On the Concave One-Dimensional Random Assignment Problem: *Kantorovich Meets Young*

Dario Trevisan[1]

Università di Pisa
dario.trevisan@unipi.it

Seminars in Probability and Finance

Padova, Nov 24 2023

[1] joint with M. Goldman (CNRS), arXiv:2305.09234

Advertisement

- one two-year post-doc position open at University of Bologna (PRIN project *LeQuN*)
- Application deadline: Dec 17, 2023
- `https://bandi.unibo.it/ricerca/assegni-ricerca?id_bando=67126`
- topics: theory of machine learning and quantum information theory
- backgrounds in probability (e.g. random matrices) or optimization are welcome!
- Main contact: `giacomo.depalma@unibo.it`

Advertisement

- one two-year post-doc position open at University of Bologna (PRIN project *LeQuN*)
- Application deadline: Dec 17, 2023
- https://bandi.unibo.it/ricerca/assegni-ricerca?id_bando=67126
- topics: theory of machine learning and quantum information theory
- backgrounds in probability (e.g. random matrices) or optimization are welcome!
- Main contact: giacomo.depalma@unibo.it

Advertisement

- one two-year post-doc position open at University of Bologna (PRIN project *LeQuN*)
- Application deadline: Dec 17, 2023
- `https://bandi.unibo.it/ricerca/assegni-ricerca?id_bando=67126`
- topics: theory of machine learning and quantum information theory
- backgrounds in probability (e.g. random matrices) or optimization are welcome!
- Main contact: `giacomo.depalma@unibo.it`

Advertisement

- one two-year post-doc position open at University of Bologna (PRIN project *LeQuN*)
- Application deadline: Dec 17, 2023
- `https://bandi.unibo.it/ricerca/assegni-ricerca?id_bando=67126`
- topics: theory of machine learning and quantum information theory
- backgrounds in probability (e.g. random matrices) or optimization are welcome!
- Main contact: `giacomo.depalma@unibo.it`

Advertisement

- one two-year post-doc position open at University of Bologna (PRIN project *LeQuN*)
- Application deadline: Dec 17, 2023
- `https://bandi.unibo.it/ricerca/assegni-ricerca?id_bando=67126`
- topics: theory of machine learning and quantum information theory
- backgrounds in probability (e.g. random matrices) or optimization are welcome!
- Main contact: `giacomo.depalma@unibo.it`

Advertisement

- one two-year post-doc position open at University of Bologna (PRIN project *LeQuN*)
- Application deadline: Dec 17, 2023
- `https://bandi.unibo.it/ricerca/assegni-ricerca?id_bando=67126`
- topics: theory of machine learning and quantum information theory
- backgrounds in probability (e.g. random matrices) or optimization are welcome!
- Main contact: `giacomo.depalma@unibo.it`

Plan

The Assignment Problem

- The assignment (or matching) problem is a combinatorial optimization problem arising in many applications:
  - workers to be assigned to jobs, producers that want to meet sellers, ...
- The task is to optimally match two sets (x_i) and (y_i) via a permutation σ in order to minimize (optimize) the total cost

$$\sum_i c(x_i, y_{\sigma(i)})$$

- When $c(x, y) = c(|x - y|)$ depends on distance (e.g. $x, y \in \mathbb{R}^d$):
  - Convex c favors monotone assignments
  - Concave c yields richer structure and hierarchies with economic interpretation ([McC99])
- Let us look at some simulations.

The Assignment Problem

- The assignment (or matching) problem is a combinatorial optimization problem arising in many applications:
  - workers to be assigned to jobs, producers that want to meet sellers, ...
- The task is to optimally match two sets (x_i) and (y_i) via a permutation σ in order to minimize (optimize) the total cost

$$\sum_i c(x_i, y_{\sigma(i)})$$

- When $c(x, y) = c(|x - y|)$ depends on distance (e.g. $x, y \in \mathbb{R}^d$):
  - Convex c favors monotone assignments
  - Concave c yields richer structure and hierarchical [illegible] interpretation ([McC99])
- Let us look at some simulations.

The Assignment Problem

- The assignment (or matching) problem is a combinatorial optimization problem arising in many applications:
  - workers to be assigned to jobs, producers that want to meet sellers, . . .

- The task is to optimally match two sets (x_i) and (y_i) via a permutation σ in order to minimize (optimize) the total cost

$$\sum_i c(x_i, y_{\sigma(i)})$$

- When $c(x, y) = c(|x - y|)$ depends on distance (e.g. $x, y \in \mathbb{R}^d$):
  - Convex c favors monotone assignments
  - Concave c yields richer structure and hierarchies with economic interpretation ([McC99]).

- Let us look at some simulations.

The Assignment Problem

- The assignment (or matching) problem is a combinatorial optimization problem arising in many applications:
  - workers to be assigned to jobs, producers that want to meet sellers, ...
- The task is to optimally match two sets (x_i) and (y_i) via a permutation σ in order to minimize (optimize) the total cost

$$\sum_i c(x_i, y_{\sigma(i)})$$

- When $c(x, y) = c(|x - y|)$ depends on distance (e.g. $x, y \in \mathbb{R}^d$):
  - Convex c favors monotone assignments
  - Concave c yields richer structure and hierarchies with economic interpretation ([McC99])
- Let us look at some simulations.

The Assignment Problem

- The assignment (or matching) problem is a combinatorial optimization problem arising in many applications:
  - workers to be assigned to jobs, producers that want to meet sellers, ...
- The task is to optimally match two sets (x_i) and (y_i) via a permutation σ in order to minimize (optimize) the total cost

$$\sum_i c(x_i, y_{\sigma(i)})$$

- When $c(x, y) = c(|x - y|)$ depends on distance (e.g. $x, y \in \mathbb{R}^d$):
  - Convex c favors monotone assignments
  - Concave c yields richer structure and hierarchies with economic interpretation ([McC99])
- Let us look at some simulations.

The Assignment Problem

- The assignment (or matching) problem is a combinatorial optimization problem arising in many applications:
  - workers to be assigned to jobs, producers that want to meet sellers, ...
- The task is to optimally match two sets (x_i) and (y_i) via a permutation σ in order to minimize (optimize) the total cost

$$\sum_i c(x_i, y_{\sigma(i)})$$

- When $c(x,y) = c(|x-y|)$ depends on distance (e.g. $x, y \in \mathbb{R}^d$):
  - Convex c favors monotone assignments
  - Concave c yields richer structure and hierarchies with economic interpretation ([McC99])
- Let us look at some simulations.

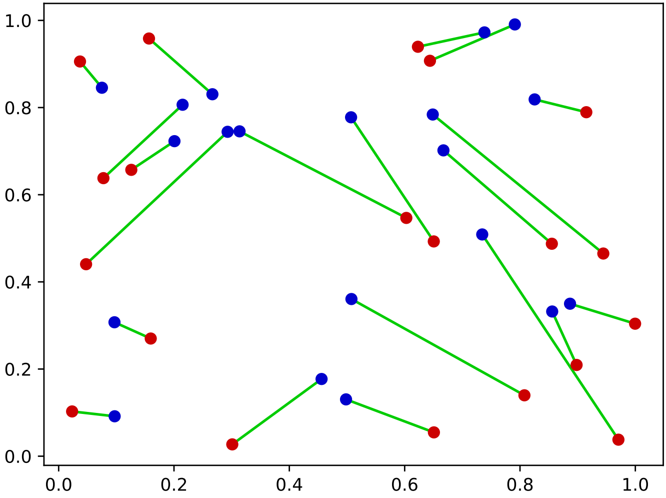
The Assignment Problem

- The assignment (or matching) problem is a combinatorial optimization problem arising in many applications:
  - workers to be assigned to jobs, producers that want to meet sellers, ...
- The task is to optimally match two sets (x_i) and (y_i) via a permutation σ in order to minimize (optimize) the total cost

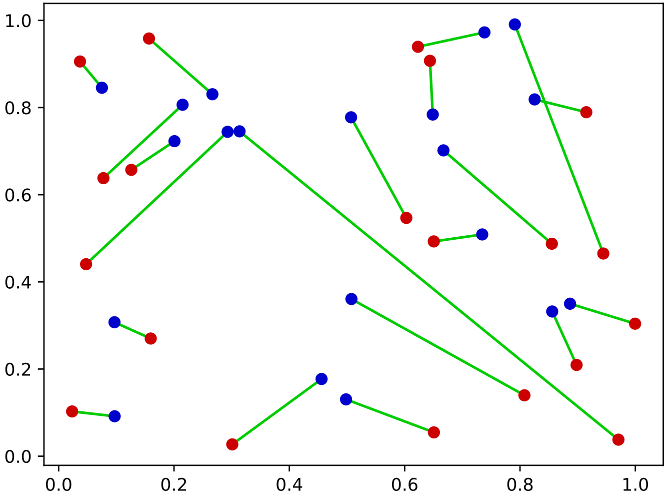
$$\sum_i c(x_i, y_{\sigma(i)})$$

- When $c(x, y) = c(|x - y|)$ depends on distance (e.g. $x, y \in \mathbb{R}^d$):
  - Convex c favors monotone assignments
  - Concave c yields richer structure and hierarchies with economic interpretation ([McC99])
- Let us look at some simulations.

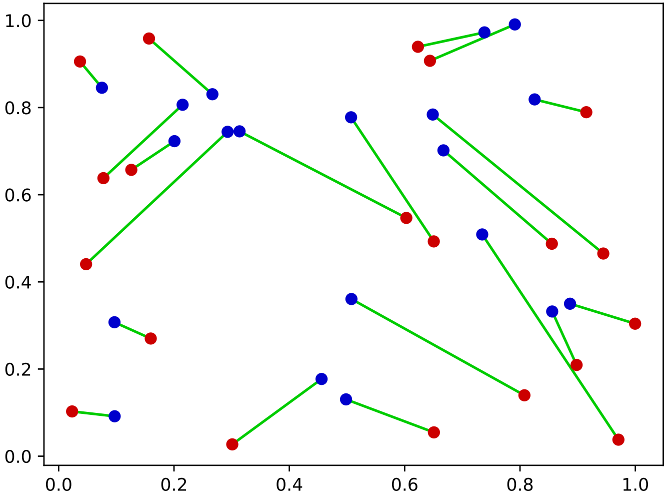
$n = 20 + 20$, $c(x, y) = |x - y|^p$, $p = 1$



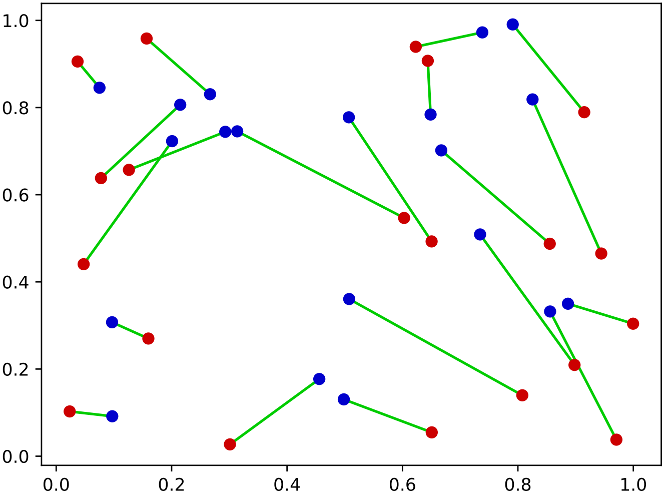
$n = 20 + 20$, $c(x, y) = |x - y|^p$, $p = 0.1$



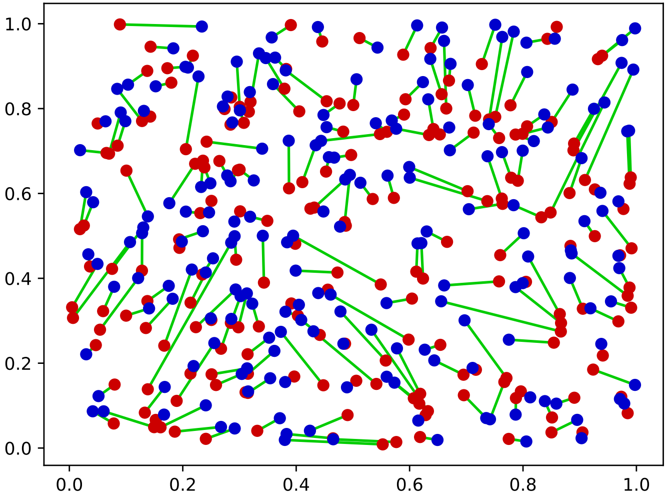
$n = 20 + 20$, $c(x, y) = |x - y|^p$, $p = 1$



