Lezione 17

Derivabilità della soluzione di SDE

$$dX_t = b_t(X_t)\,dt + \sigma_t(X_t)\,dW_t \qquad X_0 = x$$

$$(x,\omega) \mapsto X_t(x,\omega) \quad \begin{cases} \text{I)} & \partial_x X_t(x,\omega) \\ \text{II)} & D_s X_t(x,\omega) \quad s \le t \end{cases}$$

Assumiamo $b_t(x)$, $\sigma_t(x)$ $\mathcal{C}_b^\infty(\mathbb{R}^d)$ unif in t $\quad \boxed{d=1}$

I) Deriviamo formalmente l'eq.

$$d X_t(x,\omega) = b_t(X_t(x,\omega))\,dt + \sigma_t(X_t(x,\omega))\,dW_t(\omega)$$

$$\begin{cases} \partial_x\, d X_t(x,\omega) = d\,\boxed{\partial_x X_t}(x,\omega) \\ \partial_x\big(b_t \circ X_t(x,\omega)\cdot dt\big) = (\partial_x b_t)(X_t(x,\omega))\cdot \boxed{\partial_x X_t}(x,\omega)\,dt \\ \partial_x\big(\sigma_t(X(x,\omega))\cdot dW_t\big) = \partial_x\sigma_t(X(t,x))\,\boxed{\partial_x X_t}(x,\omega)\cdot dW_t \end{cases}$$

$t \mapsto \partial_x X_t$ soddisfa l'eq della variazione prima

$$(VP)\quad \begin{cases} d J_t^x = \underbrace{\partial_x b_t(X_t(x,\omega))}_{\tilde b_t}\cdot J_t^x\,dt + \underbrace{\partial_x\sigma_t(X(t,x))}_{\tilde\sigma_t}\cdot J_t^x\,dW_t \\ \boxed{J_0^x = \partial_x x = 1} \end{cases}$$

Problema 1) Buona positura dell'eq. var. prima.

2) $J_t^x = \partial_x X_t(x,\omega)$

Oss L'eq. (VP) è lineare nella soluzione J

1) In $d=1$ possiamo scrivere esplicitamente una (la) soluzione

$$dJ_t = \tilde{b}_t \, J_t \, dt + \tilde{\sigma}_t \, J_t \, dW_t$$

con $(\hat{b}_t)_t, (\tilde{\sigma}_t)_t \in \Lambda^2([0,T])$.

Allora

$$\boxed{\tilde{J}_t = \exp\left(\int_0^t \tilde{b}_s \, ds \right) \exp\left(\int_0^t \tilde{\sigma}_s \, dW_s - \frac{1}{2}\int_0^t \tilde{\sigma}_s^2 \, ds \right)}$$

$\vee$
0

è una soluzione. Formula di Itô

Itô

$$d\tilde{J}_t = \left(\hat{b}_t \, dt + \tilde{\sigma}_t \, dW_t - \cancel{\frac{1}{2} \hat{\sigma}_t^2 \, dt} + \cancel{\frac{1}{2} \hat{\sigma}_t^2 \, dt} \right) \tilde{J}_t$$

è l'unica soluzione. Sia $(J_t)_t$ una soluzione

Itô $\mapsto$ $d\tilde{J}_t^{-1} = \left(-\tilde{b}_t + \frac{1}{2}\tilde{\sigma}_t^2 + \frac{1}{2}\hat{\sigma}_t^2 \right) \tilde{J}_t^{-1} - \hat{\sigma}_t J_t^{-1} \, dW_t$

$$d\left(J_t \tilde{J}_t^{-1}\right) = (dJ_t)\tilde{J}_t^{-1} + dJ_t\left(\tilde{J}_t^{-1}\right) + dJ_t \, d\tilde{J}_t^{-1}$$

$$= 0$$

□

In $d>1$ NON ci sono "formule esplicite".

