Lezione 18

Derivata di Malliavin e concentrazione della misura.

$$\boxed{\text{Sia } F \in \mathbb{D}^{1,2} \quad DF = 0 \quad \mathbb{P}\text{ q.c.} \quad \text{Allora} \quad F = \mathbb{E}[F] \quad \mathbb{P}\text{-q.c.}}$$

Concentrazione: Se $DF \approx 0 \implies F \approx \mathbb{E}[F]$.

Semigruppo di Ornstein-Uhlenbeck:

$$P_t F(\omega) = \mathbb{E}^{\omega'}\left[F\left(e^{-t}\omega + \sqrt{1-e^{-2t}}\,\omega'\right)\right] \qquad \boxed{\begin{array}{l}\omega \in \mathbb{R}^d \\ \omega' \sim N(0, Id)\end{array}}$$

Vale $\boxed{\dfrac{d}{dt} P_t F = L P_t F} = -\delta D P_t F = P_t(LF)$

$D = \nabla$, $\delta G = -\sum_{i=1}^{d} \partial_{x_i} G_i + x_i G_i$, $\boxed{LF = \sum_{i=1}^{d} \partial^2_{x_i} F - x_i \partial_{x_i} F}$

$$\partial_t P_t F(\omega) = \mathbb{E}^{\omega'}\left[\partial_t F\left(e^{-t}\omega + \sqrt{1-e^{-2t}}\,\omega'\right)\right]$$

$$= \mathbb{E}^{\omega'}\left[\nabla F\left(e^{-t}\omega + \sqrt{1-e^{-2t}}\,\omega'\right)\left(-e^{-t}\omega + \underbrace{\frac{e^{-2t}}{\sqrt{1-e^{-2t}}}\,\omega'}_{\mathrm{I}}\right)\right]$$

$$D P_t F(\omega) = \mathbb{E}^{\omega'}\left[\nabla F\left(e^{-t}\omega + \sqrt{1-e^{-2t}}\,\omega'\right) e^{-t}\right]$$

$$\delta D P_t F(\omega) = \underbrace{\mathbb{E}^{\omega'}\left[\Delta F\left(e^{-t}\omega + \sqrt{1-e^{-2t}}\,\omega'\right) e^{-2t}\right]}_{\mathrm{II}} -$$

$$- \mathbb{E}^{\omega'}\left[\langle \nabla F\left(e^{-t}\omega + \sqrt{1-e^{-2t}}\,\omega'\right), \omega\rangle e^{-t}\right]$$

$$\mathrm{I} = \mathbb{E}^{\omega'}\left[(\nabla F)\left(e^{-t}\omega + \sqrt{1-e^{-2t}}\,\omega'\right) \frac{e^{-2t}}{\sqrt{1-e^{-2t}}}\,\omega' \right] =$$

$$= \mathbb{E}^{\omega'}\left[D^{\omega'}\left(F\left(e^{-t}\omega + \sqrt{1-e^{-2t}}\,\omega'\right)\right) \frac{e^{-2t}}{1-e^{-2t}}\,\omega' \right]$$

$$\mathrm{II} = \mathbb{E}^{\omega'}\left[\Delta F\left(e^{-t}\omega + \sqrt{1-e^{-2t}}\,\omega'\right) e^{-2t}\,\omega' \right] =$$

$$= \mathbb{E}^{\omega'}\left[\Delta^{\omega'} F\left(e^{-t}\omega + \sqrt{1-e^{-2t}}\,\omega'\right) \frac{e^{-2t}}{1-e^{-2t}} \right]$$

$$- \frac{e^{-2t}}{1-e^{-2t}}\, \mathbb{E}^{\omega'}\left[L^{\omega'} F\left(e^{-t}\omega + \sqrt{1-e^{-2t}}\,\omega'\right)\right] = 0$$

$$\mathbb{E}^{\omega'}\left[L^{\omega'} G \right] = \mathbb{E}\left[1 \cdot \delta D G\right] = \mathbb{E}\left[\nabla 1\, \nabla G\right] = 0$$

$$P_0 F(\omega) = F(\omega)$$

$$\lim_{t\to+\infty} P_t F(\omega) = \lim_{t\to+\infty} \mathbb{E}^{\omega'}\left[F\left(e^{-t}\omega + \sqrt{1-e^{-2t}}\,\omega'\right)\right] = \mathbb{E}^{\omega'}\left[\lim_{t\to+\infty} F\left(e^{-t}\omega + \sqrt{1-e^{-2t}}\,\omega'\right)\right]$$

$$= \mathbb{E}^{\omega'}\left[F(\omega')\right] = \mathbb{E}[F]$$

Se $DF \equiv 0$ allora $\delta D F = 0 \Rightarrow \partial_t P_t F = P_t(LF) = 0$

$\Rightarrow F(\omega) = \mathbb{E}[F]$ $\square$

<u>Invarianza</u> $\mathbb{E}^{\omega}\left[P_t F(\omega)\right] = \mathbb{E}^{\omega}\left[F(\omega)\right]$.

