Lezione 9

Disegualgliante funzionali e EDS

Siano b e σ regolari e sia $\mathcal{L}f = b\nabla f + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T\nabla^2 f)$

Es $b = -\nabla V$ $\sigma = \sqrt{2}\,\mathrm{Id}$ $\mathcal{L}f = -\nabla V \nabla f + \Delta f$

Semigruppo di evoluzione $P_t f(x) = \mathbb{E}[f(X(t,x))]$

dove $\begin{cases} dX(t,x) = b(X(t,x))\cdot + \sigma(X(t,x))dB \\ X(0,x) = x \end{cases}$

$$P_t(P_s f) = P_{s+t} f$$

$$f \in C_b^2 \Rightarrow \frac{d}{dt} P_t f = P_t(\mathcal{L}f)$$

$$\frac{d}{dt} P_t f = \mathcal{L}(P_t f)$$

perché $\frac{P_{t+\varepsilon}f - P_t f}{\varepsilon} = \frac{P_\varepsilon(\underbrace{P_t f}_{g}) - P_t f}{\varepsilon}$

$$= \frac{P_\varepsilon g - g}{\varepsilon} = \frac{d}{ds}\Big|_{s=0} g$$

$$= \mathcal{L}g$$

Notazione $\begin{cases} P_t = e^{t\mathcal{L}} \\ P_t P_s = e^{t\mathcal{L}} e^{s\mathcal{L}} = e^{t\mathcal{L}+s\mathcal{L}} = e^{(s+t)\mathcal{L}} \end{cases}$

μ mis. di prob. sia <u>invariante</u> $\quad \overbrace{\int_{\mathbb{R}^d} P_t f \, d\mu}^{\int \mathbb{E}[f(X(t,x))]\, d\mu(x)} = \overbrace{\int_{\mathbb{R}^d} f \, d\mu}$

$\rightsquigarrow$ Se risolvo l'eq. $dX = b(X)dt + \sigma(X)dB$

con $P_{X_0} \sim \mu$, allora $\underline{\underline{P_{X_t} \sim \mu}}$

<u>Oss</u> La legge marginale al tempo t della soluz.

$$dX_t = b(X_t) + \sigma(X_t)dB$$

è data da $\quad \mathbb{E}[f(X_t)] = \int_{\mathbb{R}^d} \underbrace{\mathbb{E}[f(X(t,x))]} \, d\mu_0$

$\mu_0 = P_{X_0}$

infatti il membro a dx $\quad = \int_{\mathbb{R}^d} f \, d\nu_t$

ν_t soddisfa l'eq. di Kolmogorov forward

con dato iniziale $\mu_0 \quad \underset{\text{unicità}}{\Longrightarrow} \quad \boxed{\nu_t = P_{X_t}}$

μ è <u>simmetrica</u> $\quad \int (P_t f)\, g\, d\mu = \int f (P_t g)\, d\mu$

$$\Updownarrow$$

$$\int (\mathcal{L} f)\, g\, d\mu = \int f (\mathcal{L} g)\, d\mu$$

<u>OSS</u> Segue che $\forall\ t \leq T$ se $(X_t)_{t\in[0,T]}$ risolve

l'EDS $\quad dX_t = b(X_t)dt + \sigma(X_t)dB \quad$ con $P_{X_0} = \mu$

allora

$$\boxed{\mathbb{E}[\, f(X_t) \mid X_T = x\,] = P_{T-t} f(x)} \left(= \mathbb{E}[f(X(T-t,x))] \right)$$

Infatti data $g \in \mathcal{C}_b(\mathbb{R}^d)$

$$\mathbb{E}[\, f(X_t)\, g(X_T)\,] \overset{??}{=} \mathbb{E}[(P_{T-t} f)(X_T)\, g(X_T)]$$

ha legge μ

[perché $(X_{t+u})_{u\geq 0}$ risolve EDS con legge iniziale P_{X_t} $= \mu$]

$$= \mathbb{E}[\, f(X_0)\, g(X_{T-t})\,]$$

$$= \mathbb{E}[\ \mathbb{E}[f(X_0)\, g(X_{T-t}) \mid X_0\,]\,]$$

$$= \mathbb{E}[\, f(X_0)\ \mathbb{E}[\, g(X_{T-t}) \mid X_0\,]\,]$$

$$\|$$

Vale $\mathbb{E}[g(X_{T-t}) \mid X_0 = x] = P_{T-t}\, g\,(x)$

$$= \mathbb{E}\left[f(X_0) \left(P_{T-t}\, g\right)(X_0) \right]$$

$$= \int_{\mathbb{R}^d} f \cdot \left(P_{T-t}\, g\right) \, d\mu$$

$$= \int_{\mathbb{R}^d} \left(P_{T-t} f\right)(x) \cdot g(x) \, d\mu(x)$$

$$= \mathbb{E}\left[P_{T-t} f(X_T) \; g(X_T) \right] \quad \checkmark$$

<u>Ricordiamo</u> l'eq. di Kolmogorov backward $f(t,x)$ regolare

$$\begin{cases} \partial_t f = -\mathcal{L} f & t \in [0,T] \\ f(T,x) = \bar{f}(x) \end{cases}$$