2) Sia $\varepsilon \neq 0$: $d\left(X_t(x+\varepsilon) - X_t(x)\right) = \left(b_t(X_t(x+\varepsilon)) - b_t(X_t(x))\right)dt$

Sia $p \geq 2$: $+ \left(\sigma_t(X_t(x+\varepsilon)) - \sigma_t(X_t(x))\right)dW_t$

$$\mathbb{E}\left[\sup_{s\leq t} |X_s(x+\varepsilon) - X_s(x)|^p\right] \lesssim \mathbb{E}\left[|\varepsilon|^p + \left(\int_0^t \mathrm{Lip}\, b_r |X_r(x+\varepsilon) - X_r(x)|\, dr\right)^p + \right.$$

$$\left. + \sup_{s\leq t} \left|\int_0^s \left(\sigma_r(X_r(x+\varepsilon)) - \sigma_r(X_r(x))\right) dW_r\right|^p\right]$$

$|b_r(x) - b_r(y)| \leq \mathrm{Lip}\, b_r\, |x-y|$

$$\lesssim |\varepsilon|^p + \mathbb{E}\left[\left(\int_0^t \mathrm{Lip}\, b_s\, |X_s(x+\varepsilon) - X_s(x)|\, ds\right)^p\right]$$

Burkholder

$$+ \mathbb{E}\left[\left(\int_0^t |\sigma_r(X_r(x+\varepsilon)) - \sigma_r(X_r(x))|^2\, dr\right)^{\frac{p}{2}}\right]$$

$\leq \sup_{s\leq t} |X_s(x+\varepsilon) - X_s(x)|$

$$\lesssim |\varepsilon|^p + \mathbb{E}\left[\left(\int_0^t \mathrm{Lip}\, b_s\, |X_s(x+\varepsilon) - X_s(x)|\, ds\right)^p\right]$$

$$+ \mathbb{E}\left[\left(\int_0^t (\mathrm{Lip}\, \sigma_s)^2 |X_s(x+\varepsilon) - X_s(x)|^2\, dr\right)^{\frac{p}{2}}\right]$$

$$\lesssim |\varepsilon|^p + \left(\int_0^t \mathrm{Lip}\, b_s\right)^p \cdot \mathbb{E}\left[\sup_{s\leq t} |X_s(x+\varepsilon) - X_s(x)|^p\right]$$

$$+ \left(\int_0^t (\mathrm{Lip}\, \sigma_s)^2\right)^{p/2} \cdot \mathbb{E}\left[\sup_{s\leq t} |X_s(x+\varepsilon) - X_s(x)|^p\right]$$

Se t è suff piccolo $\sim 1/4$ $\Longrightarrow$ $\boxed{\mathbb{E}\left[\sup_{s\leq t} |X_s(x+\varepsilon) - X_s(x)|^p\right]^{1/p} \lesssim |\varepsilon|}$

$$d\Big(\overbrace{X_t(x+\varepsilon) - X_t(x) - \varepsilon J_t^x}^{J_t^{(\varepsilon)}}\Big) =$$

$$= \Big(b_t(X_t(x+\varepsilon)) - b_t(X_t(x)) - \varepsilon\, \partial_x b_t(X_t(x))\cdot J_t^x\Big)dt \; +$$
$$+ \Big(\sigma_t(X_t(x+\varepsilon)) - \sigma_t(X_t(x)) - \varepsilon\, \partial_x \sigma_t(X_t(x))\, J_t^x\Big)dW_t$$

$$= \partial_x b_t(X(t,x))\Big(X_t(x+\varepsilon) - X_t(x) - \varepsilon J_t^x\Big)dt + R_t^{1,\varepsilon}dt$$
$$+ \partial_x \sigma_t(X(t,x))\Big(X_t(x+\varepsilon) - X_t(x) - \varepsilon J_t^x\Big)dW_t + R_t^{2,\varepsilon}dW_t$$

$$\begin{cases} dJ_t^{(\varepsilon)} = \tilde{b}_t J_t^{(\varepsilon)}dt + \tilde{\sigma}_t J_t^{(\varepsilon)}dW_t + R_t^{1,\varepsilon}dt + R_t^{2,\varepsilon}dW_t \\ J_0^{(\varepsilon)} = 0 \end{cases}$$

$$R_t^{1,\varepsilon} = b_t(X_t(x+\varepsilon)) - b_t(X_t(x)) - \partial_x b_t(X_t(x))\big(X_t(x+\varepsilon) - X_t(x)\big)$$

$$= \int_0^1 \Big(\partial_x b_t(\theta X_t(x+\varepsilon) + (1-\theta)X_t(x)) - \partial_x b_t(X_t(x))\Big)d\theta\,\big(X_t(x+\varepsilon) - X_t(x)\big)$$