Clark-Ocone-Karatzas

$\Downarrow$

Proposizione (Dis di Poincaré) Se $F \in \mathbb{D}^{1,2}$,

$$Var(F) = \mathbb{E}[(F - \mathbb{E}[F])^2] \leq \mathbb{E}[|DF|^2].$$

Dim

Bakry-Emery

$$|DP_tF|(\omega) = \left|\mathbb{E}^{\omega'}[\nabla F(e^{-t}\omega + \sqrt{1-e^{-2t}}\omega')]\right| e^{-t}$$

$$\leq \mathbb{E}^{\omega'}[|\nabla F|(e^{-t}\omega + \sqrt{1-e^{-2t}}\omega')]e^{-t}$$

$$\leq P_t(|DF|)\, e^{-t}$$

$$Var(F) = \mathbb{E}[F^2 - (\mathbb{E}[F])^2] =$$

$$= \mathbb{E}[(P_0F)^2 - (P_\infty F)^2] =$$

$$= \mathbb{E}\left[-\int_0^{+\infty} \partial_t (P_tF)^2\, dt\right] =$$

$$= \mathbb{E}\left[+\int_0^{+\infty} 2P_tF \cdot L(P_tF)dt\right] = 2\int_0^{+\infty} \mathbb{E}[(P_tF)L(P_tF)]dt$$

$$= 2\int_0^{+\infty} \mathbb{E}[|DP_tF|^2 dt] \leq 2\int_0^{+\infty} e^{-2t}\, \mathbb{E}[P_t(|DF|^2)]dt$$

$$\leq \mathbb{E}[|DF|^2]$$

Corollario

$$\Longrightarrow \quad \mathbb{P}(|F - \mathbb{E}[F]| > \lambda) \leq \frac{\mathbb{E}[|DF|^2]}{\lambda^2} \quad \forall \lambda > 0$$

Prop (Dis di Sobolev Logaritmica) Sia $F \in \mathbb{D}^{1,2}$ con $\mathbb{E}[F^2] = 1$

allora $$\mathbb{E}[(F^2)\log(F^2)] \le 2\,\mathbb{E}[|\nabla F|^2]$$

OSS posto $\boxed{q = F^2}$ densità di prob rispetto a $N(0,I)$

$$\mathbb{E}[q \log q] = D(q \,|\, N(0,\mathrm{Id})) \longrightarrow \text{Entropia relativa}$$

$$\mathbb{E}[|\nabla F|^2] = \mathbb{E}[|\nabla \sqrt{q}|^2] = \frac{1}{4}\mathbb{E}\left[\frac{|\nabla q|^2}{q}\right] \to \text{Inf di Fisher}$$

Dim $q_t := P_t q = P_t(F^2)$ $\qquad \mathbb{E}[q_t] = 1$

$$\mathbb{E}[q_t(\log q_t)] \quad \begin{cases} \xrightarrow{t\to+\infty} \mathbb{E}[1 \log 1] = 0 \\ \xrightarrow{t\to 0} \mathbb{E}[q \log q] \end{cases}$$

$$\left[\begin{array}{l} \mathbb{E}[\partial_t q_t] \\ \quad \shortparallel \\ \partial_t \mathbb{E}[q_t] = \partial_t 1 = 0 \end{array}\right.$$

$$\partial_t \mathbb{E}[q_t \log q_t] = \mathbb{E}\left[(\partial_t q_t)\log q_t + \cancel{q_t \frac{\partial_t q_t}{q_t}}\right]$$