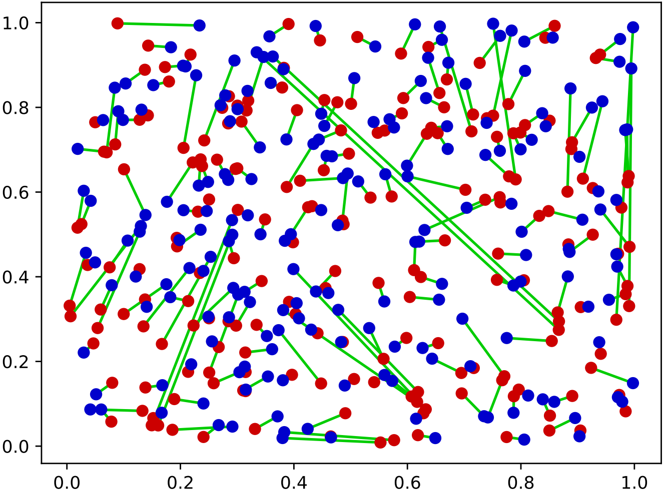
$n = 20 + 20$, $c(x, y) = |x - y|^p$, $p = 2$



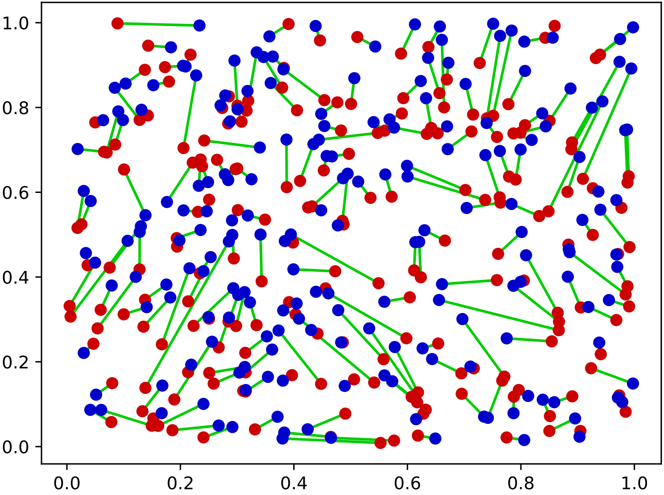
$n = 200 + 200$, $c(x, y) = |x - y|^p$, $p = 1$



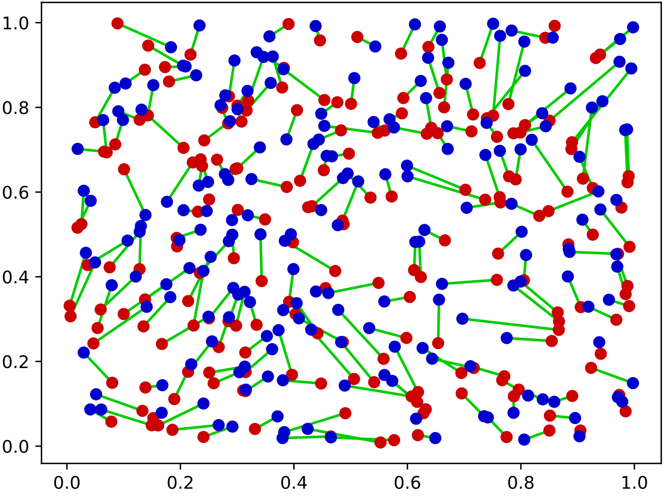
$n = 200 + 200$, $c(x, y) = |x - y|^p$, $p = 0.1$



$n = 200 + 200$, $c(x, y) = |x - y|^p$, $p = 1$



$n = 200 + 200$, $c(x, y) = |x - y|^p$, $p = 2$



Additive Euclidean Functional Theory

- Random instances of combinatorial optimization problems in Euclidean spaces are well-studied [BHH59]
- Focus on convergence results and typical behavior for large instances.
- Limitations arise for bipartite problems e.g. assignment problem.
- Local fluctuations in number of samples give rise to unexpected cost asymptotics
  - On the square [AKT84; Car+14; AST19] ...
  - but also on the line (folklore?)

Additive Euclidean Functional Theory

- Random instances of combinatorial optimization problems in Euclidean spaces are well-studied [BHH59]
- Focus on convergence results and typical behavior for large instances.
- Limitations arise for bipartite problems e.g. assignment problem.
- Local fluctuations in number of samples give rise to unexpected cost asymptotics
  - On the square [AKT84; Car+14; AST19] ...
  - but also on the line (folklore?)

Additive Euclidean Functional Theory

- Random instances of combinatorial optimization problems in Euclidean spaces are well-studied [BHH59]
- Focus on convergence results and typical behavior for large instances.
- Limitations arise for bipartite problems e.g. assignment problem.
- Local fluctuations in number of samples give rise to unexpected cost asymptotics
  - On the square [AKT84; Car+14; AST19]. . .
  - but also on the line (folklore?)

Additive Euclidean Functional Theory

- Random instances of combinatorial optimization problems in Euclidean spaces are well-studied [BHH59]
- Focus on convergence results and typical behavior for large instances.
- Limitations arise for bipartite problems e.g. assignment problem.
- Local fluctuations in number of samples give rise to unexpected cost asymptotics
  - On the square [AKT84; Car+14; AST19]. . .
  - but also on the line (folklore?)

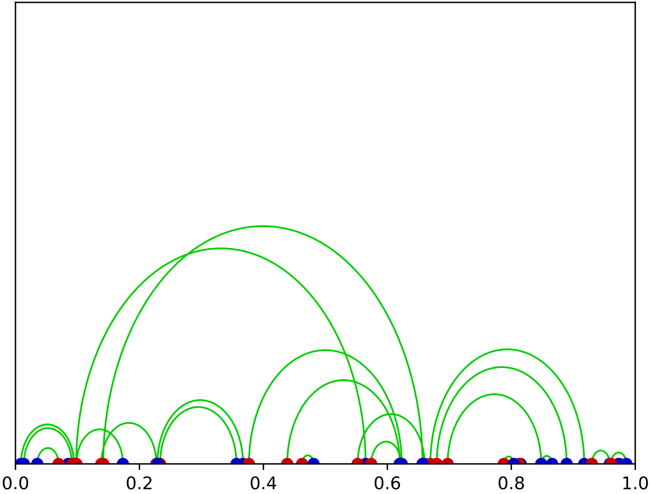
Additive Euclidean Functional Theory

- Random instances of combinatorial optimization problems in Euclidean spaces are well-studied [BHH59]
- Focus on convergence results and typical behavior for large instances.
- Limitations arise for bipartite problems e.g. assignment problem.
- Local fluctuations in number of samples give rise to unexpected cost asymptotics
  - On the square [AKT84; Car+14; AST19]. . .
  - but also on the line (folklore?)

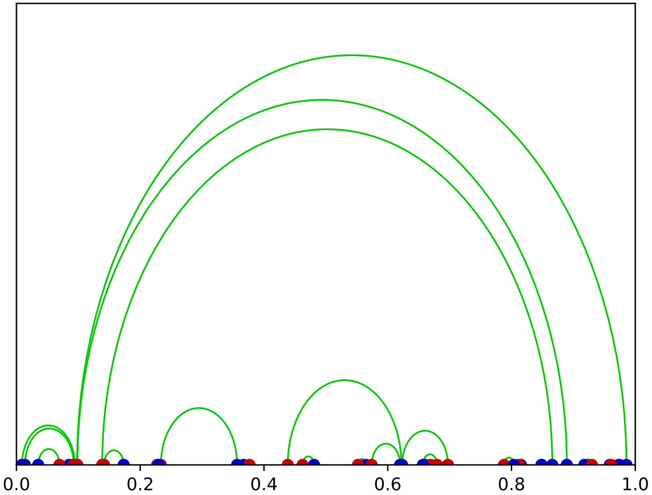
Additive Euclidean Functional Theory

- Random instances of combinatorial optimization problems in Euclidean spaces are well-studied [BHH59]
- Focus on convergence results and typical behavior for large instances.
- Limitations arise for bipartite problems e.g. assignment problem.
- Local fluctuations in number of samples give rise to unexpected cost asymptotics
  - On the square [AKT84; Car+14; AST19]...
  - but also on the line (folklore?)

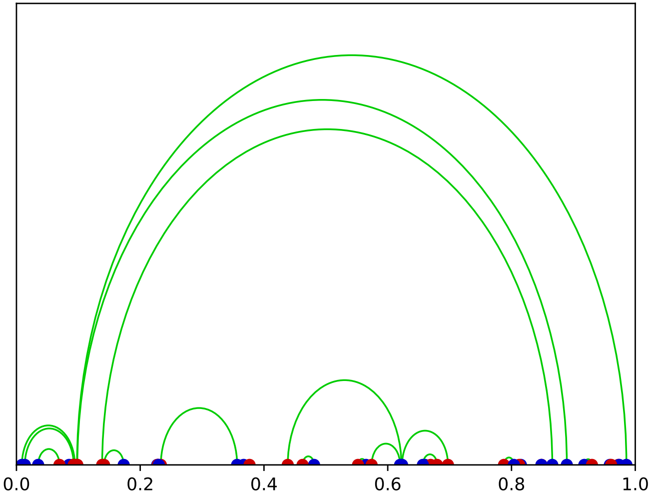
$n = 25 + 25$, $c(x, y) = |x - y|^p$, $p = 1$



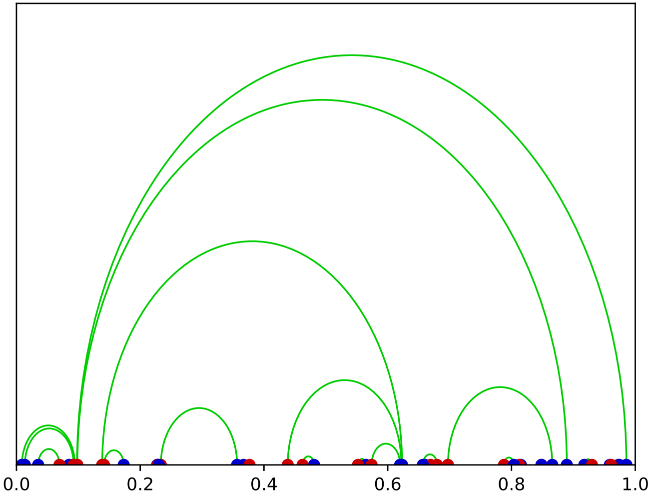
$n = 25 + 25$, $c(x, y) = |x - y|^p$, $p = 0.1$



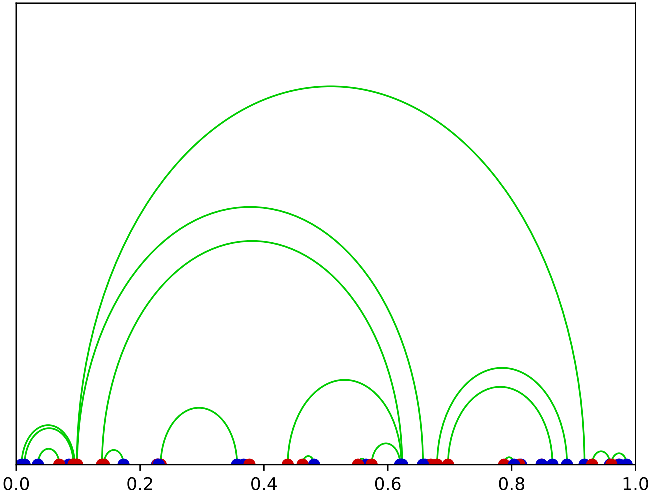
$n = 25 + 25$, $c(x, y) = |x - y|^p$, $p = 0.4$