supponendo $b_t \in \mathcal{C}_b^2$, $\sigma_t \in \mathcal{C}_b^2$

$$|R_t^{1,\varepsilon}| \lesssim |X_t(x+\varepsilon) - X_t(x)|^2$$

$$|R_t^{2,\varepsilon}| \lesssim |X_t(x+\varepsilon) - X_t(x)|^2$$

$$\mathbb{E}\Big[\sup_{s\le t} |R_s^{1,\varepsilon}|^p\Big] \lesssim \mathbb{E}\Big[\sup_{s\le t} |X_t(x+\varepsilon) - X_t(x)|^{2p}\Big] \lesssim |\varepsilon|^{2p}$$

$$\Longrightarrow \quad \mathbb{E}\Big[\sup_{s\le t}\Big|\int_0^t R_s^{2,\varepsilon}dW_s\Big|^p\Big] \lesssim |\varepsilon|^{2p}$$

$$\mathbb{E}\left[\sup_{s\le t}|J^\varepsilon_s|^p\right] \lesssim \mathbb{E}\left[\left|\int_0^t |\tilde{b}_s J^\varepsilon_s|\, ds\right|^p\right] +$$

$$+ \mathbb{E}\left[\sup_{s\le t}\left|\int_0^s \tilde{\sigma}\, J^\varepsilon\, dW\right|^p\right] + \varepsilon^{2p}$$

$$\lesssim \mathbb{E}\left[\left(\int_0^t \mathrm{Lip}\, b_s ds\right)^p \sup_{s\le t}|J^\varepsilon_s|^p\right] + \dots$$

$$\lesssim |\varepsilon|^{2p} + \left[\left(\int_0^t \mathrm{Lip} b_s\, ds\right)^p + \left(\int_0^t (\mathrm{Lip}\,\sigma_s)^2\right)^{p/2}\right]\mathbb{E}\left[\sup |J_s^{(\varepsilon)}|^p\right]$$

$\Rightarrow$ se t è suff. piccolo $\nearrow < \frac{1}{2}$

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\sup_{s\le t}\left|\frac{J^{(\varepsilon)}_s}{\varepsilon}\right|^p\right] = 0$$

<u>Teo</u> $x \mapsto (X_s(x))_{s\le t}$ come v.a. $L^p(\Omega; \mathcal{C}([0,t]), \mathbb{R}))$ è diff. ($\mathcal{C}^1$)

Cosa accade per t grande? Sappiamo $\underbrace{X_{t+s}(x) = X_s(X_t(x))}$

$$X_{t+s}(x+\varepsilon) - X_{t+s}(x) - \varepsilon \underbrace{J^x_{t+s}} = X_s(X_t(x+\varepsilon)) - X_s(X_t(x)) - \varepsilon J^{X_t(x)}_{t\mapsto s+t} J^x_{0\to t}$$

vale $\boxed{J^x_{0\to s+t} = J^{X_t(x)}_{t\mapsto s+t} \cdot J^x_{0\to t}}$

<u>Chain rule</u>

$$= J^{X_t(x)}_{t\to s+t}\Big(\underbrace{X_t(x+\varepsilon) - X_t(x)}_{\varepsilon J^x_{0\to t} + \mathrm{Resto}^2}\Big) + \mathrm{Resto}^1 - \varepsilon J^{X_t(x)}_{t\to s+t} J^x_{0\to t}$$

$$dX_t = b_t(X(t))dt + \sum_{i=1}^{d} \sigma_t^i(X_t)\,dW^i$$

<u>OSS</u> i) Se $d>1$ l'eq. (VP) non ha soluzione esplicita

$$dJ_t^x = \nabla b_t(X(t,x))\,J_t^x\,dt + \sum_i \nabla\sigma_t^i(X(t,x))\,J_t^x\,dW_t^i$$

<u>esercizio</u> $\boxed{d\tilde{J}_t = \tilde{b}_t\tilde{J}_t\,dt + \sum_i \tilde{\sigma}_t^i\tilde{J}_t\,dW_t^i}$

ha sol. $\tilde{J}_t = \exp\left(\int_0^t b_t \dots\right)$ se $\tilde{b}_t, \tilde{\sigma}_t^i$ <u>commutano</u>

ii) Possiamo considerare anche derivate successive

<u>esercizio</u> Scrivere eq. per variazione seconda $\partial_x^2 X(t,x)$

II) __Derivata di Malliavin__ Ragioniamo allo stesso modo,

deriviamo formalmente: "$D_s = \frac{\partial}{\partial dW_s}$"

$$\begin{cases} D_s\, d X_t(x,\omega) = d\, D_s X_t(x,\omega) \\ D_s\, b_t(X_t(x,\omega))\,dt = \partial_x b_t(X_t(x,\omega))\, D_s X_t(x,\omega)\, dt \\ D_s\left(\sigma_t(X_t(x,\omega))\cdot dW_t\right) = \partial_x \sigma_t(X_t(x,\omega))\, D_s X_t(x,\omega)\, dW_t + \\ \qquad + \underbrace{\sigma_t(X_t(x,\omega))\,\delta_s(t)} \end{cases}$$

