Cohomological Hall algebras of 1-dimensional sheaves and Yangians over the Bridgeland's space of stability conditions

(based on

- joint work with D.-E. Diaconescu, M. Porta, O. Schiffmann, and Eric Vasserot, arXiv: 2502.19445;

- joint work with O. Schiffmann and P. Shimpi, arXiv: 2511.08576)

1. The affine Yangian

Fix

- $Q_{fin} = (I_{fin}, \Omega_{fin})$ = finite ADE quiver (e.g. $Q = A_{N-1}$)

- $\mathfrak{g}_{fin} = \mathfrak{n}^+_{fin} \oplus \mathfrak{h}_{fin} \oplus \mathfrak{n}^-_{fin}$ = simple Lie algebra (e.g. $\mathfrak{g}_{fin} = \mathfrak{sl}(N)$)

Define the elliptic Lie algebra $\mathfrak{g}_{ell}$ as:

$$\boxed{\mathfrak{g}_{ell} := \text{universal central extension of } \mathfrak{g}_{fin} \otimes \mathbb{Q}[s^{\pm 1}, t]}$$

i.e.,

$$\mathfrak{g}_{ell} = \left(\mathfrak{g}_{fin} \otimes \mathbb{Q}[s^{\pm 1}, t]\right) \oplus \left(\bigoplus_{\ell \in \mathbb{N}} \mathbb{Q} c_\ell \oplus \bigoplus_{\substack{\ell \in \mathbb{N}, \ell \geq 1 \\ k \in \mathbb{Z}, k \neq 0}} \mathbb{Q} c_{k,\ell}\right)$$

central

with Lie bracket $[x \otimes s^k t^\ell, y \otimes s^h t^n] =$

$$\begin{cases} [x,y] \otimes t^{\ell+n} + k \cdot (x,y) \cdot c_{\ell+n} & \text{if } k+h=0 \\ [x,y] \otimes s^{k+h} t^{\ell+n} + (kh - \ell n) \cdot (x,y) \cdot c_{k+h, \ell+n} & \text{if } k+h \neq 0 \end{cases}$$

where $(-,-)$ is an invariant bilinear form on $\mathfrak{g}_{fin}$.

Let $R := \mathbb{Q}[\varepsilon_1, \varepsilon_2]$.
The affine Yangian = associative algebra $\mathbb{Y}/R$
s.t. :

$$\boxed{\mathbb{Y} = \text{a 'deformation' of } U(\mathfrak{g}_{ell})}$$

i.e., $\mathbb{Y} \otimes_R \mathbb{Q} \simeq U(\mathfrak{g}_{ell})$ and $\exists$ a filtration $F^\bullet$ of $\mathbb{Y}$ s.t.

$$gr_{F^\bullet} \mathbb{Y} \simeq U(\mathfrak{g}_{ell}) \otimes R$$

Moreover, $\mathbb{Y}$ admits a triangular decomposition

$$\mathbb{Y}^+ \otimes \mathbb{Y}^0 \otimes \mathbb{Y}^- \xrightarrow{\sim} \mathbb{Y}$$

In addition, $\mathbb{Y}^-$ is a 'deformation' of $U(\mathfrak{n}_{ell})$,
where

- $\mathfrak{n}_{ell} := \mathfrak{n}^-[t] \oplus \bigoplus_{k<0} c_{k,\ell}$
- $\mathfrak{n}^- := s^{-1}\mathfrak{g}_{fin}[s^{-1}] \oplus \mathfrak{n}^-_{fin}$ = 'standard' negative half of the affine Lie algebra $\mathfrak{g}$

Remark

- Maulik-Okounkov:
$Q = (I, \Omega)$ = quiver, $T \subseteq (\mathbb{C}^*)^{\Omega} \times \mathbb{C}^*$ torus $\rightsquigarrow \mathbb{Y}^{MO}_{Q,T}$

- Drinfeld, Guay, Nakajima, Varagnolo:
Q = quiver without edge-loops $\rightsquigarrow \mathbb{Y}^{D}_{Q,\mathbb{C}^*}$

