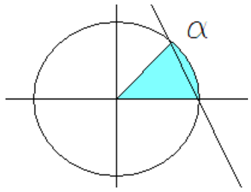
Introduzione alla Matematica

Prova scritta del 26.10.2007 _ Soluzioni [1]

1.

(a) $2 \cos x + \operatorname{sen} x - 2 \geq 0$

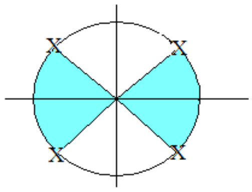


$$\begin{cases} 2X + Y - 2 \geq 0 \\ X^2 + Y^2 = 1 \end{cases}$$

Punti di intersezione $X = 1 , Y = 0$; $X = 3/5 , Y = 4/5$

$\alpha = \operatorname{arcsen} 4/5 = \arccos 3/5 = \operatorname{arctg} 4/3 > \pi/4$

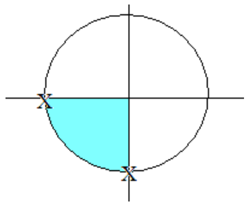
$\cos 2x > 0$



$-\pi/2 + 2k\pi < 2x < \pi/2 + 2k\pi \leftrightarrow$

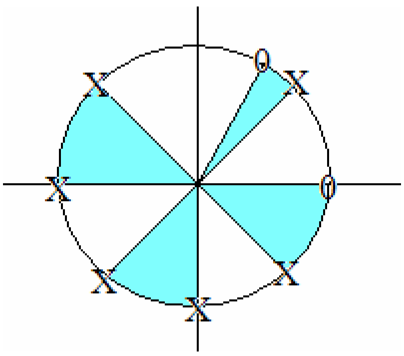
$-\pi/4 + k\pi < x < \pi/4 + k\pi$

$\operatorname{tg}(x/2 - \pi/4) > 1$



$\pi/4 + k\pi < x/2 - \pi/4 < \pi/2 + k\pi \leftrightarrow$

$\pi + 2k\pi < x < 3\pi/2 + 2k\pi$



$\pi/4 < x \leq \alpha$, $3\pi/4 < x < \pi$

$5\pi/4 < x < 3\pi/2$, $7\pi/4 < x \leq 2\pi$

$+ 2k\pi$

(b) $x < -1$ oppure $\begin{cases} x \geq -1 \\ |x^2 - x| > (x+1)^2 \end{cases}$

$x < -1$ oppure $\begin{cases} -1 \leq x \leq 0 \text{ oppure } x \geq 1 \\ x^2 - x > x^2 + 2x + 1 \end{cases}$ oppure $\begin{cases} 0 < x < 1 \\ x - x^2 > x^2 + 2x + 1 \end{cases}$

$x < -1$ oppure $-1 \leq x \;\; -1/3 \leftrightarrow x < -1/3$.

2.

(a)

$F(x) = (f \circ g \circ h)(x)$

$h(x) = \log x$, $g(x) = (1 + x)/(1 - x)$, $f(x) = \text{arcsen}\, x$.

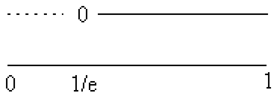
(b) C.E.

$$\begin{cases} x > 0 \ , \ x \neq e \\ \left| (1 + \log x)/(1 - \log x) \right| \leq 1 \end{cases} \Leftrightarrow$$

$$\begin{cases} x > 0 \ , \ x \neq e \\ 1 + \log^2 x + 2 \log x \leq 1 + \log^2 x - 2 \log x \end{cases} \Leftrightarrow \quad x \in (0, 1]$$

SGN

$$\begin{cases} x \in (0, 1] \\ (1 + \log x)/(1 - \log x) \geq 0 \end{cases} \Leftrightarrow \quad x \in [1/e, 1]$$