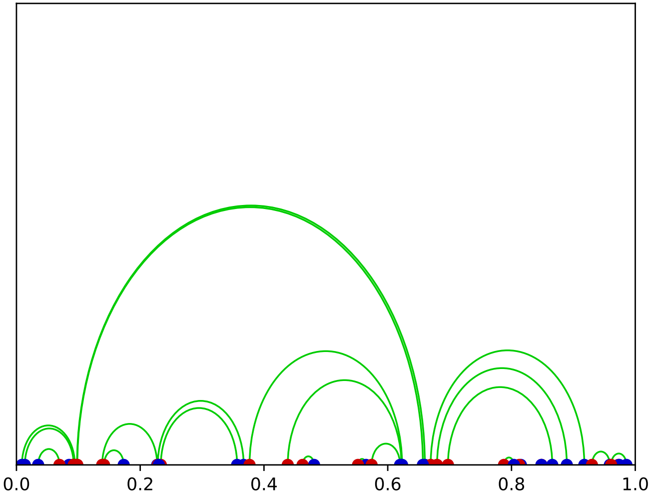
$n = 25 + 25$, $c(x, y) = |x - y|^p$, $p = 0.6$



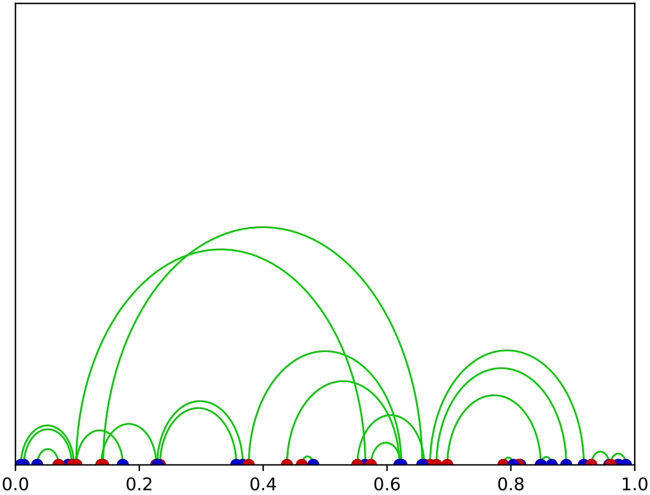
$n = 25 + 25$, $c(x, y) = |x - y|^p$, $p = 0.9$



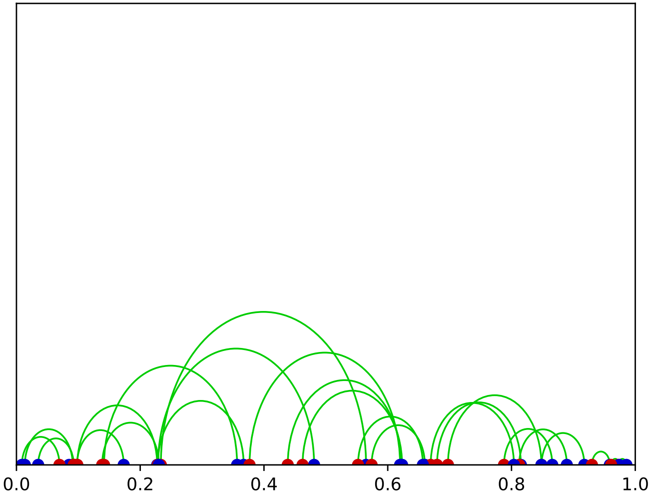
$n = 25 + 25$, $c(x, y) = |x - y|^p$, $p = 0.99$



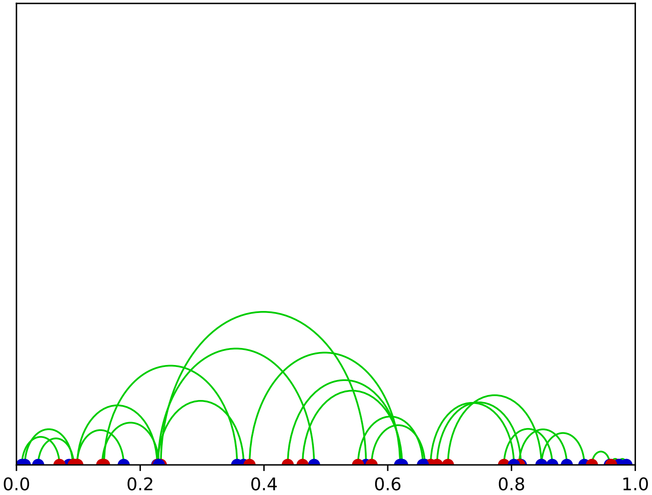
$n = 25 + 25$, $c(x, y) = |x - y|^p$, $p = 1$



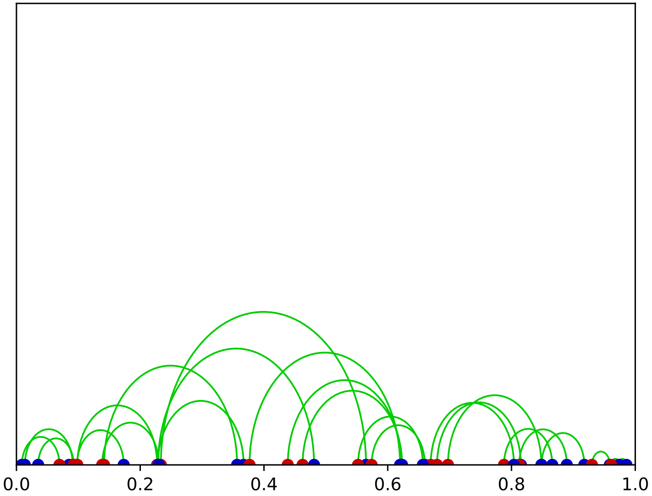
$n = 25 + 25$, $c(x, y) = |x - y|^p$, $p = 1.01$



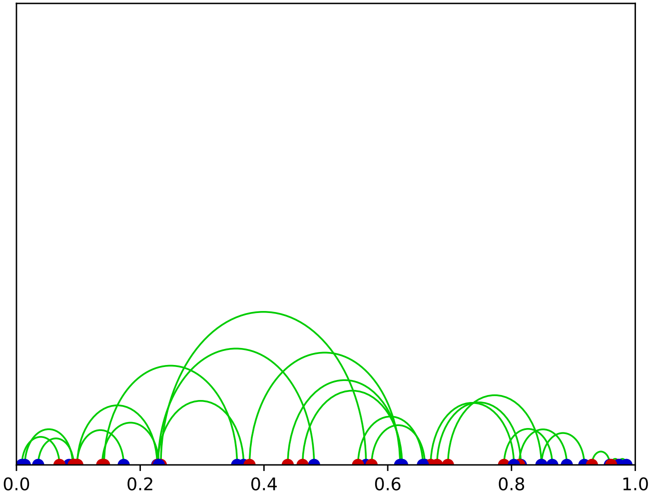
$n = 25 + 25$, $c(x, y) = |x - y|^p$, $p = 1.1$



$n = 25 + 25$, $c(x, y) = |x - y|^p$, $p = 2$



$n = 25 + 25$, $c(x, y) = |x - y|^p$, $p = 3$



Plan

Asymptotic Behavior of Assignment Costs

- We study the cost of the assignment problem over random i.i.d. points (X_i) and (Y_i) on $\mathbb{R}$ with cost $|x-y|^\alpha$:

$$\mathsf{M}_\alpha((X_i)_{i=1}^n, (Y_i)_{i=1}^n) = \min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n |X_i - Y_{\sigma(i)}|^\alpha$$

- Heuristics:
  - $0 < \alpha < 1/2 \to$ the cost scales as $n^{1-\alpha}$
  - $1/2 < \alpha < 1 \to$ local fluctuations dominate and cost scales as $\sqrt{n}$
- We prove:
  - convergence a.s. for $0 < \alpha < 1/2$
  - convergence in law for $1/2 < \alpha < 1$.
- A (new?) idea: the problem converges to an optimal transport problem with a Brownian bridge "measure" $\Rightarrow$ we propose a generalized optimal transport problem using Young integration.

Asymptotic Behavior of Assignment Costs

- We study the cost of the assignment problem over random i.i.d. points (X_i) and (Y_i) on $\mathbb{R}$ with cost $|x-y|^\alpha$:

$$\mathsf{M}_\alpha((X_i)_{i=1}^n, (Y_i)_{i=1}^n) = \min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n |X_i - Y_{\sigma(i)}|^\alpha$$

- Heuristics:
  - $0 < \alpha < 1/2 \to$ the cost scales as $n^{1-\alpha}$
  - $1/2 < \alpha < 1 \to$ local fluctuations dominate and cost scales as $\sqrt{n}$
- We prove:
  - convergence a.s. for $0 < \alpha < 1/2$
  - convergence in law for $1/2 < \alpha < 1$.
- A (new?) idea: the problem converges to an optimal transport problem with a Brownian bridge "measure" $\Rightarrow$ we propose a generalized optimal transport problem using Young integration.

Asymptotic Behavior of Assignment Costs

- We study the cost of the assignment problem over random i.i.d. points (X_i) and (Y_i) on $\mathbb{R}$ with cost $|x-y|^\alpha$:

$$\mathsf{M}_\alpha((X_i)_{i=1}^n, (Y_i)_{i=1}^n) = \min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n |X_i - Y_{\sigma(i)}|^\alpha$$

- Heuristics:
  - $0 < \alpha < 1/2 \to$ the cost scales as $n^{1-\alpha}$
  - $1/2 < \alpha < 1 \to$ local fluctuations dominate and cost scales as $\sqrt{n}$
- We prove:
  - convergence a.s. for $0 < \alpha < 1/2$
  - convergence in law for $1/2 < \alpha < 1$
- A (new?) idea: the problem converges to an optimal transport problem with a Brownian bridge "measure" $\Rightarrow$ we propose a generalized optimal transport problem using Young integration.

Asymptotic Behavior of Assignment Costs

- We study the cost of the assignment problem over random i.i.d. points (X_i) and (Y_i) on $\mathbb{R}$ with cost $|x-y|^\alpha$:

$$\mathsf{M}_\alpha((X_i)_{i=1}^n, (Y_i)_{i=1}^n) = \min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n |X_i - Y_{\sigma(i)}|^\alpha$$

- Heuristics:
  - $0 < \alpha < 1/2 \to$ the cost scales as $n^{1-\alpha}$
  - $1/2 < \alpha < 1 \to$ local fluctuations dominate and cost scales as $\sqrt{n}$
- We prove:
  - convergence a.s. for $0 < \alpha < 1/2$
  - convergence in law for $1/2 < \alpha < 1$
- A (new?) idea: the problem converges to an optimal transport problem with a Brownian bridge "measure" $\Rightarrow$ we propose a generalized optimal transport problem using Young integration.

Asymptotic Behavior of Assignment Costs

- We study the cost of the assignment problem over random i.i.d. points (X_i) and (Y_i) on $\mathbb{R}$ with cost $|x-y|^\alpha$:

$$\mathsf{M}_\alpha((X_i)_{i=1}^n, (Y_i)_{i=1}^n) = \min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n |X_i - Y_{\sigma(i)}|^\alpha$$

- Heuristics:
  - $0 < \alpha < 1/2 \to$ the cost scales as $n^{1-\alpha}$
  - $1/2 < \alpha < 1 \to$ local fluctuations dominate and cost scales as $\sqrt{n}$
- We prove:
  - convergence a.s. for $0 < \alpha < 1/2$
  - convergence in law for $1/2 < \alpha < 1$.
- A (new?) idea: the problem converges to an optimal transport problem with a Brownian bridge “measure” $\Rightarrow$ we propose a generalized optimal transport problem using Young integration.

Asymptotic Behavior of Assignment Costs

- We study the cost of the assignment problem over random i.i.d. points (X_i) and (Y_i) on $\mathbb{R}$ with cost $|x-y|^\alpha$:

$$\mathrm{M}_\alpha((X_i)_{i=1}^n, (Y_i)_{i=1}^n) = \min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n |X_i - Y_{\sigma(i)}|^\alpha$$

- Heuristics:
  - $0 < \alpha < 1/2 \to$ the cost scales as $n^{1-\alpha}$
  - $1/2 < \alpha < 1 \to$ local fluctuations dominate and cost scales as $\sqrt{n}$
- We prove:
  - convergence a.s. for $0 < \alpha < 1/2$
  - convergence in law for $1/2 < \alpha < 1$.

- A (new?) idea: the problem converges to an optimal transport problem with a Brownian bridge “measure” $\Rightarrow$ we propose a generalized optimal transport problem using Young integration.