(study of $\mathbb{Y}^{D}_{Q,\mathbb{C}^*}$ by Guay-Nakajima-Wendlandt
Guay-Regelskis-Wendlandt)

- Guay for $Q = A_N$ $(N \geq 2)$; Kodera, Bershtein-Tsymbaliuk for $Q = A_1$; Diaconescu-Porta-S.-Schiffmann-Vasserot:
Q = affine ADE quiver $\rightsquigarrow \mathbb{Y}^{D}_{Q,\mathbb{C}^*\times\mathbb{C}^*}$

- Jindal (type A), Diaconescu-Porta-S.-Schiffmann-Vasserot:
Q = affine ADE quiver :

$$\boxed{\mathbb{Y}^{D,\pm}_{Q,\mathbb{C}^*\times\mathbb{C}^*} \simeq \mathbb{Y}^{MO,\pm}_{Q,\mathbb{C}^*\times\mathbb{C}^*} \text{ as algebras}}$$

$$\Longrightarrow \mathbb{Y} := \mathbb{Y}^{D}_{Q,\mathbb{C}^*\times\mathbb{C}^*}$$

To an affine ADE quiver, we associate

- $\mathfrak{g}$ = affine Lie algebra
- W = affine Weyl group

Attention ⚠: Recall that $\exists$ halves of $\mathfrak{g}$ that are not $W \times \{\pm 1\}$-conjugate to the 'standard' halves

They were classified by Jakobsen-Kac.

Set

- Δ = root system of Q
- Δ_{fin} = root system of $Q_{fin} = \Delta^+_{fin} \cup \Delta^-_{fin}$
- α_i = simple root $\forall i \in I_{fin}$

Fix $\phi \subseteq J \subseteq I_{fin}$, $J^c := I_{fin} \setminus J$, and define

- $\Delta_{J^c} := \Delta_{fin} \cap \bigoplus_{i \in J^c} \mathbb{Z}\alpha_i$, $\quad \Delta^{\pm}_{J^c} := \Delta^{\pm}_{fin} \cap \bigoplus_{i \in J^c} \mathbb{Z}\alpha_i$

Then, we define

$$\Delta_J := \left\{ \alpha + n\delta \in \Delta : \begin{array}{l} \blacktriangleright\ \alpha \in \Delta^-_{J^c} \text{ and } n \leq 0, \text{ or} \\ \blacktriangleright\ \alpha \in \Delta^+_{J^c} \cup \{0\} \text{ and } n < 0, \text{ or} \\ \blacktriangleright\ \alpha \in \Delta^+ \setminus \Delta^+_{J^c} \text{ and } n \in \mathbb{Z} \end{array} \right\}$$

Remark (Jacobsen-Kac)

Any set of positive roots of Δ is $(W \times \{\pm 1\})$-conj. to one of the Δ_J's.

Example ($Q_{fin} = A_1 = \cdot$)

- $J = I_{fin}$: $\Delta_J = (-\mathbb{N}\delta) \cup (\Delta^+ \oplus \mathbb{Z}\delta) =$ 'nonstandard' half
- $J = \phi$: $\Delta_J = \Delta^-_{fin} \cup (\Delta_{fin} \oplus (-\mathbb{N}_{\geq 1}\delta)) \cup (-\mathbb{N}_{\geq 1}\delta) =$ 'standard' half

We also define 'nonstandard' halves of the affine Lie algebra $\mathfrak{g}$ and the elliptic Lie algebra $\mathfrak{g}_{ell}$:

$$\begin{cases} \mathfrak{n}^J := \bigoplus_{\beta \in \Delta_J} \mathfrak{g}_\beta \\ \mathfrak{n}^J_{ell} := \mathfrak{n}^J[t] \oplus \bigoplus_{k<0} \mathbb{C}_{k,\ell} \end{cases}$$

β-graded piece

Goals: geometrically construct 'deformations' of

$$U(\mathfrak{n}^J_{ell})$$

Consider Stab = Bridgeland's space of stability conditions (more precise later on)

Theorem (S.-Schiffmann-Shimpi)

$\forall x \in \text{Stab}$, $\exists\, \mathbb{Y}_x$ = 'deformation' of $\widehat{U}^x(\mathfrak{n}^x_{ell})$, where

completion

$$\mathfrak{n}^x_{ell} = b_x \cdot \mathfrak{n}^J_{ell}$$

for some $\emptyset \subseteq J \subseteq I_{fin}$ and for an element $b_x \in W$.