(c) $\text{arcsen} \dfrac{1 + \log x}{1 - \log x} = k \Rightarrow \dfrac{1 + \log x}{1 - \log x} = \text{sen}\, k$, purché $-\pi/2 \leq x \leq \pi/2$

$1 + \log x = \text{sen}\, k - \text{sen}\, k \ \log x \Rightarrow$

$\log x = (\text{sen}\, k - 1)/(\text{sen}\, k + 1)$, purché $-\pi/2 < x \leq \pi/2 \quad \Rightarrow$

$x = \exp(\ (\text{sen}\, k - 1)/(\text{sen}\, k + 1)\ )$ è soluzione accettabile se $0 < x \leq 1$ e questo accade senza ulteriori condizioni da imporre su k.

Quindi l'immagine della funzione è $(-\pi/2, \pi/2]$, la funzione è invertibile , la funzione inversa è data da $f^{-1}(k) = \exp(\ (\text{sen}\, k - 1)/(\text{sen}\, k + 1)\ )$.

(d) Proviamo che F (x) è crescente : $0 < x < y \leq 1 \quad \Rightarrow \quad F(x) < F(y)$

$F(x) < F(y) \Leftrightarrow$

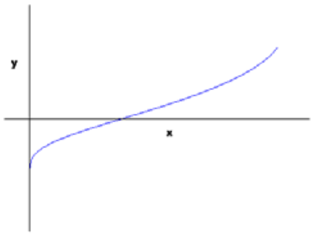
$$\frac{1 + \log x}{1 - \log x} < \frac{1 + \log y}{1 - \log y} \Leftrightarrow (1 + \log x)(1 - \log y) < (1 - \log x)(1 + \log y)$$

(il passaggio è lecito perché i denominatori per cui abbiamo moltiplicato sono positivi)

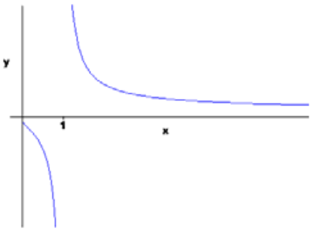
Portando avanti i calcoli, si arriva a $\log x < \log y$, che è vera essendo $x < y$.

(e) La funzione F è tale che $F : (0, 1] \rightarrow (-\pi/2, \pi/2]$ ed è crescente .

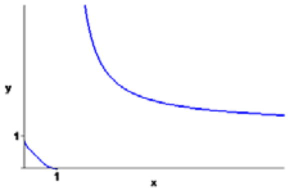
Grafico :



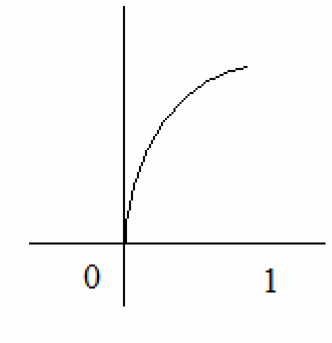
3.



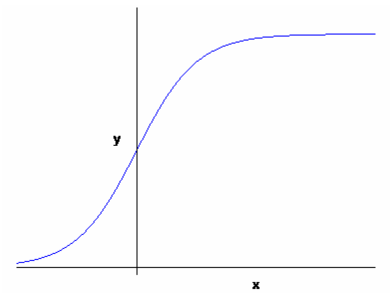
y = 1 / log x



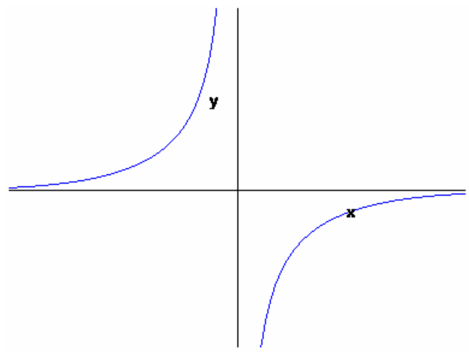
y = exp (1 / log x)



y = arccos (exp (1 / log x))