In forma integrale (siccome $D_s X_t = 0$ se $s > t$):

$$D_s X_t = \sigma_s(X_s) + \int_s^t \partial_x b_r(X_r) D_s X_r\, dr + \int_s^t \partial_x \sigma_r(X_r)\, D_s X_r\, dW_r$$

è (quasi) l'eq. VP a parte da s

$$\boxed{D_s X_t = J_{s\to t}^{\text{"}X_s(x)\text{"}} \cdot \sigma_s(X_s)}$$

Formula esplicita $(d=1)$

$$D_s X_t = \exp\left(\int_s^t \left(\partial_x b_r(X_r) - \frac{1}{2}(\partial_x \sigma_r)^2(X_r)\right) dr + \int_s^t \partial_x \sigma_r(X_r)\, dW_r\right) \sigma_s(X_s)$$

Come dimostrare che $X_t \in \mathbb{D}^{1,2}$?

Basta prendere una successione di Cauchy $(X^n_t)_n \in \mathbb{D}^{1,2}$ tale che

$$X^n_t \xrightarrow{n\to\infty} X_t \quad \text{in } L^2.$$

$$\boxed{\begin{aligned} X^{n+1}_t &= x + \int_0^t b_r(X^n_r)\,dr + \int_0^t \sigma_r(X^n_r)\,dW_r \\ X^0_t &= x \end{aligned}}$$

$$(X_s)_{s\le T} \mapsto F(X) = \left(x + \int_0^s b_r(X_r)\,dr + \int_0^s \sigma_r(X_r)\,dW_r\right)_{s\le T}$$

$$L^2(\mathcal{C}([0,T],\mathbb{R}^d)) \longrightarrow L^2(\mathcal{C}([0,T],\mathbb{R}^d))$$

contrazione se t suff piccolo.

$$\|(X_t)_{t\le T}\|_{\tilde{\mathbb{D}}^{1,2}} = \mathbb{E}\left[\sup_{t\le T}|DX_t|^2 + \sup_{t\le T}|X_t|^2\right]^{1/2}$$

$$\|F(X) - F(Y)\|_{\tilde{\mathbb{D}}^{1,2}} \qquad \left(X, Y \in M^2_{[0,T]} \text{ e } \begin{matrix}\|X\|_{\tilde{\mathbb{D}}^{1,2}} \\ \|Y\|_{\tilde{\mathbb{D}}^{1,2}}\end{matrix} < \infty\right)$$

$$\left[F(X)_t - F(Y)_t = \int_0^t \big(b_r(X_r) - b_r(Y_r)\big)\,dr + \int_0^t \big(\sigma_r(X_r) - \sigma_r(Y_r)\big)\,dW_r\right]$$

$$D_s\big(F(X)_t - F(Y)_t\big) = \int_s^t \big(\partial_x b_r(X_r)\,D_sX_r - \partial_x b_r(Y_r)\,D_sY_r\big)\,dr$$

$$= \int_0^t \partial_x b_r(X_r)\big(D_sX_r - D_sY_r\big)\,dr$$

$$+ \left(\partial_x b_r(X_r) - \partial_x b_r(Y_r)\right) D_s Y_r + \dots$$

$$\int |D_s F(X)_t - F(X)_t|^2 ds \le \int \left|\int_0^t \partial_x b_r(X_r) D_s(X_r - Y_r)\, dr\right|^2 ds$$

$$\le \int_0^s \int_0^t \|\partial_r b_r\|^2 \int |D_s(X_r - Y_r)|^2 dr\, ds$$

$$\boxed{\| F(X) - F(Y) \|_{\tilde{D}^{1,2}} \le \frac{1}{2} \|X - Y\|_{\tilde{D}^{1,2}}}$$

purché $\|Y\|_{\tilde{D}^{1,2}} \le 2$ e T suff piccolo

$(x=0)$ $\quad X^0 = 0 \quad X^1 = F(X) \quad \|X^1\| \le 1$ poiché T suff piccolo

$$\| F^{n+1}(X) - F^n(X) \|_{\tilde{D}^{1,2}} \le \lambda \| X^n - X^{n-1} \| \le \dots \le \lambda^n \|X^1 - X^0\|$$

$$\| F^{n+1}(X) \| \le \| F^{n+1} - F^n \| + \| F^n - F^{n-1} \| + \dots + \| F$$

$$\le \sum \lambda^n \|X^1 - X^0\| \le 2$$

<u>Applicazione</u> esistenza densità di X_t se $\exists c\ \sigma_t(x) > c \neq 0$