Conjecture

$\mathbb{Y}_x$ realizes a half of a completion $\widehat{\mathbb{Y}}^x$ of $\mathbb{Y}$

Attention ⚠: the proof is geometric and based on the theory of cohomological Hall algebras.

2. COHAs of surfaces: Construction

Moduli stack $\longrightarrow$ COHA $\longrightarrow$ Associative algebra

S = smooth quasi-projective surface $/\mathbb{C}$

T = (possibly trivial) torus $\curvearrowright S$

$C \subset S$ = T-invariant closed reduced subscheme

Consider

$\underline{\mathrm{Coh}}_C(S)$ = moduli stack of coherent sheaves on S set-theoretically supported on C

Example

$S = T^*X$, X = smooth projective curve $/\mathbb{C}$

$C = X \subset T^*X$ zero section

$\Longrightarrow$ $\underline{\mathrm{Coh}}_X(T^*X)$ = moduli stack of Higgs sheaves $(\mathcal{F}, \phi : \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega^1_X)$ on X, such that ϕ is nilpotent

S-Schiffmann: $S = T^*X$; Diaconescu-Porta-S-Schiffmann-Vasserot:

1. $\boxed{\exists\ HA^{(T)}_{S,C} = (T\text{-equivariant})\ \text{COHA associated to } \underline{\mathrm{Coh}}_C(S)}$

= unital associative algebra structure on

$$\underline{H}^{(T)}_*(\underline{\mathbb{R}\mathrm{Coh}}_C(S)) = \underline{H}^{(T)}_*(\underline{\mathrm{Coh}}_C(S))$$

(equivariant Borel-Moore homology)

where the multiplication $(\mathbb{R}p)_* \circ (\mathbb{R}q)^!$ is induced by

$$\underline{\mathbb{R}\mathrm{Coh}}_C(S) \times \underline{\mathbb{R}\mathrm{Coh}}_C(S) \xleftarrow{\mathbb{R}q} \underline{\mathbb{R}\mathrm{Coh}}^{ext}_C(S) \xrightarrow{\mathbb{R}p} \underline{\mathbb{R}\mathrm{Coh}}_C(S)$$

$\hookrightarrow$ (stack of extensions)

Here,

$$p: \; 0 \longrightarrow E_2 \longrightarrow E \longrightarrow E_1 \longrightarrow 0 \longmapsto E$$
$$q: \; \text{———————} \; '' \; \text{———————} \longmapsto (E_2, E_1)$$

2. $\mathbb{HA}^{(T)}_{S,C}$ depends "locally" on C, i.e., given (S_1, C_1) and (S_2, C_2) such that the formal completions $\widehat{(S_1)}_{C_1} \simeq \widehat{(S_2)}_{C_2}$ we have:

$$\mathbb{HA}^{(T)}_{S_1,C_1} \simeq \mathbb{HA}^{(T)}_{S_2,C_2}$$

3. COHA of minimal resolution of ADE singularity

- $G \subset SL(2,\mathbb{C})$ finite subgroup
  $\updownarrow$
  Q_{fin} = finite ADE quiver, Q = affine ADE quiver
- $\pi : S \longrightarrow \mathbb{C}^2/G$ Kleinian resolution of singularities
- $C := \pi^{-1}(0)$. Then, $C_{red} = C_1 \cup \cdots \cup C_e$, $C_i \simeq \mathbb{P}^1$

$(C_i \cdot C_j) = -$ Cartan matrix of Q_{fin}

- torus $T \subset GL(2,\mathbb{C})$ centralizing G
  ($T =$ trivial, $\mathbb{C}^*$, or $\mathbb{C}^* \times \mathbb{C}^*$)

Example

$G = \mathbb{Z}_2 \Longrightarrow Q_{fin} = A_1 = \cdot\ ,\ Q = A_1^{(1)} :\ \bullet \rightleftarrows \bullet$

