- <u>Disuguaglianza</u> BDG $p \geq 4$

<u>Def</u> $(X_t)_{t\in[0,T]}$ $(A_t)_{t\in[0,T]}$ $X_0 = A_0 = 0$ non decrescenti <u>adattati</u> e continui. A <u>domina</u> X se $\forall \tau : \Omega \to [0,T]$

$$\mathbb{E}[X_\tau] \leq \mathbb{E}[A_\tau]$$

<u>Prop</u> Se A domina X $\Rightarrow$ $\forall q \in (0,1)$ $\exists\, c(q) > 0$ tale che

$$\mathbb{E}[X_T^q] \leq c(q)\, \mathbb{E}[A_T^q]$$

<u>OSS</u> $X_t = (M_t^*)^p$, $A_t = [M]_t^{p/2}\, c(p)$ $p \geq 4$

A domina X e pure X domina A.

<u>Lemma</u> Se A domina X allora $\forall x > 0$ $\forall a > 0$ si ha

$$P(X_T > x \text{ e } A_T < a) \leq \frac{\mathbb{E}[A_T \wedge a]}{x}$$

<u>Dim</u> Introduciamo t.d.a. σ e τ

$$\sigma := \inf\{t \in [0,T] : X_t > x\}$$

$$\sigma: \Omega \to [0,T] \cup \{+\infty\}$$

e $\quad \tau: \inf\{ t \in [0,T] : A_t > a \}$

<u>OSS</u> Nell'evento $\{\sigma < \infty\}$ si ha

$$X_\sigma = x$$

e similmente $A_\tau = a$ nell'evento $\{\tau < \infty\}$

$$\{ X_T > x \text{ e } A_T < a \} \subseteq$$

$$\subseteq \{ \sigma < \infty , \tau = +\infty \}$$

$$\subseteq \{ \sigma \leq \tau , \sigma < +\infty \}$$

$$x P(X_T > x \text{ e } A_T < a) =$$

$$= x \, \mathbb{E}[I_{\{X_T > x \text{ e } A_T < a\}}]$$

$$\leq x \, \mathbb{E}[I_{\{\sigma \leq \tau \text{ e } \sigma < +\infty\}}]$$

$$= \mathbb{E}[X_\sigma I_{\{\sigma \leq \tau \text{ e } \sigma < +\infty\}}]$$

$$= \mathbb{E}\Big[X_{\underbrace{\sigma \wedge \tau \wedge T}_{\text{t.d.a.}}} \, I_{\{\sigma \le \tau \text{ e } \sigma < +\infty\}} \Big]$$

$$\le \mathbb{E}\big[X_{\sigma \wedge \tau \wedge T} \big]$$

$$\le \mathbb{E}\big[A_{\sigma \wedge \tau \wedge T} \big] \qquad A_{\sigma \wedge \tau} \le A_\tau$$

$$\le \mathbb{E}\big[A_{\tau \wedge T} \big]$$

$$\le \mathbb{E}\big[A_{\tau \wedge T} \wedge a \big] \le \mathbb{E}\big[A_T \wedge a \big]$$

Lemma Se A, X sono v.a. non negative tali che

$$P(X > x \text{ e } A < x) \le \frac{\mathbb{E}[\min\{A, x\}]}{x}$$

$\forall x > 0$, allora $\forall q \in (0,1)$ $\exists\, C(q)$ tale che

$$\mathbb{E}[X^q] \le C(q)\, \mathbb{E}[A^q]$$

Dim Vale $\mathbb{E}[X^q] = \int_0^{+\infty} P(X > x)\, q x^{q-1} dx$

$$\le \int_0^{+\infty} \big[P(X > x \text{ e } A < x) + P(A \ge x, \cancel{X > x}) \big] q x^{q-1} dx$$

$$\|$$

$$\le \int_0^{+\infty} \frac{\mathbb{E}[\min\{A,x\}]}{x}\, q x^{q-1}\, dx + \underbrace{\int_0^{+\infty} P(A \geq x)\, q x^{q-1}\, dx}_{= \mathbb{E}[A^q]}$$

$$\downarrow$$

$$\mathbb{E}\left[\int_0^{+\infty} \min\{A,x\}\, q x^{q-2}\, dx \right]$$

$$= \mathbb{E}\left[\int_0^{A} q x^{q-1}\, dx + \int_A^{+\infty} A\, q x^{q-2}\, dx \right]$$

$$= \mathbb{E}[A^q] + \mathbb{E}\left[\frac{q}{1-q} A^{q-1} \cdot A \right] \qquad \boxed{q-2 < -1}$$

$$= \left(1 + \frac{q}{1-q}\right) \mathbb{E}[A^q] \qquad \square$$

OSS $\forall p > 0$ BDG

$$\mathbb{E}\left[(M_T^*)^p \right]^{1/p} \le C(p) \left(\mathbb{E}\left[[M]_T^{p/2} \right] \right)^{1/p}$$