Asymptotic Behavior of Assignment Costs

- We study the cost of the assignment problem over random i.i.d. points (X_i) and (Y_i) on $\mathbb{R}$ with cost $|x-y|^\alpha$:

$$\mathrm{M}_\alpha((X_i)_{i=1}^n, (Y_i)_{i=1}^n) = \min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n |X_i - Y_{\sigma(i)}|^\alpha$$

- Heuristics:
  - $0 < \alpha < 1/2 \to$ the cost scales as $n^{1-\alpha}$
  - $1/2 < \alpha < 1 \to$ local fluctuations dominate and cost scales as $\sqrt{n}$
- We prove:
  - convergence a.s. for $0 < \alpha < 1/2$
  - convergence in law for $1/2 < \alpha < 1$.

- A (new?) idea: the problem converges to an optimal transport problem with a Brownian bridge “measure” $\Rightarrow$ we propose a generalized optimal transport problem using Young integration.

Asymptotic Behavior of Assignment Costs

- We study the cost of the assignment problem over random i.i.d. points (X_i) and (Y_i) on $\mathbb{R}$ with cost $|x-y|^\alpha$:

$$\mathrm{M}_\alpha((X_i)_{i=1}^n, (Y_i)_{i=1}^n) = \min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n |X_i - Y_{\sigma(i)}|^\alpha$$

- Heuristics:
  - $0 < \alpha < 1/2 \to$ the cost scales as $n^{1-\alpha}$
  - $1/2 < \alpha < 1 \to$ local fluctuations dominate and cost scales as $\sqrt{n}$
- We prove:
  - convergence a.s. for $0 < \alpha < 1/2$
  - convergence in law for $1/2 < \alpha < 1$.

- A (new?) idea: the problem converges to an optimal transport problem with a Brownian bridge "measure" $\Rightarrow$ we propose a generalized optimal transport problem using Young integration.

Main result

Theorem (Goldman, T., 2023)

Let $(X_i)_{i=1}^{\infty}, (Y_i)_{i=1}^{\infty} \subseteq \mathbb{R}$ be i.i.d. random variables with law μ.
Denote with f the absolutely continuous part of μ and $F(t) = \mu((-\infty, t])$.

1. If $1/2 < \alpha < 1$ and μ has bounded support, then

$$\lim_{n\to\infty} n^{-1/2} \mathrm{M}_\alpha((X_i)_{i=1}^n, (Y_i)_{i=1}^n) \stackrel{law}{\to} \|\sqrt{2} B \circ F\|_{\mathrm{W}_\alpha},$$

where $(B(t))_{t\in[0,1]}$ is a standard Brownian bridge and $\|\cdot\|_{\mathrm{W}_\alpha}$ is the Kantorovich-Young norm (defined below).

2. If $0 < \alpha < 1/2$ and $\int_{\mathbb{R}} |t|^\beta d\mu(t) < \infty$ for some $\beta > 4\alpha/(1-2\alpha)$, then

$$\lim_{n\to\infty} n^{\alpha-1} \mathrm{M}_\alpha((X_i)_{i=1}^n, (Y_i)_{i=1}^n) \stackrel{a.s.}{\to} c(\alpha) \int_{\mathbb{R}} f^{1-\alpha}(t) dt,$$

where $c(\alpha) \in (0, \infty)$.

Main result

Theorem (Goldman, T., 2023)

Let $(X_i)_{i=1}^{\infty}, (Y_i)_{i=1}^{\infty} \subseteq \mathbb{R}$ be i.i.d. random variables with law μ.
Denote with f the absolutely continuous part of μ and $F(t) = \mu((-\infty, t])$.

1. If $1/2 < \alpha < 1$ and μ has bounded support, then

$$\lim_{n\to\infty} n^{-1/2} \mathsf{M}_\alpha((X_i)_{i=1}^n, (Y_i)_{i=1}^n) \overset{law}{\to} \|\sqrt{2}B \circ F\|_{\mathsf{W}_\alpha},$$

   where $(B(t))_{t\in[0,1]}$ is a standard Brownian bridge and $\|\cdot\|_{\mathsf{W}_\alpha}$ is the Kantorovich-Young norm (defined below).

2. If $0 < \alpha < 1/2$ and $\int_{\mathbb{R}} |t|^\beta d\mu(t) < \infty$ for some $\beta > 4\alpha/(1-2\alpha)$, then

$$\lim_{n\to\infty} n^{\alpha-1} \mathsf{M}_\alpha((X_i)_{i=1}^n, (Y_i)_{i=1}^n) \overset{a.s.}{\to} c(\alpha) \int_{\mathbb{R}} f^{1-\alpha}(t)dt,$$

   where $c(\alpha) \in (0, \infty)$.

Main result

Theorem (Goldman, T., 2023)

Let $(X_i)_{i=1}^{\infty}, (Y_i)_{i=1}^{\infty} \subseteq \mathbb{R}$ be i.i.d. random variables with law μ.
Denote with f the absolutely continuous part of μ and $F(t) = \mu((-\infty, t])$.

1. If $1/2 < \alpha < 1$ and μ has bounded support, then

$$\lim_{n\to\infty} n^{-1/2} \mathsf{M}_\alpha((X_i)_{i=1}^n, (Y_i)_{i=1}^n) \xrightarrow{law} \|\sqrt{2} B \circ F\|_{\mathsf{W}_\alpha},$$

   where $(B(t))_{t\in[0,1]}$ is a standard Brownian bridge and $\|\cdot\|_{\mathsf{W}_\alpha}$ is the Kantorovich-Young norm (defined below).

2. If $0 < \alpha < 1/2$ and $\int_{\mathbb{R}} |t|^\beta d\mu(t) < \infty$ for some $\beta > 4\alpha/(1-2\alpha)$, then

$$\lim_{n\to\infty} n^{\alpha-1} \mathsf{M}_\alpha((X_i)_{i=1}^n, (Y_i)_{i=1}^n) \xrightarrow{a.s.} c(\alpha) \int_{\mathbb{R}} f^{1-\alpha}(t) dt,$$

   where $c(\alpha) \in (0, \infty)$.

Main result

Theorem (Goldman, T., 2023)

Let $(X_i)_{i=1}^{\infty}, (Y_i)_{i=1}^{\infty} \subseteq \mathbb{R}$ be i.i.d. random variables with law μ.
Denote with f the absolutely continuous part of μ and $F(t) = \mu((-\infty, t])$.

1. If $1/2 < \alpha < 1$ and μ has bounded support, then
$$\lim_{n\to\infty} n^{-1/2}\mathrm{M}_\alpha((X_i)_{i=1}^n, (Y_i)_{i=1}^n) \overset{law}{\longrightarrow} \|\sqrt{2}B \circ F\|_{\mathrm{W}_\alpha},$$
where $(B(t))_{t\in[0,1]}$ is a standard Brownian bridge and $\|\cdot\|_{\mathrm{W}_\alpha}$ is the Kantorovich-Young norm (defined below).
2. If $0 < \alpha < 1/2$ and $\int_{\mathbb{R}} |t|^\beta d\mu(t) < \infty$ for some $\beta > 4\alpha/(1-2\alpha)$, then
$$\lim_{n\to\infty} n^{\alpha-1}\mathrm{M}_\alpha((X_i)_{i=1}^n, (Y_i)_{i=1}^n) \overset{a.s.}{\to} c(\alpha) \int_{\mathbb{R}} f^{1-\alpha}(t)dt,$$
where $c(\alpha) \in (0, \infty)$.

Strategy for $1/2 < \alpha < 1$: Kantorovich-Young Problem

- We define a variational problem for functions g with finite q-variation:

$$\|g\|_{\mathsf{W}_\alpha} = \sup \left\{ \int_I f dg \, : \, [f]_{C^\alpha} \leq 1 \right\},$$

  where $\alpha + 1/q > 1$.
- It recovers usual optimal transport if g has bounded variation.
- We investigate some basic properties of this problem.

- In the Brownian bridge (random) case $g(t) = B(t)$:
  - B has finite q-variation only if $q > 2$
  - $\Rightarrow$ the problem is only meaningful if $\alpha > 1/2$.
- It explains the “phase transition” at $\alpha = 1/2$ (the same that leads leads to Rough Paths theory [FV10]).

Strategy for $1/2 < \alpha < 1$: Kantorovich-Young Problem

- We define a variational problem for functions g with finite q-variation:

$$\|g\|_{\mathsf{W}_\alpha} = \sup \left\{ \int_I f dg \, : \, [f]_{C^\alpha} \leq 1 \right\},$$

  where $\alpha + 1/q > 1$.
- It recovers usual optimal transport if g has bounded variation.
- We investigate some basic properties of this problem.
- In the Brownian bridge (random) case $g(t) = B(t)$:
  - B has finite q-variation only if $q > 2$
  - $\Rightarrow$ the problem is only meaningful if $\alpha > 1/2$
- It explains the "phase transition" at $\alpha = 1/2$ (the same that leads leads to Rough Paths theory [FV10]).

Strategy for $1/2 < \alpha < 1$: Kantorovich-Young Problem

- We define a variational problem for functions g with finite q-variation:

$$\|g\|_{W_\alpha} = \sup \left\{ \int_I f dg \, : \, [f]_{C^\alpha} \leq 1 \right\},$$

  where $\alpha + 1/q > 1$.
- It recovers usual optimal transport if g has bounded variation.
- We investigate some basic properties of this problem.

- In the Brownian bridge (random) case $g(t) = B(t)$:
  - B has finite q-variation only if $q > 2$
  - $\Rightarrow$ the problem is only meaningful if $\alpha > 1/2$
- It explains the “phase transition” at $\alpha = 1/2$ (the same that leads leads to Rough Paths theory [FV10]).

Strategy for $1/2 < \alpha < 1$: Kantorovich-Young Problem

- We define a variational problem for functions g with finite q-variation:

$$\|g\|_{\mathsf{W}_\alpha} = \sup\left\{ \int_I f dg \ : \ [f]_{C^\alpha} \leq 1 \right\},$$

  where $\alpha + 1/q > 1$.
- It recovers usual optimal transport if g has bounded variation.
- We investigate some basic properties of this problem.

- In the Brownian bridge (random) case $g(t) = B(t)$:
  - B has finite q-variation only if $q > 2$
  - $\Rightarrow$ the problem is only meaningful if $\alpha > 1/2$
- It explains the "phase transition" at $\alpha = 1/2$ (the same that leads leads to Rough Paths theory [FV10]).

Strategy for $1/2 < \alpha < 1$: Kantorovich-Young Problem

- We define a variational problem for functions g with finite q-variation:

$$\|g\|_{\mathsf{W}_\alpha} = \sup \left\{ \int_I f dg \, : \, [f]_{C^\alpha} \le 1 \right\},$$

  where $\alpha + 1/q > 1$.
- It recovers usual optimal transport if g has bounded variation.
- We investigate some basic properties of this problem.

- In the Brownian bridge (random) case $g(t) = B(t)$:
  - B has finite q-variation only if $q > 2$
  - $\Rightarrow$ the problem is only meaningful if $\alpha > 1/2$
- It explains the "phase transition" at $\alpha = 1/2$ (the same that leads leads to Rough Paths theory [FV10]).

Strategy for $1/2 < \alpha < 1$: Kantorovich-Young Problem

- We define a variational problem for functions g with finite q-variation:

$$\|g\|_{\mathsf{W}_\alpha} = \sup \left\{ \int_I f dg \,:\, [f]_{C^\alpha} \le 1 \right\},$$

where $\alpha + 1/q > 1$.

- It recovers usual optimal transport if g has bounded variation.
- We investigate some basic properties of this problem.

- In the Brownian bridge (random) case $g(t) = B(t)$:
  - B has finite q-variation only if $q > 2$
  - $\Rightarrow$ the problem is only meaningful if $\alpha > 1/2$
- It explains the "phase transition" at $\alpha = 1/2$ (the same that leads leads to Rough Paths theory [FV10]).

Strategy for $1/2 < \alpha < 1$: Kantorovich-Young Problem

- We define a variational problem for functions g with finite q-variation:

$$\|g\|_{\mathsf{W}_\alpha} = \sup\left\{ \int_I f dg \, : \, [f]_{C^\alpha} \le 1 \right\},$$

  where $\alpha + 1/q > 1$.
- It recovers usual optimal transport if g has bounded variation.
- We investigate some basic properties of this problem.