$\Longrightarrow S = T^*\mathbb{P}^1 \supset C = \mathbb{P}^1 =$ zero section

Recall the derived McKay correspondence:

$$\tau : D^b(\mathrm{Coh}(S)) \xrightarrow{\ \sim\ } D^b(\mathrm{Mod}(\Pi_Q))$$

where

- $\Pi_Q =$ preprojective algebra of Q

  $:= \mathrm{End}($tilting bundle on S inducing $\tau)$

- $\mathrm{Mod}(\Pi_Q) =$ finite-dimensional right Π_Q-modules derived McKay correspondence

$$\Downarrow$$

$$\begin{cases} K_0(\mathrm{Coh}_C(S)) \xrightarrow{\sim} K_0(\mathrm{nilp}(\Pi_Q)) \simeq \text{root lattice } Y \text{ of } Q \\ \mathrm{Pic}(S) \xrightarrow{\sim} \text{coweight lattice } \check{X}_{\mathrm{fin}} \text{ of } Q_{\mathrm{fin}} \end{cases}$$

$\Big\lfloor = (\Pi_Q\text{-repr.s})$

where $\mathrm{nilp}(\Pi_Q) \subset \mathrm{Mod}(\Pi_Q)$ is the subcategory of nilpotent repr.s of Π_Q.

Set $\mathfrak{n}_{\mathrm{ell}}^+ := \mathfrak{n}_{\mathrm{ell}}^{I_f}$ for $J = I_{\mathrm{fin}}$.

Theorem (Diaconescu-Porta-S-Schiffmann-Vasserot)

- $\exists$ a canonical algebra isomorphism

$$\mathbb{HA}_{S,C} \simeq \hat{U}(\mathfrak{n}_{\mathrm{ell}}^+)$$

(completion)

- $\exists$ a canonical algebra isomorphism

$$\mathbb{HA}_{S,C}^{T} \simeq \mathbb{Y}_{S,C}$$

where $\mathbb{Y}_{S,C}$ is a 'deformation' of $\hat{U}(n_{ell}^+)$

Conjecture
$HA^T_{S,C} \simeq \mathbb{Y}_{S,C}$ realizes a half of a completion of $\mathbb{Y}$

4. Proof of the Theorem

Note that derived McKay correspondence

$$\tau : D^b(Coh_c(S)) \xrightarrow{\sim} D^b(nilp(\Pi_Q))$$

is not t-exact w.r.t. the standard t-structures, i.e.,

$$Coh_c(S) \cancel{\xrightarrow{\sim}} nilp(\Pi_Q)$$

$$\Downarrow$$

$\underline{Coh}_c(S) \;\; \cancel{\simeq} \;\; \Lambda_Q$ = stack of nilpotent repr.s of Π_Q

$$\Downarrow$$

$$HA^T_{S,C} \;\; \cancel{\simeq} \;\; COHA_Q^{T,nil}$$

where $\mathrm{COHA}_Q^{T,nil}$ = Schiffmann-Vasserot's nilpotent 2d COHA of the quiver Q

Attention ⚠: The relation between $\mathrm{Coh}_c(S)$ and $\mathrm{nilp}(\Pi_Q)$ is more subtle:

$$\boxed{\mathrm{Coh}_c(S) = \text{"limit" of } \mathrm{nilp}(\Pi_Q)}$$

More precisely,

- hearts of bounded t-structures on $\mathcal{C} = D^b(\mathrm{nilp}(\Pi_Q))$ form a partial ordered set:

$$\boxed{H_1 := \mathcal{C}_1^{\geq 0} \cap \mathcal{C}_1^{\leq 0} \leq H_2 := \mathcal{C}_2^{\geq 0} \cap \mathcal{C}_2^{\leq 0} \iff \mathcal{C}_1^{\geq 0} \subseteq \mathcal{C}_2^{\geq 0}}$$

- $\check{\theta} \in \check{X}_{fin}$ strictly dominant (i.e., $\check{\theta}(\alpha_i) > 0, \forall i$) $\rightsquigarrow \mathcal{L}_{\check{\theta}}$ π-ample

