# Some explicit constructions of skew braces

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# Overview on skew braces

# Skew braces

Definition ([Rump, 2007, Guarnieri and Vendramin, 2017])

A *skew brace* is a triple $(G, \cdot, \circ)$, where $(G, \cdot)$, $(G, \circ)$ are groups and

$$g \circ (h \cdot k) = (g \circ h) \cdot g^{-1} \cdot (g \circ k).$$

Here $g^{-1}$ denotes the inverse of $g$ in $(G, \cdot)$.

Example

- If $(G, \cdot)$ is a group, then $(G, \cdot, \cdot)$ is a trivial skew brace.
- $(\mathbb{Z}, +, \circ)$ is a skew brace, where $a \circ b = a + (-1)^a b$.

# Connections

Skew braces are connected with

- radical rings;
- set-theoretic solutions of the Yang–Baxter equation;
- regular subgroups of holomorphs of groups;
- Hopf–Galois structures.

# Regular subgroups of the holomorph

Let $(G,\cdot)$ be a group.

Definition

The *holomorph* of $(G,\cdot)$ is the semidirect product

$$\mathrm{Hol}(G,\cdot) = (G,\cdot) \rtimes \mathrm{Aut}(G,\cdot).$$

A subgroup $A$ of $\mathrm{Hol}(G,\cdot)$ is *regular* if for all $g \in G$ there exists a unique $(x,\alpha) \in A$ such that $x \cdot \alpha(g) = 1$.

Theorem ([Guarnieri and Vendramin, 2017])

*The following data are equivalent:*

- *a regular subgroup of* $\mathrm{Hol}(G,\cdot)$*;*
- *an operation* $\circ$ *such that* $(G,\cdot,\circ)$ *is a skew brace.*

# Hopf–Galois structures

Let $L/K$ be a finite Galois extension with Galois group $(G, \cdot)$.

**Definition**

A Hopf–Galois structure on $L/K$ consists of a $K$-Hopf algebra $H$ acting in a suitable way on $L$.

**Theorem ([Greither and Pareigis, 1987, Bachiller, 2016, Smoktunowicz and Vendramin, 2018, LS and Trappeniers, 2022])**

*The following data are equivalent:*

- *a Hopf–Galois structure on* $L/K$*;*
- *an operation* $\circ$ *such that* $(G, \circ, \cdot)$ *is a skew brace.*

# Bi-skew braces

**Definition ([Childs, 2019])**

A *bi-skew brace* is a skew brace $(G, \cdot, \circ)$ such that also $(G, \circ, \cdot)$ is a skew brace.

**Question**

*Let* $(G, \cdot)$ *be a group. Can we construct explicit examples of bi-skew braces* $(G, \cdot, \circ)$*?*

# Explicit constructions of bi-skew braces

# Koch's construction

Let $(G, \cdot)$ be a group, and let $\psi$ be an *abelian map*, that is, $\psi \in \mathrm{End}(G, \cdot)$ such that $\psi(G)$ is abelian. Define

$$g \circ h = g \cdot \psi(g) \cdot h \cdot \psi(g)^{-1}.$$

**Theorem ([Koch, 2021])**

*$(G, \cdot, \circ)$ is a bi-skew brace.*

# A first generalisation

Let $(G, \cdot)$ be a group with centre $Z$, and let $\psi \in \mathrm{End}(G, \cdot)$. Define

$$g \circ h = g \cdot \psi(g) \cdot h \cdot \psi(g)^{-1}.$$

**Theorem ([Caranti and LS, 2021])**

*The following are equivalent:*

- $(G, \cdot, \circ)$ *is a bi-skew brace.*
- $[\psi(G), \psi(G)] \subseteq Z$.

**Example**

Suppose that $(G, \cdot)$ has nilpotency class two (that is, $[G, G] \subseteq Z$), and let $\psi \in \mathrm{End}(G, \cdot)$. Then $(G, \cdot, \circ)$ is a bi-skew brace, where

$$g \circ h = g \cdot \psi(g) \cdot h \cdot \psi(g)^{-1}.$$

# A final generalisation

Let $(G, \cdot)$ be a group with centre $Z$, and let $\psi\colon G \to G/Z$ be a group homomorphism with abelian image. Define

$$g \circ h = g \cdot \psi^{\uparrow}(g) \cdot h \cdot (\psi^{\uparrow}(g))^{-1},$$

where $\psi^{\uparrow}\colon G \to G$ is any lifting of $\psi$.

Theorem ([Caranti and LS, 2022, LS and Trappeniers, 2023])

- *The operation $\circ$ is well-defined.*
- $(G, \cdot, \circ)$ *is a bi-skew brace.*

# Examples in class two

Let $(G, \cdot)$ be a group of nilpotency class two with centre $Z$.

Here $G/Z$ is abelian, so every group homomorphism $G \to G/Z$ can be used for the construction. For example, for all $n \in \mathbb{Z}$ the map

$$\psi \colon G \to G/Z, \quad g \mapsto g^n Z$$

is a group homomorphism, so $(G, \cdot, \circ)$ is a bi-skew brace, where

$$g \circ h = g \cdot g^n \cdot h \cdot g^{-n} = g \cdot h \cdot [g, h]^n.$$

As application, in [Caranti and LS, 2023] we constructed skew braces that do not come from Rota–Baxter operators.

# Example with the norm

Let $(G, \cdot)$ be a group with centre $Z$, and let $N$ be the *norm* of $(G, \cdot)$, that is, the intersection of the normalisers of the subgroups of $G$.

## Theorem ([Schenkman, 1960])

*The quotient $N/Z$ is abelian.*

In particular, every group homomorphism $\psi\colon G \to N/Z$ (with lifting $\psi^{\uparrow}$) yields a bi-skew brace $(G, \cdot, \circ)$, where

$$g \circ h = g \cdot \psi^{\uparrow}(g) \cdot h \cdot (\psi^{\uparrow}(g))^{-1}.$$

## Fact ([LS and Trappeniers, 2022])

*This construction allows us to obtain Hopf–Galois structures with a desirable but apparently rare property: the Hopf–Galois correspondence is bijective.*

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