$$= -\mathbb{E}[(L q_t)\log q_t]$$

$$= -\mathbb{E}[(Dq_t)(D\log q_t)]$$

$$= -\mathbb{E}\left[\frac{|Dq_t|^2}{q_t}\right]$$

$$\mathbb{E}[q \log q] = \int_0^{+\infty} \mathbb{E}\left[\frac{|Dq_t|^2}{q_t}\right] \le \int_0^{+\infty} e^{-2t}\,\mathbb{E}\left[\frac{(P_t|Dq|)^2}{P_t q}\right] dt$$

$$\boxed{\mathbb{R}^+ \times \mathbb{R}^+ \ni (x,y) \longmapsto \Phi(x,y) = \frac{x^2}{y} \text{ è convessa}}$$

$$\Rightarrow \quad \frac{\left(P_t|Dq|(\omega)\right)^2}{P_t q(\omega)} = \Phi\left(P_t|Dq|(\omega), P_t q(\omega)\right) \le P_t\left(\Phi(|Dq|, q)\right)$$

Jensen

$$\Rightarrow \quad \boxed{\mathbb{E}[q \log q] \le \frac{1}{2}\, \mathbb{E}\left[\frac{|\nabla q|^2}{q}\right]}$$

Varianti con $|DF| \in L^p$ $\qquad p \in [1, \infty]$

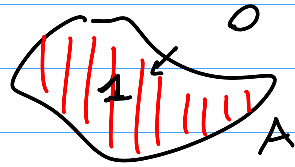
I) $p = 1$ Isoperimetria Gaussiana

II) $p \to +\infty$ Concentrazione Gaussiana

$$DF = (D_1 F, D_2 F, \dots, D_d F)$$

I) $\mathbb{D}^{1,1} \subseteq BV = \{DF$ è una misura (vettoriale) $\|DF\|_{TV} < \infty\}$

Esempio fondamentale $\qquad F = \chi_A \in BV$ "A ha perimetro finito"



"$D\chi_A = \vec{n}\left(\mathcal{H}^{d-1} \llcorner \partial A\right) \frac{e^{-|x|^2/2}}{(\sqrt{2\pi})^d}$"

$$(IBP) \Rightarrow \mathbb{E}[F \cdot \delta G] = \mathbb{E}[DF \cdot G]$$

$$\int_A \delta G \, d\mathbb{P} = \int_{\partial A} G \cdot \vec{n}\, |D\chi_A| \qquad \text{Gauss-Green}$$

$$\left[\|D\chi_A\|_{TV} = Per(A)\right]$$

Dis Poincaré $\forall F \in BV$: $\mathbb{E}\big[\,|F - \mathbb{E}[F]|\,\big] \le c\,\|DF\|_{TV}$ $\quad (c>0)$

Sia $G \in L^\infty$, $\|G\|_\infty \le 1$:

$$\mathbb{E}\Big[\big(F - \mathbb{E}[F]\big)\,G\Big] = \mathbb{E}\Big[-\Big(\int_0^{+\infty} \partial_t P_t F\Big) G\Big] =$$

$$\mathbb{E}[(P_t F)G] = \mathbb{E}[F(P_t G)]$$

$$= \int_0^{+\infty} \mathbb{E}\big[(LF)\,(P_t G)\big]\,dt$$

$$= \int_0^{+\infty} \mathbb{E}\big[DF \cdot DP_t G\big]\,dt$$

Sia $h \in \mathcal{H}$

$$\langle h, DP_t G(\omega)\rangle = \langle D\,\mathbb{E}^{\omega'}\big[G(e^{-t}\omega + \sqrt{1-e^{-2t}}\,\omega')\big], h\rangle$$

$$= \mathbb{E}^{\omega'}\big[\langle DG(e^{-t}\omega + \sqrt{1-e^{-2t}}\,\omega'), h\rangle e^{-t}\big]$$

$$= \mathbb{E}^{\omega'}\big[\langle D^{\omega'} G(e^{-t}\omega + \sqrt{1-e^{-2t}}\,\omega'), h\rangle\big]\frac{e^{-t}}{\sqrt{1-e^{-2t}}}$$