Set

$$\tilde{L}_{\check{\theta}} := \tau \circ (\mathcal{L}_{\check{\theta}} \otimes -) \circ \tau^{-1} : D^b(\mathrm{nilp}(\Pi_Q)) \longrightarrow D^b(\mathrm{nilp}(\Pi_Q))$$

- Shimpi: $\boxed{\inf_{n\geq 0} \tilde{L}_{\check{\Theta}}^{-n}\left(\mathrm{nilp}(\Pi_Q)\right) = \mathrm{Coh}_C(S)}$

Now we translate Shimpi's result in the theory of COHAs.

We will use:

- Bridgeland stability conditions
- braid group actions on bounded derived cats

Now, fix a strictly dominant coweight $\check{\Theta} \in \check{X}_{fin} \hookrightarrow \check{X}$.

$\check{\Theta}$ defines a King's stability condition on $\mathrm{nilp}(\Pi_Q)$:

$$\begin{aligned} Z_{\check{\Theta}} : K_0\left(\mathrm{nilp}(\Pi_Q)\right) \simeq \mathbb{Z} I &\longrightarrow \mathbb{C} \\ \underline{d} &\longmapsto -(\check{\Theta}, \underline{d}) + (\check{\rho}, \underline{d})\, i \end{aligned}$$

$\check{\rho}(\alpha_i) = 1$

$$\Longrightarrow \boxed{\exists \left(Z_{\check{\theta}}, \mathcal{P}_{\check{\theta}}\right) = \text{Bridgeland's stability condition on } D^b\left(\mathrm{nilp}\left(\Pi_Q\right)\right)}$$

Here, $\mathcal{P}_{\check{\theta}}$ = slicing = family of full additive subcats

$$\mathcal{P}_{\check{\theta}}(\phi) \subset D^b\left(\mathrm{nilp}\left(\Pi_Q\right)\right) \quad \forall \phi \in \mathbb{R}$$

satisfying certain conditions.

Set $\mathcal{P}_{\check{\theta}}(I) := \langle \mathcal{P}_{\check{\theta}}(\phi) : \phi \in I \rangle$ $\forall$ interval $I \subset \mathbb{R}$

Lemma

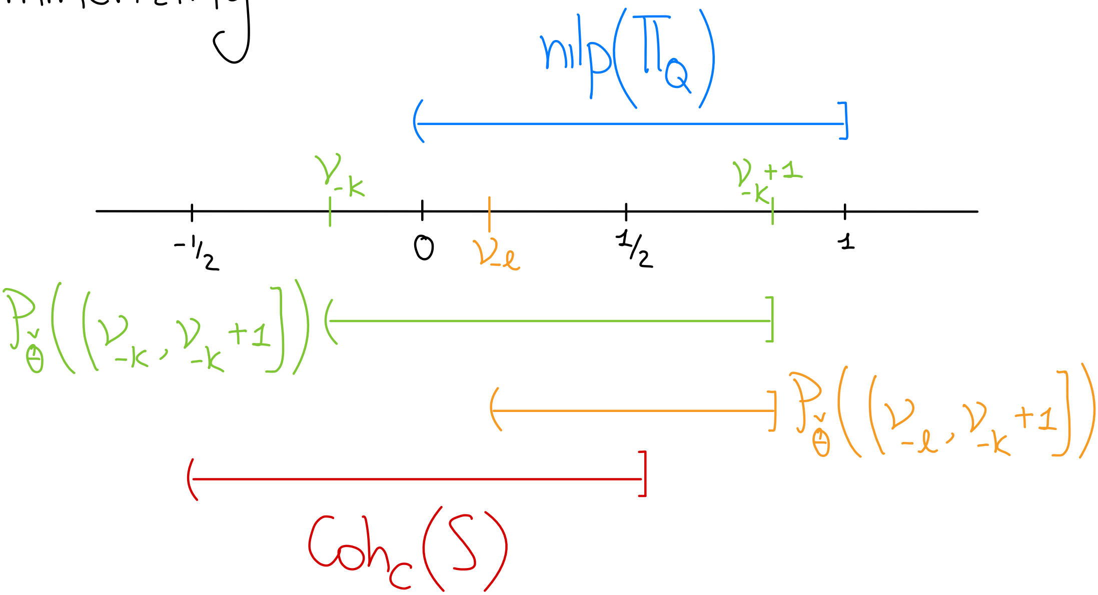
1. $\forall k \in \mathbb{Z}$, $\mathcal{L}_{\check{\theta}}^{-2k} : \mathrm{nilp}\left(\Pi_Q\right) \xrightarrow{\sim} \mathcal{P}_{\check{\theta}}\left(\left(\nu_{-k}, \nu_{-k}+1\right]\right)$

