Disegualianze di Burkholder - Davis - Gundy

Integrale di Wiener $M_t = B_t$ $H_t = h_t$ deterministico

$$\int_0^T h_s \, dB_s \quad \text{è una v.a. Gaussiana}$$

$$N\left(0, \int_0^T h_s^2 \, ds\right)$$

Per le v.a. gaussiane $(p \geq 1)$

$$\mathbb{E}\left[\left| \int_0^T h_s \, dB_s \right|^p \right] \leq C(p) \left(\int_0^T h_s^2 \, ds \right)^{p/2}$$

Teorema (BDG) Sia $(M_t)_{t \in [0,T]}$ (T finito) martingala continua con $M_0 = 0$. Allora $\forall p > 0$ valgono

$$\mathbb{E}\left[|M_T^*|^p \right] \leq C(p) \, \mathbb{E}\left[[M]_T^{p/2} \right] \qquad (1)$$

e

$$\mathbb{E}\left[[M]_T^{p/2} \right] \leq C(p) \, \mathbb{E}\left[|M_T^*|^p \right] \qquad (2)$$

dove $C(p)>0$ dipende solo da $p>0$.

In particolare se $\underline{p>1}$ Doob garantisce che

$$\mathbb{E}\left[[M]_T^{p/2} \right] \leq C'(p)\, \mathbb{E}\left[|M_T|^p \right]$$

<u>Applicazione</u> Se $M_T = \int_0^T H_s \, dB_s$ con $\left(\sup_{0 \leq s \leq T} |H_s| \in L^\infty(\mathbb{P}) \right)$ $\leq \sigma^2$

allora $[M]_T = \int_0^T H_s^2 \, ds \leq T \sigma^2$

Di conseguenza $\forall p \geq 2$

$$\mathbb{P}\left(\left| \int_0^T H_s \, dB_s \right| > \lambda \right) \leq \frac{\mathbb{E}\left[\left| \int_0^T H_s \, dB_s \right|^p \right]}{\lambda^p}$$

$$\leq \frac{C(p)}{\lambda^p} \mathbb{E}\left[\left(\int_0^T H_s^2 \, ds \right)^{p/2} \right]$$

$$\leq \frac{C(p)\, T^{p/2} \sigma^p}{\lambda^p}$$

Dim

- (1) per $p \geq 2$
- (2) per $p \geq 4$
- dominazione per abbassare p -

Per dimostrare la dis. (1) sia $f(x) = |x|^p$ (con $p \geq 2$)

$f \in \mathcal{C}^2 \Rightarrow$ Itô

$$d|M_t|^p = p|M_t|^{p-1} \operatorname{sgn}(M_t)\, dM_t + \frac{1}{2} p(p-1)|M_t|^{p-2} d[M]_t$$

$$|M_t|^p = \int_0^t p|M_s|^{p-1} \operatorname{sgn}(M_s)\, dM_s + \frac{1}{2} p(p-1) \int_0^t |M_s|^{p-2} d[M]_s$$

OSS è martingala locale: fissato $n \in \mathbb{N}$

$$\tau_n := \inf\{t \in [0,T] : |M_t| \geq n \text{ e } [M]_t \geq n\}$$

dimostriamo la tesi (1) per $M^{|\tau_n}$ e

poi $n \to \infty$ e otteniamo (1) per M perché

$$(M^{|\tau_n})^*_T \uparrow M^*_T \qquad [M^{|\tau_n}]_T = [M]_{T \wedge \tau_n} \uparrow [M]_T$$

per Beppo-Levi.

Passando al valore atteso $(t = T)$

$$\mathbb{E}\left[|M_T|^p \right] = \frac{p(p-1)}{2} \mathbb{E}\left[\int_0^T |M_t|^{p-2} d[M]_t \right]$$

$$\leq \frac{p(p-1)}{2} \mathbb{E}\left[\underbrace{\sup_{0 \leq t \leq T} |M_t|^{p-2}} \int_0^T d[M]_t \right]$$

$$\leq C(p) \ \mathbb{E}\left[(M_T^*)^{p-2} [M]_T \right]$$

(Hölder) $$\leq C(p) \ \mathbb{E}\left[(M_T^*)^p \right]^{\frac{p-2}{p}} \mathbb{E}\left[[M]_T^{p/2} \right]^{2/p}$$

$\frac{p}{2}$ e $\frac{\frac{p}{2}}{\frac{p}{2}-1} = \frac{p}{p-2}$

$\frac{2}{p} + \frac{1}{q} = 1$

Per Doob con esponente $(p \geq 2)$

$$\mathbb{E}\left[(M_T^*)^p \right] \leq C(p) \ \mathbb{E}\left[|M_T|^p \right]$$

$$\leq C'(p) \ \mathbb{E}\left[(M_T^*)^p \right]^{1-\frac{2}{p}} \mathbb{E}\left[[M]_T^{p/2} \right]^{\frac{2}{p}}$$

dividiamo ambo i membri per $\mathbb{E}\left[(M_T^*)^p \right]^{1-\frac{2}{p}}$

$$\mathbb{E}\left[(M_T^*)^p\right]^{\frac{2}{p}} \leq C'(p)\ \mathbb{E}\left[\ [M]_T^{p/2}\ \right]^{\frac{2}{p}}$$

eleviamo alla $\frac{p}{2}$ ambo i membri $\Rightarrow$ (1).