(IBP)

$$= \mathbb{E}^{\omega'}\Big[G(e^{-t}\omega + \sqrt{1-e^{-2t}}\,\omega')\,W(h)\,\frac{e^{-t}}{\sqrt{1-e^{-2t}}}\Big]$$

$$\le \|G\|_\infty \frac{e^{-t}}{\sqrt{1-e^{-2t}}}\,\mathbb{E}[|W(h)|] \le |h|\,\|G\|_\infty \frac{e^{-t}}{\sqrt{1-e^{-2t}}}$$

$$\boxed{|DP_t G(\omega)| = \sup_{\|h\|\le 1} \langle h, DP_t G(\omega)\rangle \le \|G\|_\infty \frac{e^{-t}}{\sqrt{1-e^{-2t}}}} \approx \begin{cases} \dfrac{1}{\sqrt{t}} & \text{se } t \to 0 \\ e^{-t} & \text{se } t \to \infty \end{cases}$$

$$\Rightarrow \int_0^{+\infty} \mathbb{E}[DF\, DP_tG] \le \int_0^{+\infty} \mathbb{E}[|DF|]\, \underbrace{\|G\|_\infty}_{\le 1} \frac{e^{-t}}{\sqrt{1-e^{-2t}}}\, dt$$

$$\le C\, \mathbb{E}[|DF|] \qquad \left[C = \int_0^{+\infty} \frac{e^{-t}}{\sqrt{1-e^{-2t}}}\, dt \right.$$

per approssimazione si ottiene anche il caso $F \in BV$.

In particolare se $F = \chi_A$ $\quad \mathbb{E}[\chi_A] = |A| = P(A)$

$$\mathbb{E}[|\chi_A - |A||] = |A|(1-|A|) + (1-|A|)|A|$$

$$\Rightarrow 2|A||A^c| \le C\, Per(A)$$

"se $Per(A) \approx 0 \Rightarrow A$ oppure A^c sono piccoli"

$\boxed{\text{Problema Isoperimetrico}}$ Fissato $Per(A) = p$, trovare A con $|A|$ massimo

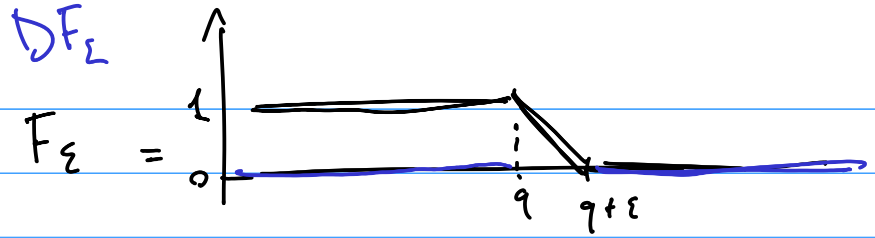
<u>Soluzione</u> (Levy, Sudakov-Borell) in $(\mathbb{R}^d, \gamma)$ sono <u>semispazi</u>

se $A = \{x_1 \le q\}$ $\quad \bullet\ |A| = \Phi(q) = \int_{-\infty}^{q} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx$

$\bullet\ Per(A) = \|D\chi_A\|_{TV}$

$$= \lim_{\varepsilon \to 0} \mathbb{E}[|D F_\varepsilon|]$$

$$= \lim_{\varepsilon \to 0} \int_q^{q+\varepsilon} \frac{1}{\varepsilon} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx = \frac{e^{-\frac{q^2}{2}}}{\sqrt{2\pi}} = \Phi'(q)$$

$F_\varepsilon =$ 

$$DF_\varepsilon = -\frac{1}{\varepsilon} I_{[q,q+\varepsilon]}$$

<u>Dis isoperimetrica</u>: Fissato $Per(A) = \Phi'(q)$, ah

$|A|$ è max se A è semipiano $\Rightarrow |A| = \Phi(q) \geq 1/2$

$$\Rightarrow \boxed{|A| \leq \Phi\left(\left(\Phi'\right)^{-1}\left(Per(A)\right)\right)} \quad \text{se } |A| \geq 1/2$$

$$\max\{|A|, |A^c|\} \leq \mathcal{U}(Per(A))$$

<u>OSS</u> $\mathcal{U}(\varepsilon) << \varepsilon$ per $\varepsilon \to 0$ migliora Poincaré

<u>Es</u> $\displaystyle\lim_{\varepsilon\to 0} \frac{\mathcal{U}(\varepsilon)}{\varepsilon/\sqrt{|\log \varepsilon|}}$ esiste finito $\square$