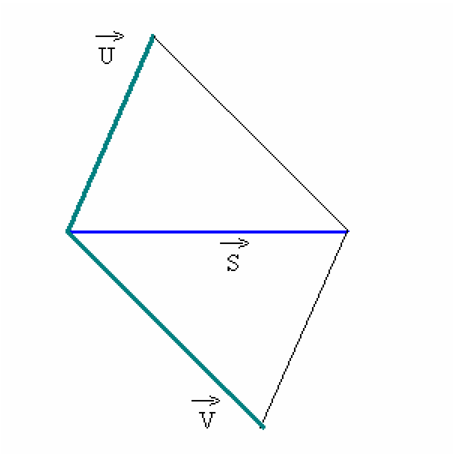


$$y = \frac{\pi}{1 + e^{-x}}$$



$$y = \operatorname{tg} \frac{\pi}{1 + e^{-x}}$$

4.



$$\frac{S}{\text{sen } 75^0} = \frac{U}{\text{sen } 45^0} = \frac{V}{\text{sen } 60^0}$$

U = 7,319 V = 8,965

5.

- per n = 1 è vera : ¼ = ¼

- supponiamola vera per n e deduciamola vera per n + 1 , proviamo cioè che

$$\sum_{k=1}^{n} \frac{3 - 2^k}{4^k} = \frac{2^n - 1}{4^n} \quad \Rightarrow \quad \sum_{k=1}^{n+1} \frac{3 - 2^k}{4^k} = \frac{2^{n+1} - 1}{4^{n+1}} .$$

$$\sum_{k=1}^{n+1} \frac{3 - 2^k}{4^k} = \sum_{k=1}^{n} \frac{3 - 2^k}{4^k} + \frac{3 - 2^{n+1}}{4^{n+1}} = \frac{2^n - 1}{4^n} + \frac{3 - 2^{n+1}}{4^{n+1}}$$

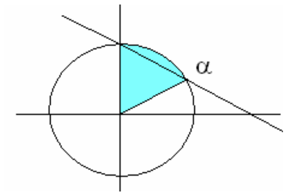
$$= \frac{4 \cdot 2^n - 4 + 3 - 2 \cdot 2^n}{4^{n+1}} = \frac{2 \cdot 2^n - 1}{4^{n+1}} = \frac{2^{n+1} - 1}{4^{n+1}}.$$

Introduzione alla Matematica

Prova scritta del 26.10.2007 _ Soluzioni [2]

1.

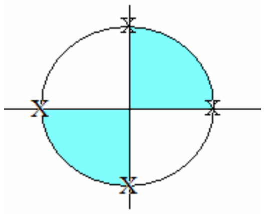
(a) $2 \operatorname{sen} x + \cos x - 2 \geq 0$



$$\begin{cases} 2Y + X - 2 \geq 0 \\ X^2 + Y^2 = 1 \end{cases}$$

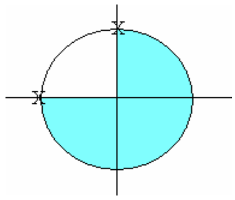
Punti di intersezione $X = 0$, $Y = 1$; $X = 4/5$, $Y = 3/5$
$\alpha = \arcsin 3/5 = \arccos 4/5 = \operatorname{arctg} 3/4$

$\operatorname{sen} 2x > 0$

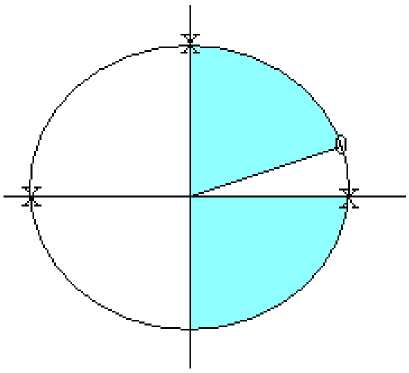


$2k\pi < 2x < \pi + 2k\pi \leftrightarrow$
$k\pi < x < \pi/2 + k\pi$

$\operatorname{tg}(x/2 + \pi/4) > -1$



$-\pi/4 + k\pi < x/2 + \pi/4 < \pi/2 + k\pi \leftrightarrow$
$-\pi + 2k\pi < x < \pi/2 + 2k\pi$



