COHOMOLOGICAL HALL ALGEBRAS OF ONE-DIMENSIONAL SHEAVES ON SURFACES: FOUNDATIONS

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ABSTRACT. This paper develops a framework for systematically studying cohomological "Hecke operators" associated with modifications of coherent sheaves on a smooth surface X along a fixed proper curve $Z \subset X$ (possibly singular and reducible), using the theory of cohomological Hall algebras.

More precisely, we develop the necessary geometric foundations in order to define the (motivic, T-equivariant) cohomological Hall algebra $\mathbf{HA}^{\mathbf{D},T}_{X,Z}$ of the moduli stack of coherent sheaves on X with *set-theoretic* support on Z, in the setting of a general motivic formalism $\mathbf{D}$ (in the sense of Khan – see [Kha21] and references therein). The algebra $\mathbf{HA}^{\mathbf{D},A}_{X,Z}$ is functorial with respect to closed immersions $Z' \subset Z$ and transformations of the motivic formalism $\mathbf{D}$, and only depends on the formal neighborhood $\widehat{X}_Z$ of Z in X.

CONTENTS

1. INTRODUCTION

The main aim of the present paper is to lay the foundations for *nilpotent cohomological Hall algebras*. In order to describe the main results and the challenges that have to be solved, start by fixing a smooth quasi-projective complex surface X and a closed subscheme $Z \subset X$. For simplicity, in this introduction we take the additional assumption that Z is proper. We are then interested in:

(1) define the moduli stack $\mathfrak{Coh}(\widehat{X}_Z)$ of coherent sheaves on X *set-theoretically* supported on Z;

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(2) define the Borel-Moore homology of such moduli stack;

(3) prove that the Borel-Moore homology of such moduli stack carries a canonical cohomological Hall algebra structure.

The first challenge comes from the set-theoretic condition: this essentially says that the moduli stack in question cannot be an algebraic stack (locally) of finite type. From the point of view of the moduli of objects, we are working with the category of coherent sheaves on the formal completion $\widehat{X}_Z$, and the difficulty is that this category is not of finite type. The second challenge is to make sense of the Borel-Moore homology of $\mathfrak{Coh}(\widehat{X}_Z)$. Both these difficulties are addressed by the following:

Theorem A (Theorem 4.65 and Corollary 4.66)**.** The functor of points

$$\mathfrak{Coh}(\widehat{X}_Z)\colon \mathsf{Aff}_k^{\mathsf{op}} \longrightarrow \mathsf{Spc}$$

that sends $S \in \mathsf{Aff}_k$ to the groupoid of S-flat coherent sheaves on $X \times S$ set-theoretically supported on $Z \times S$ is an indgeometric stack, that is it is a filtered colimit of Artin stacks where the transition maps are closed immersions. Furthermore:

(1) its reduced ${}^{\mathsf{red}}\mathfrak{Coh}(\widehat{X}_Z)$ is a quasi-separated (but not quasi-compact) Artin stack locally of finite type (in the underived sense);

(2) there is a canonical morphism

$$^{\mathsf{red}}\mathfrak{Coh}(\widehat{X}_Z) \longrightarrow \mathfrak{Coh}_{\mathsf{ps}}(X)$$

which is a closed immersion, and $\mathfrak{Coh}(\widehat{X}_Z)$ coincides with the formal completion of $\mathfrak{Coh}_{\mathsf{ps}}(X)$ at this closed substack.

Example 1.1. Let C be a smooth projective curve and set $X := T^*C$ be its cotangent bundle. Take $Z := C$ to be the curve embedded as its zero section. In this case, $\mathfrak{Coh}_{\mathsf{ps}}(T^*C)$ is the stack of Higgs sheaves on C, and ${}^{\mathsf{red}}\mathfrak{Coh}_{\mathsf{ps}}(\widehat{T^*C}_C)$ coincides with the stack of nilpotent Higgs sheaves (with reduced structure). Therefore $\mathfrak{Coh}(\widehat{T^*C}_C)$ is the formal completion of the stack of Higgs sheaves at the closed substack given by nilpotent Higgs sheaves. △