Es $F = W(h)$ $\quad DF = h \in L^\infty(P)$

II) Caso $p = \infty$ - Tecnica a "Poincaré" se $\|DF\|_{L^\infty(P)} < \infty$

$$\mathbb{E}[e^{P_t F}] \begin{cases} e^{\mathbb{E}[F]} & t \to \infty \\ \mathbb{E}[e^F] & t = 0 \end{cases}$$

$$\partial_t \mathbb{E}[e^{P_t F}] = -\mathbb{E}[e^{P_t F} L P_t F]$$

$$= -\mathbb{E}[D P_t F \cdot D e^{P_t F}]$$

$$= -\mathbb{E}[|D P_t F|^2 e^{P_t F}]$$

"smart path"

$$\log\left(\frac{\mathbb{E}[e^F]}{e^{\mathbb{E}[F]}}\right) = -\int_0^{+\infty} \partial_t \log \mathbb{E}[e^{P_t F}]\, dt$$

$$= \int_0^{+\infty} \frac{\mathbb{E}[|D P_t F|^2 e^{P_t F}]}{\mathbb{E}[e^{P_t F}]} dt \le \int_0^{+\infty} e^{-2t} dt\, \|DF\|_\infty^2$$

$$\boxed{P_t |DF| \le P_t \|DF\|_\infty \le \|DF\|_\infty} \qquad \le \frac{\|DF\|_\infty^2}{2}$$

$$\Longrightarrow \mathbb{E}[e^{(F - \mathbb{E}[F])}] \le \exp\left(\frac{1}{2}\|DF\|_\infty^2\right)$$

$$\xrightarrow{\text{Markov}} P(|F - \mathbb{E}[F]| > \lambda) \le 2\exp\left(-\frac{\lambda^2}{2\|DF\|_\infty^2}\right)$$

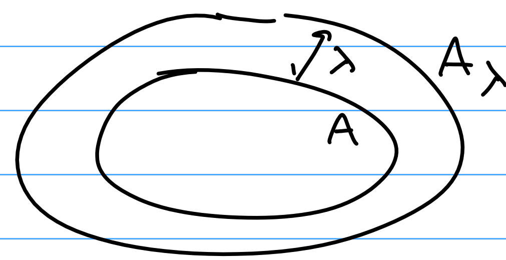
Confronta con: $F = \int_0^t \sigma_s dW_s \qquad \|\int \sigma_s^2\|_{L^\infty}$ $\square$

Talagrand isoperimetria $\longleftrightarrow$ concentrazione

Sia $A \subseteq \mathbb{R}^d$ e consideriamo $A_\lambda = \{ x \in \mathbb{R}^d : d(x, A) \leq \lambda \}$

OSS $|A_\lambda| - |A| \approx \lambda \, Per(A)$

se λ piccolo



Talagrand se $|A| \geq \frac{1}{2}$, $|A_\lambda|$ copre "quasi tutto"

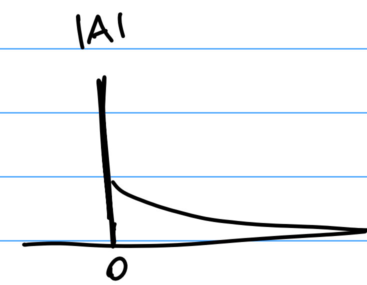
$$1 - |A_\lambda| = \underbrace{P(d(\cdot, A) > \lambda)} \leq \exp\left(-\frac{1}{8}\lambda^2\right) \qquad \lambda > 2$$

$$|A_\lambda| \geq 1 - \exp\left(-\frac{\lambda^2}{8}\right)$$

Vale $|D\, d(\cdot, A)| \leq 1$

$$d(\cdot, A) = \inf_{y \in A} d(x, y)$$

OSS $m = 0$ è mediana per $d(\cdot, A)$



$$|\mathbb{E}[X] - \cancel{\text{mediana}}\, X| \leq \mathbb{E}[|X - \text{mediana}\, X|]$$

Poincaré

$$\leq \mathbb{E}[|X - \mathbb{E}(X)|] \leq 1$$

$$0 \leq \mathbb{E}[d(\cdot, A)] \leq 1$$

Se $\lambda > 2$, $\quad (\lambda - 1) > \frac{\lambda}{2}$

$$P(d(\cdot, A) > \lambda) \leq P(d(\cdot, A) - \mathbb{E}[d(\cdot, A)] > \lambda - 1)$$

Markov componendo con exp

$$\leq \exp\left(-\frac{1}{2}(\lambda-1)^2\right)$$

$$\leq \exp\left(-\frac{1}{8}\lambda^2\right)$$

$$\boxed{|A_\lambda| \geq 1 - \exp\left(-\frac{\lambda^2}{8}\right)}$$

$\square$