Skew braces and Rota–Baxter operators

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Skew braces

Definition ([Guarnieri and Vendramin, 2017])

A *skew brace* is a triple $(A, +, \circ)$, where $(A, +)$ and $(A, \circ)$ are groups such that for all $a, b, c \in A$,

$$a \circ (b + c) = (a \circ b) - a + (a \circ c).$$

Skew braces are related with

- radical rings;
- set-theoretic solutions of the Yang–Baxter equation;
- regular subgroups of holomorphs of groups;
- Hopf–Galois structures;
- ...

Lambda map

A skew brace $(A, +, \circ)$ is associated with a function

$$\lambda\colon (A, \circ) \to \mathrm{Aut}(A, +), \quad a \mapsto (b \mapsto -a + (a \circ b)).$$

In particular, for all $a, b \in A$,

$$a \circ b = a + \lambda_a(b).$$

Definition

A skew brace $(A, +, \circ)$ is *inner* if $\lambda_a \in \mathrm{Inn}(A, +)$ for all $a \in A$.

This means that λ_a is equal to conjugation-by-$\psi(a)$ for some $\psi\colon A \to A$, and therefore we can write

$$a \circ b = a + \psi(a) + b - \psi(a).$$

Examples

Let $(A, +)$ be a group. Here are some examples of inner skew braces $(A, +, \circ)$, where we give explicitly the map $\psi \colon A \to A$ such that $a \circ b = a + \psi(a) + b - \psi(a)$.

- The trivial skew brace $(A, +, +)$; here, $\psi(a) = 0$.
- Suppose that $(A, +) = B + C$ is an exact factorisation, and define

  $$a \circ a' = (b + c) \circ a' = b + a' + c.$$

  Then $(A, +, \circ)$ is an inner skew brace, with $\psi(b + c) = -c$.
- Take $\psi \colon (A, +) \to (A, +)$ is a homomorphism with abelian image, and define

  $$a \circ b = a + \psi(a) + b - \psi(a).$$

  Then $(A, +, \circ)$ is an inner skew brace; [Koch, 2021]

Let $(A, +)$ be a group.

Definition ([Guo et al., 2021])

A *Rota–Baxter* operator on $(A, +)$ is a map $\beta \colon A \to A$ such that

$$\beta(a + \beta(a) + b - \beta(a)) = \beta(a) + \beta(b).$$

Idea: smooth Rota–Baxter operators on Lie groups $\rightsquigarrow$ Rota–Baxter operators on Lie algebras.

Proposition ([Bardakov and Gubarev, 2022])

Let β be a Rota–Baxter operator on $(A, +)$, and define

$$a \circ b = a + \beta(a) + b - \beta(a).$$

Then $(A, +, \circ)$ is an inner skew brace.

A characterisation

Let $(A, +)$ be a group, and consider a map $\psi \colon A \to A$. Define

$$a \circ_\psi b = a + \psi(a) + b - \psi(a).$$

Proposition ([Caranti and LS, 2023])

The following are equivalent:

- $(A, +, \circ_\psi)$ *is a skew brace.*
- ψ *satisfies*

$$\psi(a + \psi(a) + b - \psi(a)) \equiv \psi(a) + \psi(b) \pmod{Z(A, +)}.$$

Examples of Rota–Baxter operators

Let $(A, +)$ be a group. Some examples of inner skew braces are:

- The trivial skew brace $(A, +, +)$; here, $\psi(a) = 0$.
- Suppose that $(A, +) = B + C$ is an exact factorisation, and define
  $$a \circ a' = (b + c) \circ a' = b + a' + c.$$
  Then $(A, +, \circ)$ is an inner skew brace, with $\psi(b + c) = -c$.
- Take $\psi \colon (A, +) \to (A, +)$ is a homomorphism with abelian image; then by [Koch, 2021] $(A, +, \circ)$ is an inner skew brace, where
  $$a \circ b = a + \psi(a) + b - \psi(a).$$

All of these maps ψ are examples of Rota–Baxter operators!

A natural question

Definition

An inner skew brace $(A, +, \circ)$ *comes* from a Rota–Baxter operator if there exists a Rota–Baxter operator β on $(A, +)$ such that

$$a \circ b = a + \beta(a) + b - \beta(a).$$

Question

Do all inner skew braces $(A, +, \circ)$ come from Rota–Baxter operators?

The approach

Let $(A, +, \circ)$ be an inner skew brace. Then

$$a \circ b = a + \psi(a) + b - \psi(a),$$

where $\psi\colon A \to A$ satisfies

$$\kappa(a, b) + \psi(a \circ b) = \psi(a) + \psi(b)$$

for a suitable $\kappa\colon A \times A \to Z(A, +)$.

Theorem ([Caranti and LS, 2023])

- *κ is a 2-cocycle for the trivial $(A, \circ)$-module $Z(A, +)$, whose cohomology class in $\mathbf{H}^2((A, \circ), Z(A, +))$ does not depend on the choice of ψ.*
- *$(A, +, \circ)$ comes from a Rota–Baxter operator if and only if the cohomology class of κ is trivial.*

An example

Let p be an odd prime, and let $(A, +)$ be the Heisenberg group of order p^3. For all $n \in \{0, \dots, p-1\}$, define

$$a \circ_n b = a + na + b - na.$$

Then $(A, +, \circ_n)$ is an inner skew brace [Caranti and LS, 2022].

Proposition ([Caranti and LS, 2023])

- *If* $n \neq (p-2)^{-1}$*, then* $(A, +, \circ_n)$ *comes from a Rota–Baxter operator, which can be computed explicitly.*
- *If* $n = (p-2)^{-1}$*, then* $(A, +, \circ_n)$ *does not come from a Rota–Baxter operator.*

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