with $\nu_\ell := \frac{1}{\pi} \arctan\left(2h\ell\right)$

Coxeter number

2. $\mathcal{P}_{\check{\theta}}\left(\left(-\frac{1}{2}, \frac{1}{2}\right]\right) \simeq \mathrm{Coh}_c(S)$

Summarizing:



For $\ell, k \in \mathbb{N}$, $k \geq \ell$, set

$$\boxed{\Lambda_Q^{\ell,k} := \text{derived moduli stack of objects in } \mathcal{P}_{\check\theta}((\nu_{-\ell},\nu_{-k}+1])}$$

Attention:

1. Let $\Lambda_Q^{>\nu_{-\ell+k}} \subset \Lambda_Q$ = usual Harder-Narasimhan stratum. Then

$$\tilde{\mathbb{L}}_{\check\theta}^{2k} : \Lambda_Q^{\ell,k} \xrightarrow{\sim} \Lambda_Q^{\ell-k,0} \simeq \Lambda_Q^{>\nu_{-\ell+k}}$$

2. $\tilde{\mathbb{L}}_{\check\theta}^{2k}$ is a "braid group operator".

Recall that

- the extended affine braid group

$$\boxed{B_{ex} \simeq \left(B_{fin} \cup \{L_{\lambda^\vee} : \lambda \in \check{X}_{fin}\}\right) / \text{rel.s}}$$

Lemma

$\exists$ a group homomorphism

$$\rho : B_{ex} \longrightarrow \mathrm{Aut}(D^b(\mathrm{nilp}(\Pi_Q)))$$

such that $\rho(L_{\lambda^\vee}) = \tilde{L}_{\lambda^\vee}$ $\forall \lambda \in \check{X}_{fin}$

We have:

Lemma

1. The vector space

$$\boxed{HA^{(T)}_{\check{\Theta}} := \lim_{\ell} \operatorname*{colim}_{k \geq \ell} H^{(T)}_*(\Lambda^{\ell,k}_Q)}$$

has the structure of an unital associative algebra with multiplication induced from that of $COHA^{(T),nil}_Q$

2. $\exists$ an algebra isomorphism $\mathrm{HA}^{(T)}_{S,C} \simeq \mathrm{HA}^{(T)}_{\check{\theta}}$

Corollary

$\mathrm{HA}^{(T)}_{\check{\theta}}$ does not depend on the specific choice of strictly dominant finite coweight.

Now, we relate $\mathrm{HA}^{T}_{\check{\theta}} := \lim\limits_{\ell} \operatorname*{colim}\limits_{k \geq \ell} H^{T}_{*}(\Lambda^{\ell,k}_{Q})$ to Yangians.
Recall that

$$\mathrm{COHA}^{T,nil}_{Q} \simeq \mathbb{Y}^{-}$$

(as proved by Schiffmann-Vasserot).

Then, as a vector space

$$\operatorname*{colim}_{k \geq \ell} H^{T}_{*}(\Lambda^{\ell,k}_{Q}) \simeq T^{2\ell}_{\check{\theta}}(\mathbb{Y}^{-}) \Big/ T^{2\ell}_{\check{\theta}}(J)$$

braid op.

where $J := \sum\limits_{(\check{\theta},\underline{d}) \leq 0} \mathbb{Y}^{-}_{-\underline{d}} \cdot \mathbb{Y}^{-}$.