se risolta permetteva di costruire la martingala

$$d f(t, X_t) = \cancel{\left(\partial_t f + \mathcal{L} f\right)}(t, X_t)\, dt + $$

$$+ \underbrace{\sigma(X_t) \nabla f(t, X_t)\, dB}_{\text{martingala}}$$

formula per la soluz^{ne} è

$$f(t,x) = \mathbb{E}\left[\bar{f}\left(X(T-t,x)\right)\right]$$
$$\overset{!}{=} \left(P_{T-t}\bar{f}\right)(x)$$

<u>infatti</u> $\partial_t f(t,x) = \partial_t\left(P_{T-t} f\right)(x) =$

$$= \partial_t e^{(T-t)\mathcal{L}} f$$
$$= -\mathcal{L}\, e^{(T-t)\mathcal{L}} f$$

Disuguaglianza funzionale $\rightsquigarrow$ EDS $\rightsquigarrow$ <u>dis martingale</u>

<u>Teorema</u> Dis massimale per semigruppi (Rota)

Sia μ misura invariante e simmetrica e sia $f \in L^p(\mu)$

<u>$p > 1$</u> . Allora $\sup_{t \geq 0} |P_t f|(x) = P^* f(x) \in L^p(\mu)$

con $$\int_{\mathbb{R}^d} |P^* f|^p \, d\mu \leq C(p) \int_{\mathbb{R}^d} |f|^p \, d\mu$$

con $C = C(p)$ dipende solo da $p > 1$

Esempi $\mu = e^{-V}$ $\quad V \in C^2$

Dim basta dimostrarlo per $t \in [0, 2T]$ e anche per $f \geq 0$

$$\underbrace{P^*_{2T} f(x)} = \sup_{t\in[0,2T]} P_t f(x) = \sup_{t\in[0,T]} P_{2(T-t)} f(x)$$

$$= \sup_{t\in[0,T]} P_{T-t}\Big(\underbrace{P_{T-t}} f\Big)(x)$$

"reversibilità"

$$= \sup_{t\in[0,T]} \mathbb{E}\Big[\big(P_{T-t} f\big)(X_t) \,\Big|\, X_T = x \Big]$$

$$\leq \mathbb{E}\Big[\sup_{t\in[0,T]} \underbrace{\big(P_{T-t} f\big)(X_t)} \,\Big|\, X_T = x \Big]$$

Kolmogorov back

martingala !

$$\int_{\mathbb{R}^d} |P^*_{2T} f(x)|^p \, d\mu(x) = \mathbb{E}\Big[|P^*_{2T} f|^p (X_T) \Big]$$

$$\leq \mathbb{E}\Big[\Big| \mathbb{E}\Big[\Big|\sup_{t\in[0,T]} \big(P_{T-t} f\big)(X_t)\Big| \,\Big|\, X_T \Big] \Big|^p \Big]$$

$$= \mathbb{E}\Big[\big| \underbrace{\sup_{t\in[0,T]} (P_{T-t}f)(X_t)}_{} \big|^p \Big]$$

$$M_t = (P_{T-t}f)(X_t)$$

martingala Doob

$$\leq C(p)\, \mathbb{E}\left[|f(X_T)|^p \right]$$

$$= C(p) \int_{\mathbb{R}^d} |f|^p \, d\mu$$

Doob $\mathbb{E}\left[|M^*|^p \right] \leq C(p)\, \mathbb{E}\left[|M_T|^p \right]$

$$P(M^* > \lambda) \leq \frac{\mathbb{E}\left[|M_T| \right]}{\lambda} \quad ??$$

$\Downarrow$??

$$\mu\Big(\sup_{t\in[0,T]} |P_t f| > \lambda \Big) \overset{??}{\leq} c \frac{\int |f| \, d\mu}{\lambda}$$

Dis di Poincaré Diciamo che vale Poincaré rispetto a μ mis di prob. su $\mathbb{R}^d$ se $\exists\, c>0$ tale che

$$\int_{\mathbb{R}^d} f^2 \, d\mu - \left(\int_{\mathbb{R}^d} f \, d\mu\right)^2 \le c \int_{\mathbb{R}^d} |\nabla f|^2 d\mu$$

oss Se X ha legge μ $\mathrm{Var}(f\circ X) \le c\, \mathbb{E}[\,|\nabla f|^2 \circ X]$