- In the Brownian bridge (random) case $g(t) = B(t)$:
  - B has finite q-variation only if $q > 2$
  - $\Rightarrow$ the problem is only meaningful if $\alpha > 1/2$
- It explains the “phase transition” at $\alpha = 1/2$ (the same that leads leads to Rough Paths theory [FV10]).

Strategy for $0 < \alpha < 1/2$: Boundary Functional

- We modify the assignment allowing for a reservoir of points in $\{0, 1\}$ of $[0, 1]$ (export/import OT).
- $\Rightarrow$ boundary functional in random combinatorial optimization theory [Ste97; Yuk06].
- A widely open question is whether the asymptotic cost of the boundary functional equals that of the assignment problem.

In our case $d = 1$, $\alpha \in (0, 1)$:

- Due to the concave cost, optimal assignments have the no-crossing property
- $\Rightarrow$ the boundary functional cost indeed is close $\lesssim n^{1/2}$ the original assignment problem cost
- By known results in [BB13] it follows convergence for $\alpha < 1/2$.

Strategy for $0 < \alpha < 1/2$: Boundary Functional

- We modify the assignment allowing for a reservoir of points in $\{0, 1\}$ of $[0, 1]$ (export/import OT).
- $\Rightarrow$ boundary functional in random combinatorial optimization theory [Ste97; Yuk06].
- A widely open question is whether the asymptotic cost of the boundary functional equals that of the assignment problem.

In our case $d = 1$, $\alpha \in (0, 1)$:

- Due to the concave cost, optimal assignments have the no-crossing property
- $\Rightarrow$ the boundary functional cost indeed is close $\lesssim n^{1/2}$ the original assignment problem cost
- By known results in [BB13] it follows convergence for $\alpha < 1/2$.

Strategy for $0 < \alpha < 1/2$: Boundary Functional

- We modify the assignment allowing for a reservoir of points in $\{0, 1\}$ of $[0, 1]$ (export/import OT).
- $\Rightarrow$ boundary functional in random combinatorial optimization theory [Ste97; Yuk06].
- A widely open question is whether the asymptotic cost of the boundary functional equals that of the assignment problem.

In our case $d = 1$, $\alpha \in (0, 1)$:

- Due to the concave cost, optimal assignments have the no-crossing property
- $\Rightarrow$ the boundary functional cost indeed is close $\lesssim n^{1/2}$ the original assignment problem cost.
- By known results in [BB13] it follows convergence for $\alpha < 1/2$.

Strategy for $0 < \alpha < 1/2$: Boundary Functional

- We modify the assignment allowing for a reservoir of points in $\{0, 1\}$ of $[0, 1]$ (export/import OT).
- $\Rightarrow$ boundary functional in random combinatorial optimization theory [Ste97; Yuk06].
- A widely open question is whether the asymptotic cost of the boundary functional equals that of the assignment problem.

In our case $d = 1$, $\alpha \in (0, 1)$:

- Due to the concave cost, optimal assignments have the no-crossing property
- $\Rightarrow$ the boundary functional cost indeed is close $\lesssim n^{1/2}$ the original assignment problem cost
- By known results in [BB13] it follows convergence for $\alpha < 1/2$

Strategy for $0 < \alpha < 1/2$: Boundary Functional

- We modify the assignment allowing for a reservoir of points in $\{0, 1\}$ of $[0, 1]$ (export/import OT).
- $\Rightarrow$ boundary functional in random combinatorial optimization theory [Ste97; Yuk06].
- A widely open question is whether the asymptotic cost of the boundary functional equals that of the assignment problem.

In our case $d = 1$, $\alpha \in (0, 1)$:

- Due to the concave cost, optimal assignments have the no-crossing property
- $\Rightarrow$ the boundary functional cost indeed is close $\lesssim n^{1/2}$ the original assignment problem cost
- By known results in [BB13] it follows convergence for $\alpha < 1/2$.

Strategy for $0 < \alpha < 1/2$: Boundary Functional

- We modify the assignment allowing for a reservoir of points in $\{0, 1\}$ of $[0, 1]$ (export/import OT).
- $\Rightarrow$ boundary functional in random combinatorial optimization theory [Ste97; Yuk06].
- A widely open question is whether the asymptotic cost of the boundary functional equals that of the assignment problem.

In our case $d = 1$, $\alpha \in (0, 1)$:

- Due to the concave cost, optimal assignments have the no-crossing property
- $\Rightarrow$ the boundary functional cost indeed is close $\lesssim n^{1/2}$ the original assignment problem cost
- By known results in [BB13] it follows convergence for $\alpha < 1/2$.

Strategy for $0 < \alpha < 1/2$: Boundary Functional

- We modify the assignment allowing for a reservoir of points in $\{0, 1\}$ of $[0, 1]$ (export/import OT).
- $\Rightarrow$ boundary functional in random combinatorial optimization theory [Ste97; Yuk06].
- A widely open question is whether the asymptotic cost of the boundary functional equals that of the assignment problem.

In our case $d = 1$, $\alpha \in (0, 1)$:

- Due to the concave cost, optimal assignments have the no-crossing property
- $\Rightarrow$ the boundary functional cost indeed is close $\lesssim n^{1/2}$ the original assignment problem cost
- By known results in [BB13] it follows convergence for $\alpha < 1/2$.

Plan

Hölder and Variation Norms

- Hölder seminorm of exponent $\alpha \in (0,1)$:

$$[f]_{C^\alpha} = \sup_{s \neq t} \frac{|f(t) - f(s)|}{|t-s|^\alpha}$$

- p-variation seminorm (for $p \geq 1$):

$$[f]_{p-\mathrm{var}} = \sup_{\{t_i\}} \left\{ \left(\sum |f(t_i) - f(t_{i-1})|^p \right)^{1/p} \right\}$$

- For any $\alpha \in (0,1)$,

$$[f]_{1/\alpha-\mathrm{var}} \leq |I|^\alpha [f]_{C^\alpha}.$$

- Total variation is 1-variation. Functions of bounded variation can be represented by measures.

Hölder and Variation Norms

- Hölder seminorm of exponent $\alpha \in (0,1)$:

$$[f]_{C^\alpha} = \sup_{s \neq t} \frac{|f(t) - f(s)|}{|t-s|^\alpha}$$

- p-variation seminorm (for $p \geq 1$):

$$[f]_{p-\mathrm{var}} = \sup_{\{t_i\}} \left\{ \left(\sum |f(t_i) - f(t_{i-1})|^p \right)^{1/p} \right\}$$

- For any $\alpha \in (0,1)$,

$$[f]_{1/\alpha-\mathrm{var}} \leq |I|^\alpha [f]_{C^\alpha}.$$

- Total variation is 1-variation. Functions of bounded variation can be represented by measures.

Hölder and Variation Norms

- Hölder seminorm of exponent $\alpha \in (0,1)$:

$$[f]_{C^\alpha} = \sup_{s \neq t} \frac{|f(t) - f(s)|}{|t-s|^\alpha}$$

- p-variation seminorm (for $p \geq 1$):

$$[f]_{p-\mathrm{var}} = \sup_{\{t_i\}} \left\{ \left(\sum |f(t_i) - f(t_{i-1})|^p \right)^{1/p} \right\}$$

- For any $\alpha \in (0,1)$,

$$[f]_{1/\alpha-\mathrm{var}} \leq |I|^\alpha [f]_{C^\alpha}.$$

- Total variation is 1-variation. Functions of bounded variation can be represented by measures.

Hölder and Variation Norms

- Hölder seminorm of exponent $\alpha \in (0,1)$:

$$[f]_{C^\alpha} = \sup_{s \neq t} \frac{|f(t) - f(s)|}{|t-s|^\alpha}$$

- p-variation seminorm (for $p \geq 1$):

$$[f]_{p-\text{var}} = \sup_{\{t_i\}} \left\{ \left(\sum |f(t_i) - f(t_{i-1})|^p \right)^{1/p} \right\}$$

- For any $\alpha \in (0,1)$,

$$[f]_{1/\alpha-\text{var}} \leq |I|^\alpha [f]_{C^\alpha}.$$

- Total variation is 1-variation. Functions of bounded variation can be represented by measures.

Young Integration

The Riemann-Stieltjes integral $\int fdg$ exists if

$$\text{both } [f]_{C^0} \text{ and } [g]_{1-\mathrm{var}} \text{ are finite.}$$

Theorem (L.-C. Young, 1936)

If $p, q \geq 1$ such that $1/p + 1/q > 1$, then:

- *$\int_a^b fdg$ exists for f and g with no common discontinuity points and*

$$\text{both } [f]_{p-var} \text{ and } [g]_{q-var} \text{ are finite.}$$

- *The following bound holds:*

$$\left| \int_a^b fdg - f(a)(g(b) - g(a)) \right| \leq C(p,q)[f]_{p-var}[g]_{q-var}$$

$\Rightarrow$ If $f \in C^\alpha$ with $\alpha + 1/q > 1$ and $g(b) = g(a) = 0$, then

$$\left| \int_I fdg \right| \leq C(1/\alpha, q)|I|^\alpha [f]_{C^\alpha}[g]_{q-\mathrm{var}}.$$

Young Integration

The Riemann-Stieltjes integral $\int fdg$ exists if

$$\text{both } [f]_{C^0} \text{ and } [g]_{1-\text{var}} \text{ are finite.}$$

Theorem (L.-C. Young, 1936)

If $p, q \geq 1$ such that $1/p + 1/q > 1$, then:

- *$\int_a^b fdg$ exists for f and g with no common discontinuity points and*

  $$\textit{both } [f]_{p-var} \textit{ and } [g]_{q-var} \textit{ are finite.}$$

- *The following bound holds:*

  $$\left| \int_a^b fdg - f(a)(g(b) - g(a)) \right| \leq C(p,q)[f]_{p-var}[g]_{q-var}$$

$\Rightarrow$ If $f \in C^\alpha$ with $\alpha + 1/q > 1$ and $g(b) = g(a) = 0$, then

$$\left| \int_I fdg \right| \leq C(1/\alpha, q)|I|^\alpha [f]_{C^\alpha} [g]_{q-\text{var}}.$$

Young Integration

The Riemann-Stieltjes integral $\int fdg$ exists if

$$\text{both } [f]_{C^0} \text{ and } [g]_{1-\mathrm{var}} \text{ are finite.}$$

Theorem (L.-C. Young, 1936)

If $p, q \geq 1$ such that $1/p + 1/q > 1$, then:

- *$\int_a^b fdg$ exists for f and g with no common discontinuity points and*

$$\text{both } [f]_{p-var} \text{ and } [g]_{q-var} \text{ are finite.}$$

- *The following bound holds:*

$$\left| \int_a^b fdg - f(a)(g(b) - g(a)) \right| \leq C(p,q)[f]_{p-var}[g]_{q-var}$$

$\Rightarrow$ If $f \in C^\alpha$ with $\alpha + 1/q > 1$ and $g(b) = g(a) = 0$, then

$$\left| \int_I fdg \right| \leq C(1/\alpha, q)|I|^\alpha [f]_{C^\alpha}[g]_{q-\mathrm{var}}.$$

Wasserstein Distance

Given positive Borel measures μ and ν on $(\mathcal{X}, \mathsf{d})$ with finite q-th moments:

- Optimal transport cost of order q:

$$\inf_{\pi \in \Gamma(\mu,\lambda)} \int_{\mathcal{X}\times\mathcal{X}} \mathsf{d}(x,y)^q \pi(dx, dy), \tag{1}$$

- For $q \in (0,1]$, it induces a distance. Otherwise take its q-th root.
- This yields Wasserstein distance $\mathsf{W}_q(\mu,\nu)$.
- The Wasserstein distance enjoys the Kantorovich dual formulation, for $q \in (0,1]$:

$$\mathsf{W}_q(\mu,\nu) = \sup_f \left\{ \int_{\mathcal{X}} f d(\mu - \nu) \, : \, |f(x) - f(y)| \le \mathsf{d}(x,y)^q \quad \forall x, y \in \mathcal{X} \right\}. \tag{2}$$

Wasserstein Distance

Given positive Borel measures μ and ν on $(\mathcal{X}, \mathsf{d})$ with finite q-th moments:

- Optimal transport cost of order q:

$$\inf_{\pi \in \Gamma(\mu,\lambda)} \int_{\mathcal{X}\times\mathcal{X}} \mathsf{d}(x,y)^q \pi(dx,dy), \tag{1}$$