Se $p \to \infty$ la $C(p)$ diverge

Dis massimale di Doob Se $(M_t)_{t\in[0,T]}$ mart. continua

allora valgono le dis. $(T<\infty)$

i) $P(M^*_T > x) \le \frac{E[|M_T|]}{x}$

ii) $\forall p>1$ $E[|M^*_T|^p] \le C(p)\, E[|M_T|^p]$

Dim i) $\sigma := \inf\{t\in[0,T] : |M_t| > x\}$

$|E[M_{\sigma\wedge T}]| \le E[|M_T|]$

ii) La i) implica che $\forall x>0$

$P(M^*_T > x) \le E[|M_T|\, I_{\{|M_T| > \frac{x}{2}\}}]$

$M_t = E[M_T | \mathcal{F}_t]$

$E[(M^*_T)^p] = \int_0^{+\infty} P(M^*_T > x)\, p\, x^{p-1}\, dx$

$\le \int_0^{+\infty} E[|M_T|\, I_{\{|M_T| > \frac{x}{2}\}}\, p\, x^{p-2}]\, dx$

$$= \mathbb{E}\left[|M_T| \int_0^{2|M_T|} p\, x^{p-2}\, dx \right]$$

$$= \mathbb{E}\left[|M_T| \frac{2|M_T|^{p-1}}{p-1} p \right] \quad \square$$

<u>Cosa accade</u> se $p = +\infty$?

Torniamo all'integrale di Wiener $M_t = \int_0^t h_s \, dB_s$

$\sigma^2 = \int_0^T h_s^2 \, ds < \infty$ $\qquad (x > 0)$

$$P\left(\left| \int_0^T h_s \, dB_s \right| > x \right) = 2 \int_x^{+\infty} e^{-\frac{y^2}{2\sigma^2}} \frac{dy}{\sqrt{2\pi\sigma^2}}$$

$$\leq C \, e^{-\frac{x^2}{\underline{2\sigma^2}}}$$

"code" gaussiane.

<u>Idea</u> sostituire σ^2 con $\| [M]_T \|_{L^\infty(\mathbb{P})}$.

<u>OSS</u> Se X v.a. tale che $\mathbb{E}[X] = 0$

$$\mathbb{E}[\exp(\lambda X)] \leq \exp\left(\frac{\lambda^2 \sigma^2}{2} \right) \quad \forall \lambda \in \mathbb{R}$$

per qualche costante σ^2 allora

$$P(|X| > x) \leq 2\exp\left(-\frac{x^2}{2\sigma^2}\right) \quad \forall x > 0$$

Infatti $P(X > x) = P(\exp(\lambda X) > \exp(\lambda x))$
$(\lambda > 0)$

$$\leq \mathbb{E}[\exp(\lambda X)]\exp(-\lambda x)$$

$$\leq \exp\left(\frac{\lambda^2\sigma^2}{2} - \lambda x\right)$$

$\min_{\lambda > 0} \frac{\lambda^2\sigma^2}{2} - \lambda x \Rightarrow \lambda = \frac{x}{\sigma^2}$

$$\exp\left(\frac{x^2}{2\sigma^2} - \frac{x^2}{\sigma^2}\right)$$

$$= \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

<u>Teorema</u> Sia $(M_t)_{t\in[0,T]}$ mart continua con $M_0 = 0$ e

$[M]_T \leq \sigma^2$ $\mathbb{P}$-q.c. per qualche costante σ^2

Allora

$$\mathbb{P}(|M_T| > x) \leq 2\exp\left(-\frac{x^2}{2\sigma^2}\right) \quad \forall x > 0$$

$$\rightarrow \mathbb{P}(M_T^* > x) \leq C\exp\left(-\frac{x^2}{\sigma^2}\right)$$

<u>Dim</u> Introduciamo l'esponenziale stocastico

$$\mathcal{E}(M)_t = \exp\left(M_t - \frac{1}{2}[M]_t \right)$$

<u>Itô</u> $$d\,\mathcal{E}(M)_t = \mathcal{E}(M)_t\, d\left(M_t - \cancel{\frac{1}{2}[M]_t} \right) + \cancel{+\frac{1}{2}\mathcal{E}(M)_t\, d[M]_t}$$

$$= \mathcal{E}(M)_t\, dM_t$$

$$\boxed{\mathcal{E}(M)_t = 1 + \int_0^t \mathcal{E}(M)_s\, dM_s}$$

martingala locale $\underline{\geq 0}$.