Se $d\mu = e^{-V(x)}dx$ con $V \in \mathcal{C}^2$ e $\nabla^2 V(x) \geq K\, Id$ $\forall x \in \mathbb{R}^d$

dove $K>0$ è una costante

allora vale Poincaré con $c = \frac{2}{K}$

$$\boxed{V(x) = \frac{|x|^2}{2} + \text{cost}}$$

Come dimostrare Poincaré per queste misure? Sia $\boxed{\mathcal{L}f = -\nabla V \nabla f + \sqrt{2}\,\Delta f}$

Dis di Bakry - Emery (K) $\forall f \in \mathcal{C}^1_b$

$\forall t \geq 0\ \forall x \in \mathbb{R}^d$ $\quad |\nabla P_t f|^2(x) \le e^{-2Kt} P_t\left(|\nabla f|^2\right)(x)$

Infatti $BE(K) \Rightarrow$ Poincaré perché se $(X_t)_{t\in[0,T]}$

la martingala $M_t = (P_{T-t} f)(X_t)$ $\quad M_0 = (P_T f)(X_0)$

ha variazione quadratica $d[M]_t = 2\,|\nabla P_{T-t} f|^2(X_t)\,dt$

l'isometria di Ito

$$\mathbb{E}\left[(M_T - M_0)^2\right] = \mathbb{E}\left[\int_0^T 2\,|\nabla P_{T-t} f|^2(X_t)\,dt\right]$$

$$\mathbb{E}\left[(M_T - M_0)^2\right] = \mathbb{E}\left[M_T^2 - 2M_T M_0 + M_0^2\right] = \mathbb{E}[M_T^2] - \mathbb{E}[M_0^2] = \mathbb{E}[f^2(X_T)] - \mathbb{E}\left[(P_T f)^2 \circ X_0\right]$$

$$\mathbb{E}\left[\int_0^T 2\,|\nabla P_{T-t} f|^2(X_t)\,dt\right] = \int_0^T 2 \int_{\mathbb{R}^d} |\nabla P_{T-t} f|^2\,d\mu \overset{BE(K)}{\leq} \int_0^T \int_{\mathbb{R}^d} e^{-2K(T-t)}\,\cancel{P_{T-t}}|\nabla f|^2\,d\mu$$

$$\int f^2\,d\mu - \int (P_T f)^2\,d\mu \leq \boxed{2\int_0^T e^{-2kt}\,dt} \int |\nabla f|^2\,d\mu$$

$T \to \infty$

$$\int f^2\,d\mu - \left(\int f\right)^2 \leq \frac{1}{k} \int |\nabla f|^2\,d\mu$$

OSS Se vale $BE(k)$ $|\nabla P_t f|^2 \leq e^{-2kt} P_t |\nabla f|^2$

e $f \in \mathcal{C}^1_b$ il membro a dx $\to 0$ se $t \to \infty$

$\Rightarrow$ se $t \to \infty$ $P_t f \to P_\infty f$ costante

Come si dimostra $BE(k)$?

Consideriamo il processo

$$d \underbrace{|\nabla P_{T-t} f|^2(X(t,x))}_{g(t, X(t,x))} = \Big(\underbrace{\partial_t |\nabla P_{T-t} f|^2 + \mathcal{L} |\nabla P_{T-t} f|^2}\Big) \circ X(t,x)$$

$$+ \, dM_t$$

martingala

poniamo $\boxed{h(t,x) = \partial_t |\nabla P_{T-t} f|^2 + \mathcal{L} |\nabla P_{T-t} f|^2}$

e si mostra che $\underline{h(t,x) \geq 2K |\nabla P_{T-t} f|^2}$

passando al valore atteso

$$\mathbb{E}\Big[|\nabla P_{T-t} f|^2 (X(t,x)) \Big] \Big|_0^T \geq \cancel{\mathbb{E}\Big[\int_0^T 2K |\nabla P_{T-t} f|^2 \circ (X_t) \Big]}$$

$$\mathbb{E}\left[|\nabla f|^2(X(T,x))\right] - \cancel{\mathbb{E}}\left[|\nabla P_T f|^2(x)\right] \geq 0$$

$$\parallel$$

ossia $$\boxed{P_T|\nabla f|^2(x) \geq |\nabla P_T f|^2(x)}$$

$$h(t,x) = \partial_t |\nabla P_{T-t} f|^2 + \mathcal{L}|\nabla P_{T-t} f|^2$$

$$= -2\nabla \underbrace{P_{T-t}f}_{=a(x)} \nabla \mathcal{L} \underbrace{P_{T-t}f}_{=a(x)} + \mathcal{L}|\nabla \underbrace{P_{T-t}f}_{=a(x)}|^2$$

$$= -2\nabla a \nabla \mathcal{L} a + \mathcal{L}|\nabla a|^2 = \qquad \mathcal{L}f = -\nabla V \nabla f + \sqrt{2}\Delta f$$

$\downarrow$ conto

$$= 2\left(\cancel{\|\nabla^2 a\|^2} + \langle \nabla^2 V \nabla a, \nabla a\rangle\right)$$

$$\geq 2K|\nabla a|^2$$

$\square$