- Sia $p \geq 4$ e usiamo Itô con $p=2$

$$d M_t^2 = 2 M_t\, dM_t + d[M]_t$$

$$\downarrow$$

martingala

$$[M]_T = -2\int_0^T M_t\, dM_t + M_T^2$$

eleviamo ambo i membri alla $\frac{p}{2}$

$$[M]_T^{p/2} = \left(-2\int_0^T M_t\, dM_t + M_T^2\right)^{p/2}$$

Se $x, y \in \mathbb{R}$ $\quad |x+y|^{p/2} \leq C(p)\left(|x|^{p/2} + |y|^{p/2}\right)$

$\frac{p}{2} \geq 1$

$$\Rightarrow \quad [M]_T^{p/2} \leq C(p)\left|2\int_0^T M_t\, dM_t\right|^{p/2} + C(p)\left|M_T\right|^p$$

Il termine $\left|\int_0^T M_t\, dM_t\right|^{p/2}$ in valore atteso è

controllato da $\left[\int_0^{\cdot} M\, dM\right]_T^{p/4} = \left(\int_0^T M_t^2\, d[M]_t\right)^{p/4}$

perché $\frac{p}{2} \geq 2$

$$\mathbb{E}\left[\left|\int_0^T M_t\, dM_t\right|^{p/2}\right] \leq C(p)\, \mathbb{E}\left[\left(\int_0^T M_t^2\, d[M]_t\right)^{p/4}\right]$$

$$\leq C(p)\, \mathbb{E}\left[\left(M_T^*\right)^{p/2} [M]_T^{p/4}\right]$$

(Cauchy-Schwarz) $\leq C(p)\, \mathbb{E}\left[\left(M_T^*\right)^p\right]^{1/2} \mathbb{E}\left[[M]_T^{p/2}\right]^{1/2}$

Tornando alla dis. sopra e passando al valore atteso:

$$\mathbb{E}\left[[M]_T^{p/2}\right] \leq C(p)\mathbb{E}\left[\left|2\int_0^T M_t\, dM_t\right|^{p/2}\right] + C(p)\mathbb{E}\left[|M_T|^p\right]$$

$$\leq C'(p)\, \underbrace{\mathbb{E}\left[\left(M_T^*\right)^p\right]^{1/2}}_{x}\, \underbrace{\mathbb{E}\left[[M]_T^{p/2}\right]^{1/2}}_{y} +$$

$$+ C(p)\, \mathbb{E}\left[\left(M_T^*\right)^p\right]$$

Notiamo che $\forall \varepsilon > 0 \quad x, y \in \mathbb{R}$

$$|xy| \leq \frac{x^2}{2\varepsilon} + \varepsilon \frac{y^2}{2}$$

Se poniamo $\varepsilon = \dfrac{1}{C'(p)}$ nella dis sopra

$$C'(p) \underbrace{\mathbb{E}\left[(M^*_T)^p\right]^{1/2}}_{x} \underbrace{\mathbb{E}\left[[M]_T^{p/2}\right]^{1/2}}_{y} \leq$$

$$\leq \frac{(C'(p))^2}{2} \mathbb{E}\left[(M^*_T)^p\right] + \frac{1}{2} \mathbb{E}\left[[M]_T^{p/2}\right]$$

Per $p \geq 4$ valgono (1) e (2).

Come passare ad esponenti $p < 4$?

Def Dati $(X_t)_{t \in [0,T]}$ $(A_t)_{t \in [0,T]}$ processi continui e adattati e crescenti e non negativi. Diciamo che **A domina X** se $\forall \tau$ t.d.a. vale

$$\mathbb{E}[X_\tau] \leq \mathbb{E}[A_\tau]$$

<u>Proposizione</u> Se A domina X, per ogni $q \in (0,1)$

vale $\mathbb{E}[X_T^q] \leq C(q)\, \mathbb{E}[A_T^q]$

dove $C(q) > 0$ dipende solo da q.

Torniamo alla dis (1)

$$\mathbb{E}[|M_T^*|^p] \leq C(p)\, \mathbb{E}[[M]_T^{p/2}]$$

Se $(M_t)_{t \in [0,T]}$ τ t.d.a. applichiamola a $M^{|\tau}$

$$(M^{|\tau})_T^* = (M^*)_{\tau \wedge T} = (M^*)_T^{|\tau}$$

$$[M^{|\tau}]_T = [M]_{T \wedge \tau} = [M]_T^{|\tau}$$

Posto $X_t = (M_t^*)^p$ e $A_t = [M]_t^{p/2}$

vale che <u>A domina X</u>.