Finally, we have:

Lemma

1. $\exists$ a unital associative algebra structure on:

$$\mathbb{Y}_{S,C} := \lim_{\ell} T^{2\ell}_{\check{\Theta}}(\mathbb{Y}^{-}) \Big/ T^{2\ell}_{\check{\Theta}}(J)$$

2. $\exists$ an algebra isomorphism $\mathcal{HA}^{T}_{\check{\Theta}} \xrightarrow{\sim} \mathbb{Y}_{S,C}$.

$\square$

Let's address what happens when we drop "strictly" from $\check{\Theta}$. Fix $\emptyset \subseteq J \subseteq I_{fin}$ and let $\check{\Theta} \in \check{X}_{fin}$ be s.t.:

$$\check{\Theta}(\alpha_i) = 0 \quad \forall i \in I_f \setminus J \quad \text{and} \quad \check{\Theta}(\alpha_i) > 0 \quad \forall i \in J$$

Shimpi: $\inf\limits_{n \geq 0} \mathcal{L}^{-n}_{\check{\Theta}}(\mathrm{nilp}(\Pi_Q)) = P_C(S/S_J)$

where

- contraction of C_i, for $i \in I_{fin} \setminus J$:

$$\begin{tikzcd}
S \arrow[rr, "\pi_J"] \arrow[dr, "\pi"'] & & S_J \arrow[dl] \\
& \mathbb{C}^2/G &
\end{tikzcd}$$

- $\mathcal{P}_c(S/S_J) \subset D^b(\mathrm{Coh}_c(S))$ heart of the tilted t-structure with respect to

$$\begin{cases} \mathcal{T}_J := \left\{ \mathcal{E} \in \mathrm{Coh}_c(S) : \mathbb{R}^1\pi_{J*}(\mathcal{E}) = 0,\ \mathrm{Hom}(\mathcal{E}, \mathcal{F}) = 0 \ \ \forall \mathcal{F} \text{ s.t. } \mathbb{R}^1\pi_{J*}(\mathcal{F}) = 0 \right\} \\ \mathcal{F}_J := \left\{ \mathcal{E} \in \mathrm{Coh}_c(S) : \mathbb{R}^0\pi_{J*}(\mathcal{E}) = 0 \right\} \end{cases}$$

($\mathcal{P}_c(S/S_J)$ = Van der Bergh's abelian category of perverse coherent sheaves set-theoretically supported on C)

$\Longrightarrow \exists$ COHA$^{(T)}$ of $\mathcal{P}_c(S/S_J) := \mathbf{HA}_J^{(T)}$

Theorem (S-Schiffmann-Shimpi)

- $\exists$ a canonical algebra isomorphism

$$\mathbf{HA}_J \simeq \widehat{U}(\mathfrak{n}_{ell}^J)$$

- $\exists$ a canonical algebra isomorphism

$$\mathbf{HA}_J^T \simeq \mathbb{Y}_J$$

where $\mathbb{Y}_J$ is a 'deformation' of $\widehat{U}(\mathfrak{g}^J_{ell})$

Now, let

- $\mathrm{Stab}(S) = \mathrm{Stab}(D^b(\mathrm{Coh}_c(S)))$ = space of stability conds
- $\mathrm{Stab}^\circ(S) \subset \mathrm{Stab}(S)$ be its distinguished connected component.

Lemma

$\forall\, x \in (Z, P) \in \mathrm{Stab}^\circ(S)$ and $\forall\, I \subset \mathbb{R}$ interval of lenght 1 $\exists\, \phi \subseteq J \subseteq I_{fin}$ and an element b in the affine braid group s.t.

$$\mathcal{P}(I) \simeq b \cdot \mathcal{P}_c(S/S_J) \quad \text{(up to shifts)}$$

Therefore, we get:

Theorem (S-Schiffmann-Shimpi)

$\forall\, x = (\mathcal{P}, Z) \in \mathrm{Stab}^\circ(S)\ \exists\, HA^{(T)}_x$ associated to $\mathcal{P}(0,1]$:

$$\mathbb{Y}_x := HA^T_x = \text{'deformation' of } \widehat{U}^x(\mathfrak{n}^x_{ell})$$