$\alpha \leq x < \pi/2$, $\pi < x < 2\pi$

$+ 2k\pi$

(b) $x < 1$ oppure $\begin{cases} x \geq 1 \\ |x^2 - 2x| > (x-1)^2 \end{cases}$

$x < 1$ oppure $\begin{cases} x \geq 2 \\ x^2 - 2x > x^2 - 2x + 1 \end{cases}$ oppure $\begin{cases} 1 \leq x < 2 \\ 2x - x^2 > x^2 - 2x + 1 \end{cases}$

$x < 1$ oppure $1 \leq x < 1 + 1/\sqrt{2} \leftrightarrow x < 1 + 1/\sqrt{2}$.

2.

(a)

$F(x) = (f \circ g \circ h)(x)$

$h(x) = \log x \quad , \quad g(x) = (1 - x)/(1 + x) \quad , \quad f(x) = \text{arcsen}\, x$.

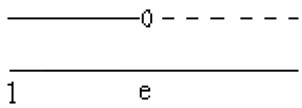
(b) C.E.

$$\begin{cases} x > 0 \ , \ x \neq 1/e \\ \left| (1 - \log x)/(1 + \log x) \right| \leq 1 \end{cases} \Leftrightarrow$$

$$\begin{cases} x > 0 \ , \ x \neq 1/e \\ 1 + \log^2 x - 2 \log x \leq 1 + \log^2 x + 2 \log x \end{cases} \Leftrightarrow \quad x \in [1, +\infty)$$

SGN

$$\begin{cases} x \geq 1 \\ (1 - \log x)/(1 + \log x) \geq 0 \end{cases} \Leftrightarrow \quad x \in [1, e]$$



(c) $\text{arcsen} \dfrac{1 - \log x}{1 + \log x} = k \Rightarrow \dfrac{1 - \log x}{1 + \log x} = \text{sen}\, k$, purché $-\pi/2 \leq x \leq \pi/2$

$1 - \log x = \text{sen}\, k + \text{sen}\, k \ \log x \Rightarrow$

$\log x = (1 - \text{sen}\, k)/(1 + \text{sen}\, k)$, purché $-\pi/2 < x \leq \pi/2 \qquad \Rightarrow$

$x = \exp(\ (1 - \text{sen}\, k)/(1 + \text{sen}\, k)\ )$ è soluzione accettabile se $x \geq 1$ e questo accade senza ulteriori condizioni da imporre su k.

Quindi l'immagine della funzione è $(-\pi/2, \pi/2]$, la funzione è invertibile , la funzione inversa è data da $f^{-1}(k) = \exp(\ (1 - \text{sen}\, k)/(1 + \text{sen}\, k)\ )$.

(d) Proviamo che F (x) è decrescente : $1 \leq x < y \quad \Rightarrow \quad F(x) > F(y)$

$F(x) > F(y) \Leftrightarrow$

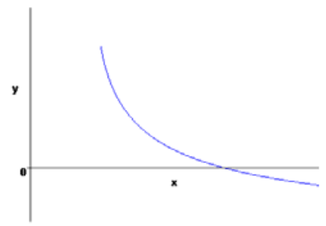
$$\frac{1 - \log x}{1 + \log x} > \frac{1 - \log y}{1 + \log y} \Leftrightarrow (1 - \log x)(1 + \log y) > (1 + \log x)(1 - \log y)$$

(il passaggio è lecito perché i denominatori per cui abbiamo moltiplicato sono positivi)

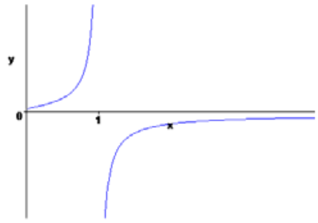
Portando avanti i calcoli, si arriva a $\log x < \log y$, che è vera essendo $x < y$.