$$\mathbb{E}[\mathcal{E}(M)_T] \overset{??}{=} \mathbb{E}[\mathcal{E}(M)_0] = 1$$

però ci sono t.d.a. $\tau \uparrow T$ per cui $\mathcal{E}(M)^{|\tau}$ è vera martingala. $\mathcal{E}(M)^{|\tau} = \mathcal{E}(M^{|\tau})$ quindi

$$\mathbb{E}[\mathcal{E}(M)_\tau] = \mathbb{E}[\mathcal{E}(M)_0] = 1$$

Il lemma di Fatou implica che

$$\boxed{\mathbb{E}[\mathcal{E}(M)_T] \leq 1}$$

Abbiamo $\mathbb{E}\left[\exp\left(M_T - \frac{1}{2}[M]_T \right)\right] \leq 1$

Pertanto dato $\lambda \in \mathbb{R}$ vale analogamente

$$\mathbb{E}\left[\exp\left(\lambda M_T - \frac{1}{2}\lambda^2 [M]_T \right)\right] \leq 1$$

quindi

$$\mathbb{E}\left[\exp(\lambda M_T) \right] =$$

$$= \mathbb{E}\left[\exp\left(\lambda M_T - \frac{1}{2}\lambda^2 [M]_T\right) \underbrace{\exp\left(\frac{\lambda^2 [M]_T}{2}\right)} \right]$$

$$\leq \mathbb{E}\left[\exp\left(\lambda M_T - \frac{1}{2}\lambda^2 [M]_T\right)\right] \exp\left(\frac{\lambda^2 \sigma^2}{2}\right)$$

$$\leq \exp\left(\frac{\lambda^2\sigma^2}{2}\right)$$

$$\mathbb{E}\left[\exp\left(\sup_{t\in[0,T]} M_t \right)\right] =$$

$$= \mathbb{E}\left[\sup_{t\in[0,T]} \exp\left(M_t - \frac{1}{2}[M]_t \right) \exp\left(\frac{1}{2}[M]_t\right)\right]$$

$$\leq \mathbb{E}\left[\sup_{t\in[0,T]} \mathcal{E}(M)_t \right] \exp\left(\frac{\sigma^2}{2}\right)$$

$$\leq \mathbb{E}\left[\sup_{t\in[0,T]} \mathcal{E}(M)_t^2 \right]^{1/2} \exp\left(\frac{\sigma^2}{2}\right)$$

$$\overset{(\text{Doob})}{\leq} C\, \mathbb{E}\left[\mathcal{E}(M)_T^2 \right]^{1/2} \exp\left(\frac{\sigma^2}{2}\right)$$

$$\leq C\, \mathbb{E}\left[\exp\left(2M_T - [M]_T\right)\right]^{1/2} e^{\frac{\sigma^2}{2}}$$

$$\leq C\, \mathbb{E}\left[\exp\left(\underbrace{2M_T - \frac{1}{2}[2M]_T}\ + \underbrace{[M]_T}\right)\right]^{1/2} e^{\frac{\sigma^2}{2}}$$

$$\leq C\, \mathbb{E}\left[\mathcal{E}(2M)_T \right]^{1/2} e^{\sigma^2}$$

$$\leq C\, e^{\sigma^2}$$

$\square$

<u>Torniamo</u> all'esempio $M_t = \int_0^t H_s\, dB_s$

Se $\sup_{s \in [0,T]} |H_s|^2 \leq \sigma^2$ $\mathbb{P}$-q.c.

$$[M]_T = \int_0^T |H_s|^2 ds \leq T\sigma^2$$

$$\mathbb{P}\left(\left| \int_0^T H_s\, dB_s \right| > x \right) \leq 2 \exp\left(- \frac{x^2}{2T\sigma^2} \right)$$

Se $H_s = g(B_s)$ con g limitata allora

$\int_0^T g(B_s)\, dB_s$ ha code gaussiane

Se $g = f'$ allora con la formula di Itô

$$\underbrace{f(B_T) - f(0) - \frac{1}{2}\int_0^T f''(B_s)\,ds}_{\Downarrow\ \text{ha code gaussiane}} = \underbrace{\int_0^T f'(B_s)\,dB_s}$$

Con il calcolo di Malliavin otteniamo un risultato simile ad esempio però la concentrazione di $\int_0^T \underbrace{g(B_s)}\,dB_s$ richiederà la regolarità di g —

$\square$