The above theorem allows to define the Borel-Moore homology of $\mathfrak{Coh}(\widehat{X}_Z)$: indeed, Borel-Moore homology only depends on the underlying reduced stack, and therefore it is sensible to set as a definition

$$\mathsf{H}_*^{\mathsf{BM}}(\mathfrak{Coh}(\widehat{X}_Z);\mathbb{Q}) := \mathsf{H}_*^{\mathsf{BM}}({}^{\mathsf{red}}\mathfrak{Coh}(\widehat{X}_Z);\mathbb{Q})\ ,$$

where the latter makes sense thanks to the work of A. Khan [Kha19]. It is worth remarking that some extra care needs to be applied since ${}^{\mathsf{red}}\mathfrak{Coh}(\widehat{X}_Z)$ is not quasi-compact; as already explained in [PS23, Appendix A], this leads to consider these Borel-Moore homology groups as *topological* vector spaces, with a topology induced by quasi-compact open exhaustions of ${}^{\mathsf{red}}\mathfrak{Coh}(\widehat{X}_Z)$.

Nevertheless, this is not sufficient to endow $\mathsf{H}_*^{\mathsf{BM}}(\mathfrak{Coh}(\widehat{X}_Z);\mathbb{Q})$ with a Hall multiplication. The reason is, as usual, that the canonical map

$$\partial_0 \times \partial_2\colon \mathfrak{Coh}^{\mathsf{ext}}(\widehat{X}_Z) \longrightarrow \mathfrak{Coh}(\widehat{X}_Z) \times \mathfrak{Coh}(\widehat{X}_Z)$$

is too singular to allow to define on the nose a refined Gysin pullback. However, it turns out that the square

$$\begin{array}{ccc} \mathfrak{Coh}^{\mathsf{ext}}(\widehat{X}_Z) & \longrightarrow & \mathfrak{Coh}^{\mathsf{ext}}_{\mathsf{ps}}(X) \\ \big\downarrow{\scriptstyle \partial_0\times\partial_2} & & \big\downarrow{\scriptstyle \partial_0\times\partial_2} \\ \mathfrak{Coh}(\widehat{X}_Z) \times \mathfrak{Coh}(\widehat{X}_Z) & \longrightarrow & \mathfrak{Coh}(X) \times \mathfrak{Coh}(X) \end{array}$$

is a pullback. As explained in [PS23], the right vertical map admits a canonical derived enhancement that is derived lci. The theory developed by A. Khan [Kha19] allows therefore to define a refined Gysin pullback. One *could* try to exploit this to define the Hall multiplication for $\mathsf{H}_*^{\mathsf{BM}}(\mathfrak{Coh}(\widehat{X}_Z))$ by hand, but the problem is proceeding in this way, the Hall multiplication would a priori depend on the ambient surface X.

Since in explicit computations is useful to reduce to 'local' computations depending only on the formal completion $\widehat{X}_Z$ of X along Z (as done in [DPS⁺25, §??]), we take a more fundamental approach. Namely, we define a derived enhancement $\mathbf{Coh}(\widehat{X}_Z)$ of $\mathfrak{Coh}(\widehat{X}_Z)$, and more precisely we construct a full 2-Segal derived stack $\mathcal{S}_\bullet\mathbf{Coh}(\widehat{X}_Z)$ encoding the Hall multiplication as an algebra structure in correspondences. Our construction makes more generally sense for coherent sheaves on formal schemes, and does not require a priori the embedding in an ambient variety. In particular, the Hall algebra structure encoded by $\mathcal{S}_\bullet\mathbf{Coh}(\widehat{X}_Z)$ only depends, in a tautological way, on the formal completion $\widehat{X}_Z$. We then prove:

Theorem B (Theorem 5.2)**.**

(1) As a *derived* stack, $\mathbf{Coh}(\widehat{X}_Z)$ is the formal completion of $\mathbf{Coh}(X)$ at ${}^{\mathsf{red}}\mathfrak{Coh}(\widehat{X}_Z)$.

(2) