Describing Hopf–Galois structures via skew braces

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The classical structure

Let L/K be a finite Galois extension with Galois group G.

The group algebra

$$K[G] = \left\{ \sum_{\sigma \in G} k_\sigma \sigma \mid k_\sigma \in K \right\}$$

acts naturally on L:

$$\left(\sum_{\sigma \in G} k_\sigma \sigma \right) \cdot x = \sum_{\sigma \in G} k_\sigma \sigma(x).$$

Fact

The group algebra $K[G]$ is a K-Hopf algebra.

Hopf–Galois structures

Definition ([Chase and Sweedler, 1969])

A *Hopf–Galois structure* $(H, \star)$ on L/K consists of

- a K-Hopf algebra H;
- an action $\star$ of H on L that "mimics" the action $\cdot$ of $K[G]$.

We may have more Hopf–Galois structures on L/K, other than the *classical structure* $(K[G], \cdot)$.

Motivation

In [Byott, 2002], it is shown that in some cases, successful results in Galois module theory may be found only employing Hopf–Galois structures different from the classical one:

Known descriptions

Problem

Find an effective description of Hopf–Galois structures.

- Group theoretic description in [Greither and Pareigis, 1987].
- Connection with skew braces in the appendix of Byott and Vendramin in [Smoktunowicz and Vendramin, 2018].

Both give very few results in the study of the Hopf–Galois correspondence.

Skew braces

Definition ([Guarnieri and Vendramin, 2017])

A skew brace is a triple $(A, +, \circ)$, where $(A, +), (A, \circ)$ are groups, and

$$a \circ (b + c) = (a \circ b) - a + (a \circ c).$$

- Given a group $(A, \circ)$, $(A, \circ, \circ)$ is a *trivial* skew brace.
- Given a skew brace $(A, +, \circ)$, $(A, \circ)$ acts on $(A, +)$:

$$\lambda \colon (A, \circ) \to \mathrm{Aut}(A, +), \quad a \mapsto \lambda_a \colon b \to -a + (a \circ b).$$

- The *left ideals* of $(A, +, \circ)$ are the subgroups B of $(A, +)$ and $(A, \circ)$ such that $\lambda_a(B) \subseteq B$ for all $a \in A$.

A new version of the connection

Let L/K be a finite Galois extension with Galois group $(G, \circ)$.

Theorem ([LS and Trappeniers, 2023])

There exists a bijection between

- *Hopf–Galois structures on L/K;*
- *operations $+$ such that $(G, +, \circ)$ is a skew brace.*

Explicitly, $(G, +, \circ) \leftrightarrow H = L[G, +]^{(G,\circ)}$, where $(G, \circ)$ acts on L via Galois action and on $(G, +)$ via the map λ of $(G, +, \circ)$.

Moreover, $L[G, +]^{(G,\circ)}$ acts on L as follows:

$$\left(\sum_{g \in G} \ell_g g\right) \star x = \sum_{g \in G} \ell_g g(x).$$

Example

The classical structure is associated with the trivial skew brace.

An example of skew brace...

Let L/K be a Galois extension with Galois group $(G, \circ) \cong C_{2n}$, where $n \geq 3$ is odd, written as

$$G = \{\sigma^i \tau^j \mid i = 0, \ldots, n-1 \text{ and } j = 0, 1\}.$$

Define

$$\sigma^i \tau^j + \sigma^a \tau^b = \sigma^{i+(-1)^j a} \tau^{j+b}.$$

Then $(G, +, \circ)$ is a skew brace, with $(G, +)$ dihedral of order $2n$.

It is easy to check that

$$\lambda_{\sigma\tau} \colon G \to G, \quad g \to g',$$

where g' denotes the inverse of g with respect to $(G, \circ)$.

... and its associated Hopf–Galois structure

Therefore $\sum_{g\in G} \ell_g g \in L[G,+]$ is in $H = L[G,+]^{(G,\circ)}$ if and only if

$$\sum_{g\in G} \ell_g g = \sum_{g\in G} \sigma\tau(\ell_g) g',$$

that is,

$$H = \left\{ \sum_{g\in G} \ell_g g \mid \sigma\tau(\ell_g) = \ell_{g'} \text{ for all } g \in G \right\} \subseteq L[G,+].$$

Moreover, H acts on L as follows:

$$\left(\sum_{g\in G} \ell_g g \right) \star x = \sum_{g\in G} \ell_g g(x).$$

The Hopf–Galois correspondence

Let L/K be a finite Galois extension with Galois group $(G, \circ)$. Given a Hopf–Galois structure $(H, \star)$ on L/K, there exists a map

$$\{K\text{-Hopf subalgebras of } H\} \to \{\text{intermediate fields of } L/K\}$$
$$J \mapsto L^J \text{ (fixed field).}$$

This map is called the *Hopf–Galois correspondence*; it is always injective, but not necessarily surjective.

Other then the classical structure, there were two classes of known examples in which this map is surjective, found in [Greither and Pareigis, 1987] and [Childs, 2017].

The Hopf–Galois correspondence via skew braces

Suppose that $(H, \star)$ is associated with the skew brace $(G, +, \circ)$.

Proposition ([LS and Trappeniers, 2023])

There exists a bijection

$$\{K\text{-Hopf subalgebras of } H\} \leftrightarrow \{\text{left ideals of } (G, +, \circ)\}.$$

Corollary ([LS and Trappeniers, 2023])

The following are equivalent:

- *The Hopf–Galois correspondence is surjective.*
- *Every subgroup of $(G, \circ)$ is a left ideal of $(G, +, \circ)$.*

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