Skew braces and the Hopf–Galois correspondence

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Hopf–Galois structures

Let L/K be a finite Galois extension with Galois group G.

Definition ([Chase and Sweedler, 1969])

A Hopf–Galois structure $(H, \star)$ on L/K consists of

- a K-Hopf algebra H;
- an action $\star$ of H on L that satisfies certain nice properties.

Example

The *classical structure* consists of the group algebra $K[G]$ with the usual Galois action.

The Hopf–Galois correspondence

Consider a Hopf–Galois structure on $(H, \star)$ on L/K.

There exists a map

$$\begin{aligned} \{K\text{-Hopf subalgebras of } H\} &\to \{\text{intermediate fields of } L/K\} \\ S &\mapsto L^S \text{ (fixed field).} \end{aligned}$$

This map is called the *Hopf–Galois correspondence* (HGC).

It is always injective [Chase and Sweedler, 1969].

Question

When (and why) is the HGC surjective?

Surjective Hopf–Galois correspondence

- When we consider the classical structure, we recover the usual Galois correspondence, which is surjective.
- If G is Hamiltonian, then there exists a Hopf–Galois structure on L/K for which the HGC is surjective [Greither and Pareigis, 1987].
- If G is cyclic of odd prime power order, then L/K satisfies *Childs's property*: the HGC is surjective for all Hopf–Galois structures [Childs, 2017].

Skew braces

Definition ([Guarnieri and Vendramin, 2017])

A skew brace is a triple $(A, +, \circ)$, where $(A, +), (A, \circ)$ are groups, and

$$a \circ (b + c) = (a \circ b) - a + (a \circ c).$$

- Given a group $(A, \circ)$, $(A, \circ, \circ)$ is a *trivial* skew brace.
- Given a skew brace $(A, +, \circ)$, $(A, \circ)$ acts on $(A, +)$:

$$\lambda \colon (A, \circ) \to \mathrm{Aut}(A, +), \quad a \mapsto \lambda_a \colon b \to -a + (a \circ b).$$

- The *left ideals* of $(A, +, \circ)$ are the subgroups B of $(A, \circ)$ such that $\lambda_a(B) \subseteq B$ for all $a \in A$.

A new version of the connection

Let L/K be a finite Galois extension with Galois group $(G,\circ)$.

Theorem ([LS and Trappeniers, 2023])

There exists a bijection between

- *Hopf–Galois structures on L/K;*
- *operations $+$ such that $(G,+,\circ)$ is a skew brace.*

Explicitly, $(G,+,\circ) \leftrightarrow L[G,+]^{(G,\circ)}$, where $(G,\circ)$ acts on L via Galois action and on $(G,+)$ via the map λ of $(G,+,\circ)$.

Moreover, $L[G,+]^{(G,\circ)}$ acts on L as follows:

$$\left(\sum_{g\in G} \ell_g g\right) \star x = \sum_{g\in G} \ell_g g(x).$$

Example

The classical structure is associated with the trivial skew brace.

The Hopf–Galois correspondence via skew braces

Consider a Hopf–Galois structure $(H, \star)$ on L/K with associated skew brace $(G, +, \circ)$.

Proposition ([LS and Trappeniers, 2023])

There exists a bijection

$$\{\textit{left ideals of } (G, +, \circ)\} \xrightarrow{\cong} \{K\textit{-Hopf subalgebras of } H\}.$$

Moreover,

$$G_1 \longmapsto H_1 \longmapsto L^{H_1} = L^{G_1} \text{ (usual Galois theory).}$$

Corollary

The HGC is surjective if and only if every subgroup of $(G, \circ)$ is a left ideal of $(G, +, \circ)$.

An example of skew brace...

Let L/K be a Galois extension with cyclic Galois group $(G, \circ)$ of order $2n$, where $n \geq 3$ is odd, written as

$$G = \{\sigma^i \tau^j \mid i = 0, \ldots, n-1 \text{ and } j = 0, 1\}.$$

Define

$$\sigma^i \tau^j + \sigma^a \tau^b = \sigma^{i+(-1)^j a} \tau^{j+b}.$$

Then $(G, +, \circ)$ is a skew brace, with $(G, +)$ dihedral of order $2n$.

It is easy to check that

$$\lambda_{\sigma\tau} \colon G \to G, \quad g \to g',$$

where g' denotes the inverse of g with respect to $(G, \circ)$.

... and its associated Hopf–Galois structure

Therefore $\sum_{g \in G} \ell_g g \in L[G,+]$ is in $H = L[G,+]^{(G,\circ)}$ if and only if

$$\sum_{g \in G} \ell_g g = \sum_{g \in G} \sigma\tau(\ell_g) g',$$

that is,

$$H = \left\{ \sum_{g \in G} \ell_g g \mid \sigma\tau(\ell_g) = \ell_{g'} \text{ for all } g \in G \right\} \subseteq L[G,+].$$

Moreover, H acts on L as follows:

$$\left(\sum_{g \in G} \ell_g g \right) \star x = \sum_{g \in G} \ell_g g(x).$$

As every subgroup of $(G,\circ)$ is a left ideal, the HGC is surjective.

Some other consequences

- We can classify all the extensions with Childs's property.

  For example, if the Galois group is a cyclic 2-group, then the HGC is surjective for all Hopf–Galois structures.
- We can construct explicitly Hopf–Galois structures such that the HGC is surjective.

  For example, we can obtain in this way on quaternion extensions 16 Hopf–Galois structures for which the HGC is surjective.

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