below):



Indeed, if the input state is $|00\rangle$, also the output will be $|00\rangle$ (all the controls are not activated). If the input is $|11\rangle$, only the first and third controls are activated, hence we apply twice an X on the target bit, hence the output is $|11\rangle$. If the input is $|01\rangle$, only the first and the second gates are activated and the output is $|10\rangle$, while if $|10\rangle$ only the second and third and the output is $|01\rangle$.

Problem 15

Given a permutation over n-elements $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$, its (column) permutation matrix $U_\sigma \in \mathbb{C}^{n \times n}$ is given by

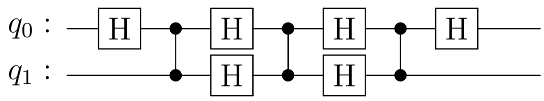
$$(U_\sigma)_{ij} = \begin{cases} 1 & \text{if } j = \sigma(i) \\ 0 & \text{otherwise.} \end{cases}$$

1. Prove that, for every permutation σ, the matrix U_σ is unitary with inverse $U_{\sigma^{-1}}$.
2. Show that every permutation matrix over 4 elements can be obtained as a suitable composition of the gates $X \otimes 1$, $1 \otimes X$ and SWAP.

3. Is every permutation matrix over 8 elements obtainable as a suitable composition of the gates $X \otimes 1^{\otimes 2}$, $1 \otimes X \otimes 1$, $1^{\otimes 2} \otimes X$ and Toffoli gates as in Problem 12?

Problem 16

Show that the following circuit implements the SWAP gate using only Hadamard and CZ gates



Problem 17

Consider a system with two qubits $(\mathbb{C}^2)^{\otimes 2}$ initialized on the state

$$|\psi\rangle = |+\rangle \otimes |+\rangle\,,$$

where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Consider the following observables

$$P_0 = \frac{1}{2}\left(1 \otimes 1 + \sigma_z \otimes \sigma_z\right),$$
$$P_1 = \frac{1}{2}\left(1 \otimes 1 - \sigma_z \otimes \sigma_z\right).$$

1. Show that P_0, P_1 are actually orthogonal projections.
2. Compute the expectations $\langle P_0 \rangle_{|\psi\rangle}$, $\langle P_1 \rangle_{|\psi\rangle}$.
3. Assume that we measure P_0 on the system and obtain the result 0. Describe the state after the measurement in the computational basis.

Problem 18

Consider a system with two qubits $(\mathbb{C}^2)^{\otimes 2}$ initialized on the Bell state

$$\left|\psi^+\right\rangle = \left(|00\rangle + |11\rangle\right)/\sqrt{2}.$$

Using only suitable gates of the form $U \otimes 1$, with $U \in \mathbb{C}^{2\times 2}$ unitary, is it possible to obtain all the other states in the Bell basis?

Problem 19

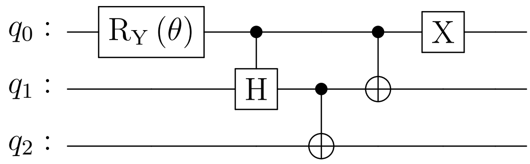
The classical Monty Hall problem consists of determining what is the best strategy in the following game. We are given three identical closed boxes, exactly one of them containing a valuable item, the other two being empty. We are asked to choose a box among the three and a game host (who knowns the boxes' contents) then reveals an empty box, among one that we did not choose. The host then asks if we wish to change our mind and swap our initial box with the other closed one. What is the choice that maximizes the probability of finding the item?

Following https://youtu.be/Hd9KhRts1uw, let us solve the same problem using a quantum circuit on three qubits $(\mathbb{C}^2)^{\otimes 3}$ and an extra qubit representing the number of the box revealed by the host. Assume that the initial state is $|\psi_0\rangle = |000\rangle \otimes |0\rangle$.

1. We prepare the system into a uniform superposition of three states (representing the fact that the item may be in each of the three boxes with uniform probability)

$$|\psi_1\rangle = \frac{1}{\sqrt{3}} \left(|100\rangle + |010\rangle + |001\rangle\right) \otimes |0\rangle$$

   Show that this can be done using the following circuit:



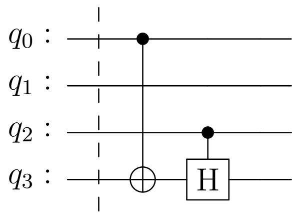
   where

$$R_y(\theta) = e^{-i\theta\sigma_y/2} = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix},$$

   for a suitable θ to be determined.

2. Is the state $|\Psi_1\rangle$ separable or entangled?

3. Assume for simplicity that we choose the box 3 (the bottom one). Since the host shows either box 0 or box 1, we use the following circuit to store on the fourth qubit the number of the box revealed:



   Write the statevector of the system after applying these gates (assuming that the state before is $|\psi_1\rangle \otimes |0\rangle$).

4. If we measure only the fourth qubit, what is the probability of measuring 0? and of measuring 1? What is the state of the system after the measurement?

5. If we measure instead all the four qubits, compute the probabilities of all the possible outcomes.

6. Does this circuit confirms that it is more convenient to swap?

Problem 20

The classical coin flipping game is described as follows. A box contains a fair coin which can display either head or tail, and initially displays head. There are two players (Alice and Bob), which alternatively can flip the coin or do nothing. It is not allowed at any time to see the state of the coin neither see the other player's moves. First Alice plays a move, then Bob and then again Alice. Finally, Alice wins if the coin displays head, otherwise Bob wins. As usual in game theory, we let a mixed strategy consist of a classical random variable $(X_1, X_2) \in \{\text{Flip}, \neg\text{Flip}\}^2$ for the strategy of Alice, and similarly $Y_1 \in \{\text{Flip}, \neg\text{Flip}\}$ for Bob. The two variables are independent. What is the best strategy for Alice? can she win with probability larger than 1/2?

The *quantum coin flipping game*[2] is a variant where Alice can perform also quantum operations, while Bob can only perform the classical coin flip. Let us encode head as $|0\rangle$ and tail as $|1\rangle$ on a single qubit system. Initially the system is on head $|0\rangle$. Alice can choose any unitary gate U_1 and U_2 on her moves, while Bob can only apply the X gate (corresponding to the classical coin flip) according to some classical probability.

1. Show that Alice has a strategy which allows her to win with probability 1.
2. What if we reverse the roles and now Alice can only play classical moves (i.e. X gates) and Bob has also quantum ones? Has Bob a winning strategy with probability 1?
3. What if both Alice and Bob can play quantum moves?
4. What if Alice can play only X or Z gates and Bob only X gates?

Problem 21

Hello Quantum (or Hello Qiskit https://qiskit.org/textbook/ch-ex/hello-qiskit.html) is a game developed by IBM and also freely available on Google Play and Apple App Store. It consists of several puzzles where one has to move a 2-qubit state to a target state using only Z, H, X and CZ gates. An additional challenge is to solve the puzzle using a minimum amount of elementary gates. Play throughout the game and write your solutions in circuit notations.

[2] https://qiskit.org/textbook/ch-demos/coin-game.html