- For $q \in (0,1]$, it induces a distance. Otherwise take its q-th root.
- This yields Wasserstein distance $\mathsf{W}_q(\mu,\nu)$.
- The Wasserstein distance enjoys the Kantorovich dual formulation, for $q \in (0,1]$:

$$\mathsf{W}_q(\mu,\nu) = \sup_f \left\{ \int_{\mathcal{X}} f d(\mu-\nu) \, : \, |f(x)-f(y)| \le \mathsf{d}(x,y)^q \quad \forall x,y \in \mathcal{X} \right\}. \tag{2}$$

Wasserstein Distance

Given positive Borel measures μ and ν on $(\mathcal{X}, \mathsf{d})$ with finite q-th moments:

- Optimal transport cost of order q:

$$\inf_{\pi \in \Gamma(\mu, \lambda)} \int_{\mathcal{X} \times \mathcal{X}} \mathsf{d}(x, y)^q \pi(dx, dy), \tag{1}$$

- For $q \in (0, 1]$, it induces a distance. Otherwise take its q-th root.
- This yields Wasserstein distance $\mathsf{W}_q(\mu, \nu)$.
- The Wasserstein distance enjoys the Kantorovich dual formulation, for $q \in (0, 1]$:

$$\mathsf{W}_q(\mu, \nu) = \sup_f \left\{ \int_{\mathcal{X}} f d\,(\mu - \nu) \, : \, |f(x) - f(y)| \leq \mathsf{d}(x, y)^q \quad \forall x, y \in \mathcal{X} \right\}. \tag{2}$$

Wasserstein Distance

Given positive Borel measures μ and ν on $(\mathcal{X}, \mathsf{d})$ with finite q-th moments:

- Optimal transport cost of order q:

$$\inf_{\pi \in \Gamma(\mu,\lambda)} \int_{\mathcal{X}\times\mathcal{X}} \mathsf{d}(x,y)^q \pi(dx,dy), \tag{1}$$

- For $q \in (0,1]$, it induces a distance. Otherwise take its q-th root.
- This yields Wasserstein distance $\mathsf{W}_q(\mu,\nu)$.
- The Wasserstein distance enjoys the Kantorovich dual formulation, for $q \in (0,1]$:

$$\mathsf{W}_q(\mu,\nu) = \sup_f \left\{ \int_{\mathcal{X}} f d\,(\mu-\nu) \,:\, |f(x)-f(y)| \leq \mathsf{d}(x,y)^q \quad \forall x,y \in \mathcal{X} \right\}. \tag{2}$$

A Kantorovich-Young Problem

Given $g : I = [a, b] \to \mathbb{R}$ with $g(b) = g(a) = 0$ and $[g]_{q-\mathrm{var}}$ finite:

- Define the Kantorovich-Young norm:

$$\|g\|_{\mathsf{W}_\alpha} = \sup_{[f]_{C^\alpha} \leq 1} \int_I f dg$$

  with $\alpha + 1/q > 1$

- This norm is finite since:

$$\|g\|_{\mathsf{W}_\alpha} \leq C(\alpha, q)|I|^\alpha [g]_{q-\mathrm{var}}$$

- Moreover, we have stability w.r.t. q-variation:

$$\left| \|g\|_{\mathsf{W}_\alpha} - \|\tilde{g}\|_{\mathsf{W}_\alpha} \right| \leq \|g - \tilde{g}\|_{\mathsf{W}_\alpha} \leq C(\alpha, q)|I|^\alpha [g - \tilde{g}]_{q-\mathrm{var}}.$$

- It extends the Kantorovich norm from functions of bounded variation to functions of finite q-variation.

A Kantorovich-Young Problem

Given $g : I = [a, b] \to \mathbb{R}$ with $g(b) = g(a) = 0$ and $[g]_{q-\mathrm{var}}$ finite:

- Define the Kantorovich-Young norm:

$$\|g\|_{\mathrm{W}_\alpha} = \sup_{[f]_{C^\alpha} \leq 1} \int_I f dg$$

  with $\alpha + 1/q > 1$
- This norm is finite since:

$$\|g\|_{\mathrm{W}_\alpha} \leq C(\alpha, q)|I|^\alpha [g]_{q-\mathrm{var}}$$

- Moreover, we have stability w.r.t. q-variation:

$$|\|g\|_{\mathrm{W}_\alpha} - \|\tilde{g}\|_{\mathrm{W}_\alpha}| \leq \|g - \tilde{g}\|_{\mathrm{W}_\alpha} \leq C(\alpha, q)|I|^\alpha [g - \tilde{g}]_{q-\mathrm{var}}.$$

- It extends the Kantorovich norm from functions of bounded variation to functions of finite q-variation.

A Kantorovich-Young Problem

Given $g : I = [a, b] \to \mathbb{R}$ with $g(b) = g(a) = 0$ and $[g]_{q-\mathrm{var}}$ finite:

- Define the Kantorovich-Young norm:

$$\|g\|_{\mathsf{W}_\alpha} = \sup_{[f]_{C^\alpha} \leq 1} \int_I f dg$$

  with $\alpha + 1/q > 1$
- This norm is finite since:

$$\|g\|_{\mathsf{W}_\alpha} \leq C(\alpha, q)|I|^\alpha [g]_{q-\mathrm{var}}$$

- Moreover, we have stability w.r.t. q-variation:

$$\|g\|_{\mathsf{W}_\alpha} - \|\tilde{g}\|_{\mathsf{W}_\alpha}| \leq \|g - \tilde{g}\|_{\mathsf{W}_\alpha} \leq C(\alpha, q)|I|^\alpha [g - \tilde{g}]_{q-\mathrm{var}}.$$

- It extends the Kantorovich norm from functions of bounded variation to functions of finite q-variation.

A Kantorovich-Young Problem

Given $g : I = [a, b] \to \mathbb{R}$ with $g(b) = g(a) = 0$ and $[g]_{q-\mathrm{var}}$ finite:

- Define the Kantorovich-Young norm:

$$\|g\|_{\mathrm{W}_\alpha} = \sup_{[f]_{C^\alpha} \leq 1} \int_I f dg$$

  with $\alpha + 1/q > 1$
- This norm is finite since:

$$\|g\|_{\mathrm{W}_\alpha} \leq C(\alpha, q)|I|^\alpha [g]_{q-\mathrm{var}}$$

- Moreover, we have stability w.r.t. q-variation:

$$|\|g\|_{\mathrm{W}_\alpha} - \|\tilde{g}\|_{\mathrm{W}_\alpha}| \leq \|g - \tilde{g}\|_{\mathrm{W}_\alpha} \leq C(\alpha, q)|I|^\alpha [g - \tilde{g}]_{q-\mathrm{var}}.$$

- It extends the Kantorovich norm from functions of bounded variation to functions of finite q-variation.

A Primal Problem

Coupling with finite energy

A positive measure π on $I \times I$ is a coupling for g with finite α-energy if:

- $\int |t-s|^\alpha \pi(ds,dt) < \infty$
- For all $f \in C^\alpha(I)$, $\quad \int (f(t)-f(s))\pi(ds,dt) = \int f dg$

Notation: $\pi \in \Gamma_\alpha(g)$.

- We seek a coupling $\pi \in \Gamma_\alpha(g)$ minimizing:

$$\int |t-s|^\alpha \pi(ds,dt)$$

$\Rightarrow$ primal characterization of $\|g\|_{W_\alpha}$.

A Primal Problem

Coupling with finite energy

A positive measure π on $I \times I$ is a coupling for g with finite α-energy if:

- $\int |t-s|^\alpha \pi(ds, dt) < \infty$
- For all $f \in C^\alpha(I)$, $\int (f(t) - f(s))\pi(ds, dt) = \int f dg$

Notation: $\pi \in \Gamma_\alpha(g)$.

- We seek a coupling $\pi \in \Gamma_\alpha(g)$ minimizing:

$$\int |t-s|^\alpha \pi(ds, dt)$$

$\Rightarrow$ primal characterization of $\|g\|_{W_\alpha}$.

A Primal Problem

Coupling with finite energy

A positive measure π on $I \times I$ is a coupling for g with finite α-energy if:

- $\int |t-s|^\alpha \pi(ds,dt) < \infty$
- For all $f \in C^\alpha(I)$, $\quad \int (f(t) - f(s))\pi(ds,dt) = \int fdg$

Notation: $\pi \in \Gamma_\alpha(g)$.

- We seek a coupling $\pi \in \Gamma_\alpha(g)$ minimizing:

$$\int |t-s|^\alpha \pi(ds,dt)$$

$\Rightarrow$ primal characterization of $\|g\|_{W_\alpha}$.

A Primal Problem

Coupling with finite energy

A positive measure π on $I \times I$ is a coupling for g with finite α-energy if:

- $\int |t-s|^\alpha \pi(ds, dt) < \infty$
- For all $f \in C^\alpha(I)$, $\quad \int (f(t) - f(s))\pi(ds, dt) = \int f dg$

Notation: $\pi \in \Gamma_\alpha(g)$.

- We seek a coupling $\pi \in \Gamma_\alpha(g)$ minimizing:

$$\int |t-s|^\alpha \pi(ds, dt)$$

$\Rightarrow$ primal characterization of $\|g\|_{\mathsf{W}_\alpha}$.

Duality

As expected we have agreement between the two problems.

Proposition (Kantorovich-Young duality)

Let $I = [a, b] \subseteq \mathbb{R}$, $q > 1$ and $g : I \to \mathbb{R}$ with finite q-variation and $g(a) = g(b)$. For every $\alpha \in (1 - 1/q, 1]$ the supremum

$$\|g\|_{W_\alpha} = \sup_{f:[f]_{C^\alpha} \leq 1} \left\{ \int_I f dg \right\}$$

is attained by some f with $[f]_{C^\alpha} = 1$ and

$$\|g\|_{W_\alpha} = \min_{\pi \in \Gamma_\alpha(g)} \int_{I \times I} |t - s|^\alpha \pi(ds, dt) < \infty.$$

- In particular, the set $\Gamma_\alpha(g)$ is not empty.

Duality

As expected we have agreement between the two problems.

Proposition (Kantorovich-Young duality)

Let $I = [a, b] \subseteq \mathbb{R}$, $q > 1$ and $g : I \to \mathbb{R}$ with finite q-variation and $g(a) = g(b)$. For every $\alpha \in (1 - 1/q, 1]$ the supremum

$$\|g\|_{\mathsf{W}_\alpha} = \sup_{f:[f]_{C^\alpha} \leq 1} \left\{ \int_I f dg \right\}$$

is attained by some f with $[f]_{C^\alpha} = 1$ and

$$\|g\|_{\mathsf{W}_\alpha} = \min_{\pi \in \Gamma_\alpha(g)} \int_{I \times I} |t - s|^\alpha \pi(ds, dt) < \infty.$$

- In particular, the set $\Gamma_\alpha(g)$ is not empty.

Rethinking Young's integral as a coupling

- Assume $I = [0,1]$, $g \in C^\beta(I)$, $g(0) = g(1)$, $f \in C^\alpha(I)$.
- If $\alpha + \beta > 1$, a dyadic summation (sewing lemma) gives

$$\int_0^1 f dg = \sum_{n=1}^{\infty} \sum_{k=0}^{2^{n-1}-1} \left(f\left((2k)2^{-n}\right) - f\left((2k+1)2^{-n}\right)\right) \cdot \\ \cdot \left(g\left((2k+2)2^{-n}\right) - g\left((2k+1)2^{-n}\right)\right),$$

- Define

$$\pi := \sum_{n=1}^{\infty} \sum_{k=0}^{2^{n-1}-1} \left(g\left((2k+2)2^{-n}\right) - g\left((2k+1)2^{-n}\right)\right)^+ \delta_{((2k)2^{-n},(2k+1)2^{-n})} \\ + \left(g\left((2k+2)2^{-n}\right) - g\left((2k+1)2^{-n}\right)\right)^- \delta_{((2k+1)2^{-n},(2k)2^{-n})}.$$

- Then π is a coupling with finite α-energy.