(e) La funzione F è tale che $F : [1, +\infty) \rightarrow (-\pi/2, \pi/2]$ ed è decrescente .

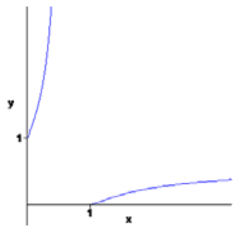
Grafico :



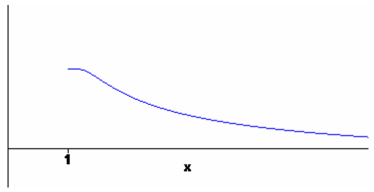
3.



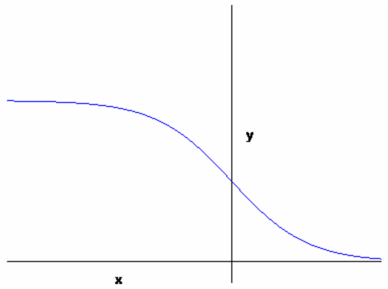
y = - 1 / log x



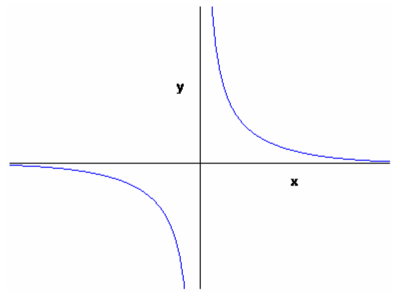
y = exp (- 1 / log x)



y = arcos (exp (- 1 / log x))

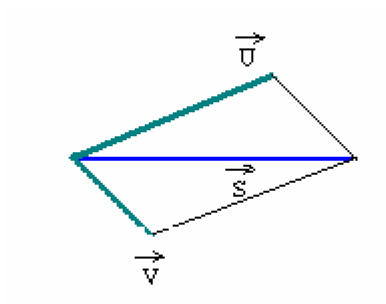


$$y = \frac{\pi}{1 + e^{-x}}$$



$$y = \operatorname{tg} \frac{\pi}{1 + e^{-x}}$$

4.



$$\frac{S}{\text{sen } 105^0} = \frac{U}{\text{sen } 45^0} = \frac{V}{\text{sen } 30^0}$$

U = 7,32 V = 5,18

5.

- per n = 1 è vera : - 1 / 9 = - 1 / 9

- supponiamola vera per n e deduciamola vera per n + 1 , proviamo cioè che

$$\sum_{k=1}^{n} \frac{3^k - 4}{9^k} = \frac{1}{2}\,\frac{1 - 3^n}{9^n} \quad \Rightarrow \quad \sum_{k=1}^{n+1} \frac{3^k - 4}{9^k} = \frac{1}{2}\,\frac{1 - 3^{n+1}}{9^{n+1}} \ .$$

$$\sum_{k=1}^{n+1} \frac{3^k - 4}{9^k} = \sum_{k=1}^{n} \frac{3^k - 4}{9^k} + \frac{3^{n+1} - 4}{9^{n+1}} = \frac{1 - 3^n}{2 \cdot 9^n} + \frac{3^{n+1} - 4}{9^{n+1}}$$

$$= \frac{9 - 9 \cdot 3^n + 2 \cdot 3^{n+1} - 8}{2 \cdot 9^{n+1}} = \frac{1 - 9 \cdot 3^n + 6 \cdot 3^n}{2 \cdot 9^{n+1}} = \frac{1}{2}\,\frac{1 - 3^{n+1}}{9^{n+1}} .$$