Ricordiamo il CRITERIO per densità: $X \in \mathbb{D}^{1,2}$ ha densità se $|DX| \neq 0$ $\mathbb{P}$-q.c.

$$|DX_t|^2 = \int_0^t |\underbrace{J_{s\to t}} \sigma_s(X_s)|^2 ds \neq 0 \quad \mathbb{P}\text{-q.c.}$$

Possiamo usare IBP? $h \in L^2([0,T])$ $\quad \frac{h}{\langle DX, h\rangle} \in \operatorname{dom} \delta$

$$\langle D X_t, h \rangle = \int_0^t D_s X_t \, h_s \, ds$$

$$\delta\left(\frac{h}{\langle D X_t, h\rangle} \right) = \frac{W(h)}{\langle DX_t, h\rangle} + \frac{\langle D^2 X_t, h \otimes h\rangle}{(\langle DX_t, h\rangle)^2} \in L^2$$

$G \in \operatorname{dom}(\delta)$ $\quad \delta(YG) = Y\delta G - \langle G, DY\rangle$

$$\underline{\langle D^2 X_t, h\otimes h\rangle} = \int_0^t D_{s'} \int_0^t D_s X_t \, h_s \, ds \; h_{s'} \, ds' = \int_0^t \int_0^t D^2_{s,s'} X_t \, h_s h_{s'} \, ds \, ds'$$

$$D_s X_t = \exp\left(\int_s^t \left(b_r'(X_r) - \frac{\sigma_r'^2(X_r)}{2}\right) dr + \frac{1}{2}\int_s^t \sigma_r'(X_r) \, dW_r\right) \sigma_s(X_s)$$

$$= \exp(H_t - H_s) \, \sigma_s(X_s)$$

$$H_s = \int_0^s b_r'(X_r) - \frac{(\sigma_r')^2(X_r)}{2} + \frac{1}{2}\int_0^s \sigma_r' \, dW_r$$

$$D_s X_t \geq \exp(H_t) \exp\left(-\sup_{r \leq t} |H_r| \right) c$$

$h \geq 0$

$\int_0^t h(s) ds = 1$

$$\int_0^t D_s X_t \, h(s) \geq \exp(H_t) \exp(- H^*_t) \, c \cdot \underbrace{\cancel{\int_0^t h(s) ds}}_{1}$$

$$\frac{1}{\langle DX_t, h\rangle} \leq \frac{\exp(2 H^*_t)}{c}$$

Ricordiamo le stime per le esponenziali di martingale con $[M]_T$

$$\mathbb{E}[\exp(p M^*_t)] \leq c_1 \exp\left(p^2 c_2 \| [M]_T \|_\infty\right)$$

OSS Se $d > 1$ la condizione è $\sigma\sigma^* \geq c\, \mathrm{Id}$ $c > 0$

ma si può indebolire (teo di Hörmander)

L'esistenza della densità di X_t si ottiene anche via PDE.

Possiamo però con Malliavin trattare altri funzionali

Es $X_t^* = \sup_{s \leq t} X_s \in \mathbb{D}^{1,2}$

Sia $\{s_i\}_i$ denso in $[0,t]$

$\max_{i \leq n} X_{s_i} \in \mathbb{D}^{1,2}$ $\boxed{D_s \max X_{s_i} = D_s X_{\bar{s}_i}}$

$\max X_{s_i} = X_{\bar{s}_i}$

per $n \to \infty$ $\max X_{s_i} \to X_t^*$ in L^2 (conv dom)

$$\sup_n \| D \max X_{s_i} \|_{L^2} < \infty$$

$\Rightarrow X_t^* \in \mathbb{D}^{1,2}$ $|DX_t^*| \neq 0 \Rightarrow X_t^*$ ha densità.

OSS $X_t^* \notin \mathbb{D}^{2,2}$ (DX_t^* è BV) $\square$