Rethinking Young's integral as a coupling

- Assume $I = [0,1]$, $g \in C^\beta(I)$, $g(0) = g(1)$, $f \in C^\alpha(I)$.
- If $\alpha + \beta > 1$, a dyadic summation (sewing lemma) gives

$$\int_0^1 f dg = \sum_{n=1}^{\infty} \sum_{k=0}^{2^{n-1}-1} \left(f\left((2k)2^{-n}\right) - f\left((2k+1)2^{-n}\right)\right) \cdot \\ \cdot \left(g\left((2k+2)2^{-n}\right) - g\left((2k+1)2^{-n}\right)\right),$$

- Define

$$\pi := \sum_{n=1}^{\infty} \sum_{k=0}^{2^{n-1}-1} \left(g\left((2k+2)2^{-n}\right) - g\left((2k+1)2^{-n}\right)\right)^+ \delta_{((2k)2^{-n},(2k+1)2^{-n})} \\ + \left(g\left((2k+2)2^{-n}\right) - g\left((2k+1)2^{-n}\right)\right)^- \delta_{((2k+1)2^{-n},(2k)2^{-n})}.$$

- Then π is a coupling with finite α-energy.

Rethinking Young's integral as a coupling

- Assume $I = [0, 1]$, $g \in C^\beta(I)$, $g(0) = g(1)$, $f \in C^\alpha(I)$.
- If $\alpha + \beta > 1$, a dyadic summation (sewing lemma) gives

$$\int_0^1 f dg = \sum_{n=1}^{\infty} \sum_{k=0}^{2^{n-1}-1} \left(f\left((2k)2^{-n}\right) - f\left((2k+1)2^{-n}\right)\right) \cdot \\ \cdot \left(g\left((2k+2)2^{-n}\right) - g\left((2k+1)2^{-n}\right)\right),$$

- Define

$$\pi := \sum_{n=1}^{\infty} \sum_{k=0}^{2^{n-1}-1} \left(g\left((2k+2)2^{-n}\right) - g\left((2k+1)2^{-n}\right)\right)^+ \delta_{((2k)2^{-n},(2k+1)2^{-n})} \\ + \left(g\left((2k+2)2^{-n}\right) - g\left((2k+1)2^{-n}\right)\right)^- \delta_{((2k+1)2^{-n},(2k)2^{-n})}.$$

- Then π is a coupling with finite α-energy.

Rethinking Young's integral as a coupling

- Assume $I = [0, 1]$, $g \in C^\beta(I)$, $g(0) = g(1)$, $f \in C^\alpha(I)$.
- If $\alpha + \beta > 1$, a dyadic summation (sewing lemma) gives

$$\int_0^1 f dg = \sum_{n=1}^{\infty} \sum_{k=0}^{2^{n-1}-1} \left(f\left((2k)2^{-n}\right) - f\left((2k+1)2^{-n}\right)\right) \cdot \\ \cdot \left(g\left((2k+2)2^{-n}\right) - g\left((2k+1)2^{-n}\right)\right),$$

- Define

$$\pi := \sum_{n=1}^{\infty} \sum_{k=0}^{2^{n-1}-1} \left(g\left((2k+2)2^{-n}\right) - g\left((2k+1)2^{-n}\right)\right)^+ \delta_{((2k)2^{-n},(2k+1)2^{-n})} \\ + \left(g\left((2k+2)2^{-n}\right) - g\left((2k+1)2^{-n}\right)\right)^- \delta_{((2k+1)2^{-n},(2k)2^{-n})}.$$

- Then π is a coupling with finite α-energy.

Plan

Sketch of proof, case $1/2 < \alpha < 1$

- Given i.i.d. $(X_i)_{i=1}^n$, $(Y_i)_{i=1}^n$, for $t \in \mathbb{R}$, define

$$F_n(t) = \frac{1}{n}\sum_{i=1}^n 1_{\{X_i \le t\}}, \quad \tilde{F}_n(t) = \frac{1}{n}\sum_{i=1}^n 1_{\{Y_i \le t\}}.$$

- By Birkhoff's theorem

$$\mathsf{M}_\alpha((X_i)_{i=1}^n, (Y_i)_{i=1}^n) = \mathsf{W}_\alpha\left(\sum_{i=1}^n \delta_{X_i}, \sum_{i=1}^n \delta_{Y_i}\right) = n\|F_n - \tilde{F}_n\|_{\mathsf{W}_\alpha}.$$

Sketch of proof, case $1/2 < \alpha < 1$

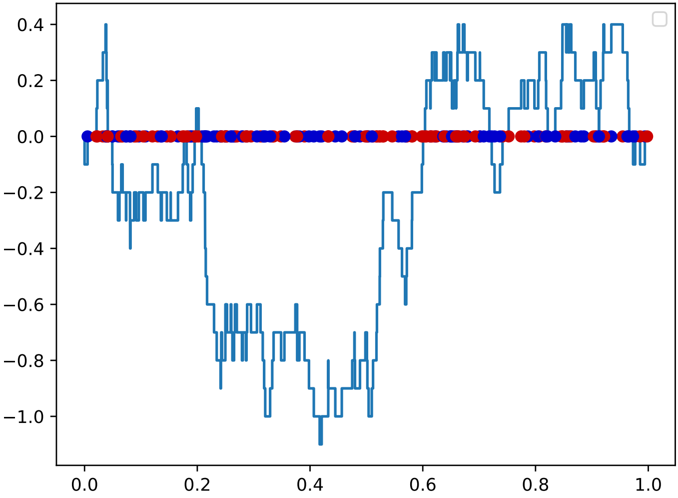
- Given i.i.d. $(X_i)_{i=1}^n$, $(Y_i)_{i=1}^n$, for $t \in \mathbb{R}$, define

$$F_n(t) = \frac{1}{n}\sum_{i=1}^n 1_{\{X_i \le t\}}, \quad \tilde{F}_n(t) = \frac{1}{n}\sum_{i=1}^n 1_{\{Y_i \le t\}}.$$

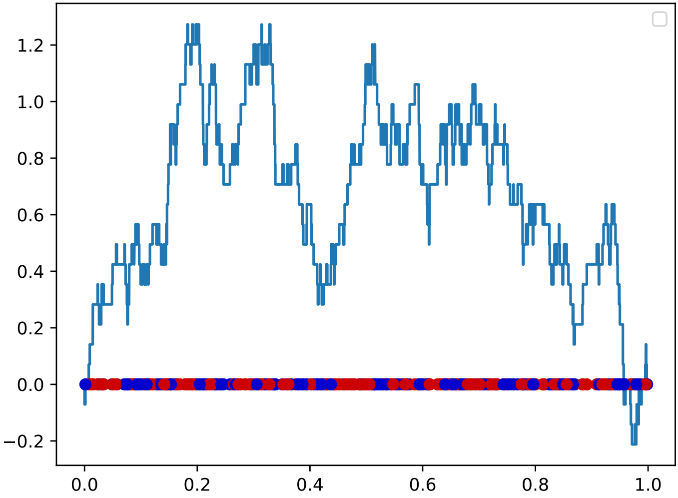
- By Birkhoff's theorem

$$\mathsf{M}_\alpha((X_i)_{i=1}^n, (Y_i)_{i=1}^n) = \mathsf{W}_\alpha\left(\sum_{i=1}^n \delta_{X_i}, \sum_{i=1}^n \delta_{Y_i}\right) = n||F_n - \tilde{F}_n\|_{\mathsf{W}_\alpha}.$$

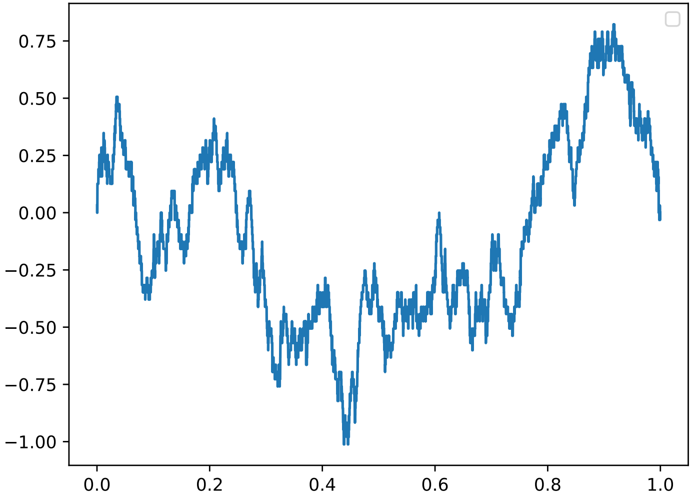
empirical CDF $n = 100$ (rescaled)



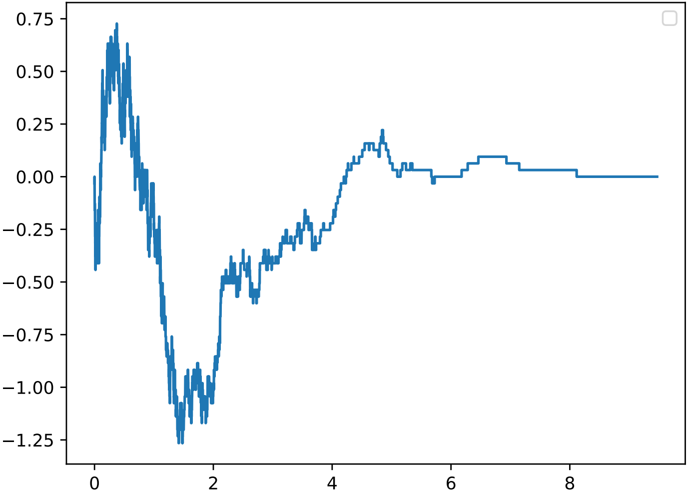
empirical CDF $n = 200$ (rescaled)



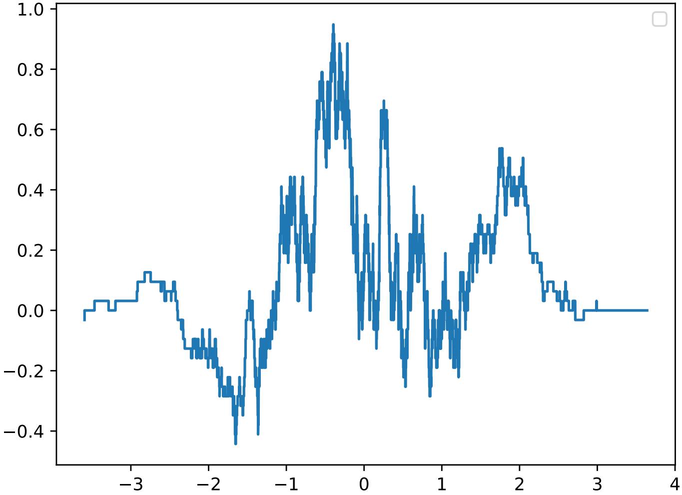
empirical CDF $n = 1000$ (rescaled)



empirical CDF $n = 1000$ exponential density (rescaled)



empirical CDF $n = 1000$ Gaussian density (rescaled)



- By a result [HD01], there exists a Brownian bridge B_n such that for every $p \geq 1$,

$$\mathbb{E}\left[[\sqrt{n}(F_n - \tilde{F}_n) - \sqrt{2} B_n \circ F]_{q-\text{var}}^p \right]^{1/p} \leq C n^{-(q-2)/2q}.$$

- Since B_n is Hölder continuous with exponent less than $1/2$, we have:

$$\mathbb{E}\left[[B_n \circ F]_{q-\text{var}}^p \right] < \infty$$

- By Kantorovich-Young stability with respect to convergence in q-variation:

$$\mathbb{E}[|\|\sqrt{n}(F_n - \tilde{F}_n)\|_{W_\alpha} - \|\sqrt{2} B_n \circ F\|_{W_\alpha}|^p]^{1/p} \leq C n^{-(q-2)/2q}.$$

- By a result [HD01], there exists a Brownian bridge B_n such that for every $p \geq 1$,

$$\mathbb{E}\left[[\sqrt{n}(F_n - \tilde{F}_n) - \sqrt{2} B_n \circ F]^p_{q-\mathrm{var}} \right]^{1/p} \leq C n^{-(q-2)/2q}.$$

- Since B_n is Hölder continuous with exponent less than $1/2$, we have:

$$\mathbb{E}\left[[B_n \circ F]^p_{q-\mathrm{var}} \right] < \infty$$

- By Kantorovich-Young stability with respect to convergence in q-variation:

$$\mathbb{E}[|\|\sqrt{n}(F_n - \tilde{F}_n)\|_{\mathsf{W}_\alpha} - \|\sqrt{2} B_n \circ F\|_{\mathsf{W}_\alpha}|^p]^{1/p} \leq C n^{-(q-2)/2q}.$$

- By a result [HD01], there exists a Brownian bridge B_n such that for every $p \geq 1$,

$$\mathbb{E}\left[[\sqrt{n}(F_n - \tilde{F}_n) - \sqrt{2}B_n \circ F]^p_{q-\mathrm{var}}\right]^{1/p} \leq Cn^{-(q-2)/2q}.$$

- Since B_n is Hölder continuous with exponent less than $1/2$, we have:

$$\mathbb{E}\left[[B_n \circ F]^p_{q-\mathrm{var}}\right] < \infty$$

- By Kantorovich-Young stability with respect to convergence in q-variation:

$$\mathbb{E}[|\|\sqrt{n}(F_n - \tilde{F}_n)\|_{\mathsf{W}_\alpha} - \|\sqrt{2}B_n \circ F\|_{\mathsf{W}_\alpha}|^p]^{1/p} \leq Cn^{-(q-2)/2q}.$$

Plan

Further Questions

1. For $1/2 < \alpha < 1$, we assume bounded support. Extending to unbounded intervals likely requires:
   - A theory of the Kantorovich-Young problem with growth conditions
   - Verifying them in the convergence towards the Brownian bridge
2. The $\alpha = 1/2$ case remains open. It is known [BL20] that:

$$\limsup_{n\to\infty} \mathbb{E}\left[\mathsf{M}_{1/2}((X_i)_{i=1}^n, (Y_i)_{i=1}^n)\right] / \sqrt{n \log n} < \infty,$$

   We prove a lower bound when μ is uniform:

$$\liminf_{n\to\infty} \mathbb{E}\left[\mathsf{M}_{1/2}((X_i)_{i=1}^n, (Y_i)_{i=1}^n)\right] / \sqrt{n \log n} > 0.$$

3. Our method extends to the bipartite Traveling Salesperson Problem. We conjecture it also applies to the bipartite $\kappa-$factor problem [BB13; GT22].

Further Questions

1. For $1/2 < \alpha < 1$, we assume bounded support. Extending to unbounded intervals likely requires:
   - A theory of the Kantorovich-Young problem with growth conditions
   - Verifying them in the convergence towards the Brownian bridge
2. The $\alpha = 1/2$ case remains open. It is known [BL20] that:

$$\limsup_{n\to\infty} \mathbb{E}\left[\mathsf{M}_{1/2}((X_i)_{i=1}^n, (Y_i)_{i=1}^n)\right] / \sqrt{n \log n} < \infty,$$

   We prove a lower bound when μ is uniform:

$$\liminf_{n\to\infty} \mathbb{E}\left[\mathsf{M}_{1/2}((X_i)_{i=1}^n, (Y_i)_{i=1}^n)\right] / \sqrt{n \log n} > 0.$$

3. Our method extends to the bipartite Traveling Salesperson Problem. We conjecture it also applies to the bipartite $\kappa-$factor problem [BB13; GT22].

Further Questions

1. For $1/2 < \alpha < 1$, we assume bounded support. Extending to unbounded intervals likely requires:
   - A theory of the Kantorovich-Young problem with growth conditions
   - Verifying them in the convergence towards the Brownian bridge
2. The $\alpha = 1/2$ case remains open. It is known [BL20] that:

$$\limsup_{n\to\infty} \mathbb{E}\left[\mathsf{M}_{1/2}((X_i)_{i=1}^n, (Y_i)_{i=1}^n)\right] / \sqrt{n \log n} < \infty,$$

   We prove a lower bound when μ is uniform:

$$\liminf_{n\to\infty} \mathbb{E}\left[\mathsf{M}_{1/2}((X_i)_{i=1}^n, (Y_i)_{i=1}^n)\right] / \sqrt{n \log n} > 0.$$

3. Our method extends to the bipartite Traveling Salesperson Problem. We conjecture it also applies to the bipartite $\kappa-$factor problem [BB13; GT22].

Further Questions

1. For $1/2 < \alpha < 1$, we assume bounded support. Extending to unbounded intervals likely requires:
   - A theory of the Kantorovich-Young problem with growth conditions
   - Verifying them in the convergence towards the Brownian bridge
2. The $\alpha = 1/2$ case remains open. It is known [BL20] that:

$$\limsup_{n\to\infty} \mathbb{E}\left[\mathsf{M}_{1/2}((X_i)_{i=1}^n, (Y_i)_{i=1}^n)\right] / \sqrt{n \log n} < \infty,$$

   We prove a lower bound when μ is uniform:

$$\liminf_{n\to\infty} \mathbb{E}\left[\mathsf{M}_{1/2}((X_i)_{i=1}^n, (Y_i)_{i=1}^n)\right] / \sqrt{n \log n} > 0.$$

3. Our method extends to the bipartite Traveling Salesperson Problem. We conjecture it also applies to the bipartite $\kappa-$factor problem [BB13; GT22].

Further Questions

1. For $1/2 < \alpha < 1$, we assume bounded support. Extending to unbounded intervals likely requires:
   - A theory of the Kantorovich-Young problem with growth conditions
   - Verifying them in the convergence towards the Brownian bridge
2. The $\alpha = 1/2$ case remains open. It is known [BL20] that:

$$\limsup_{n\to\infty} \mathbb{E}\left[\mathsf{M}_{1/2}((X_i)_{i=1}^n, (Y_i)_{i=1}^n)\right] / \sqrt{n \log n} < \infty,$$

   We prove a lower bound when μ is uniform:

$$\liminf_{n\to\infty} \mathbb{E}\left[\mathsf{M}_{1/2}((X_i)_{i=1}^n, (Y_i)_{i=1}^n)\right] / \sqrt{n \log n} > 0.$$

3. Our method extends to the bipartite Traveling Salesperson Problem. We conjecture it also applies to the bipartite $\kappa-$factor problem [BB13; GT22].

THANK YOU!!!

Lower Bound for $\alpha = 1/2$

- The Peano curve γ is $1/2$-Hölder continuous and pushes Lebesgue measure on $[0,1]$ to area measure on $[0,1]^2$
- It satisfies:

$$\mathsf{M}_1((\gamma(X_i))_{i=1}^n, (\gamma(Y_i))_{i=1}^n) \leq [\gamma]_{C^{1/2}} \mathsf{M}_{1/2}((X_i)_{i=1}^n, (Y_i)_{i=1}^n),$$

- It is known (AKT) that for i.i.d. uniform points on the square, we have:

$$\liminf_{n\to\infty} \mathbb{E}\left[\mathsf{M}_1((\tilde{X}_i))_{i=1}^n, (\tilde{Y}_i))_{i=1}^n)\right] / \sqrt{n \log n} > 0,$$

- Combining the above, we conclude:

$$\liminf_{n\to\infty} \frac{\mathbb{E}[\mathsf{M}_{1/2}((X_i)_{i=1}^n, (Y_i)_{i=1}^n)]}{\sqrt{n \log n}} > 0.$$

Lower Bound for $\alpha = 1/2$

- The Peano curve γ is $1/2$-Hölder continuous and pushes Lebesgue measure on $[0, 1]$ to area measure on $[0, 1]^2$
- It satisfies:

$$\mathsf{M}_1((\gamma(X_i))_{i=1}^n, (\gamma(Y_i))_{i=1}^n) \leq [\gamma]_{C^{1/2}} \mathsf{M}_{1/2}((X_i)_{i=1}^n, (Y_i)_{i=1}^n),$$

- It is known (AKT) that for i.i.d. uniform points on the square, we have:

$$\liminf_{n\to\infty} \mathbb{E}\left[\mathsf{M}_1((\tilde{X}_i))_{i=1}^n, (\tilde{Y}_i))_{i=1}^n)\right] / \sqrt{n \log n} > 0,$$

- Combining the above, we conclude:

$$\liminf_{n\to\infty} \frac{\mathbb{E}[\mathsf{M}_{1/2}((X_i)_{i=1}^n, (Y_i)_{i=1}^n)]}{\sqrt{n \log n}} > 0.$$

Lower Bound for $\alpha = 1/2$

- The Peano curve γ is $1/2$-Hölder continuous and pushes Lebesgue measure on $[0,1]$ to area measure on $[0,1]^2$
- It satisfies:

$$\mathsf{M}_1((\gamma(X_i))_{i=1}^n, (\gamma(Y_i))_{i=1}^n) \leq [\gamma]_{C^{1/2}} \mathsf{M}_{1/2}((X_i)_{i=1}^n, (Y_i)_{i=1}^n),$$

- It is known (AKT) that for i.i.d. uniform points on the square, we have:

$$\liminf_{n\to\infty} \mathbb{E}\left[\mathsf{M}_1((\tilde{X}_i))_{i=1}^n, (\tilde{Y}_i))_{i=1}^n)\right] / \sqrt{n \log n} > 0,$$

- Combining the above, we conclude:

$$\liminf_{n\to\infty} \frac{\mathbb{E}[\mathsf{M}_{1/2}((X_i)_{i=1}^n, (Y_i)_{i=1}^n)]}{\sqrt{n \log n}} > 0.$$

Lower Bound for $\alpha = 1/2$

- The Peano curve γ is $1/2$-Hölder continuous and pushes Lebesgue measure on $[0,1]$ to area measure on $[0,1]^2$
- It satisfies:

$$\mathsf{M}_1((\gamma(X_i))_{i=1}^n, (\gamma(Y_i))_{i=1}^n) \leq [\gamma]_{C^{1/2}} \mathsf{M}_{1/2}((X_i)_{i=1}^n, (Y_i)_{i=1}^n),$$

- It is known (AKT) that for i.i.d. uniform points on the square, we have:

$$\liminf_{n\to\infty} \mathbb{E}\left[\mathsf{M}_1((\tilde{X}_i))_{i=1}^n, (\tilde{Y}_i))_{i=1}^n)\right] / \sqrt{n \log n} > 0,$$

- Combining the above, we conclude:

$$\liminf_{n\to\infty} \frac{\mathbb{E}[\mathsf{M}_{1/2}((X_i)_{i=1}^n, (Y_i)_{i=1}^n)]}{\sqrt{n \log n}} > 0.$$

Plan

Plan

[AKT84] M. Ajtai, J. Komlós, and G. Tusnády. "On optimal matchings.". In: *Combinatorica* 4 (1984), pp. 259–264. ISSN: 0209-9683; 1439-6912/e. DOI: 10.1007/BF02579135.

[AST19] L. Ambrosio, F. Stra, and D. Trevisan. "A PDE approach to a 2-dimensional matching problem". In: *Probab. Theory Relat. Fields* 173.1-2 (2019), pp. 433–477.

[BB13] F. Barthe and C. Bordenave. "Combinatorial optimization over two random point sets.". In: *Séminaire de probabilités XLV*. Cham: Springer, 2013, pp. 483–535. ISBN: 978-3-319-00320-7/pbk; 978-3-319-00321-4/ebook. DOI: 10.1007/978-3-319-00321-4_19.

[BHH59] Jillian Beardwood, John H Halton, and John Michael Hammersley. "The shortest path through many points". In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 55. 4. Cambridge University Press. 1959, pp. 299–327.

[BL20] Sergey G Bobkov and Michel Ledoux. "Transport inequalities on Euclidean spaces for non-Euclidean metrics". In: *Journal of Fourier Analysis and Applications* 26.4 (2020), p. 60.

[Car+14] S. Caracciolo et al. "Scaling hypothesis for the Euclidean bipartite matching problem". In: *Physical Review E* 90.1 (2014).

[FV10] Peter K Friz and Nicolas B Victoir. *Multidimensional stochastic processes as rough paths: theory and applications*. Vol. 120. Cambridge University Press, 2010.

[GT22] Michael Goldman and Dario Trevisan. “Optimal transport methods for combinatorial optimization over two random point sets”. In: *arXiv preprint arXiv:2209.14615* (2022).

[HD01] Yen-Chin Huang and RM Dudley. “Speed of convergence of classical empirical processes in p-variation norm”. In: *Annals of probability* (2001), pp. 1625–1636.

[McC99] Robert J McCann. “Exact solutions to the transportation problem on the line”. In: *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences* 455.1984 (1999), pp. 1341–1380.

[Ste97] J Michael Steele. *Probability theory and combinatorial optimization*. SIAM, 1997.

[Yuk06] Joseph E Yukich. *Probability theory of classical Euclidean optimization problems*. Springer, 2006.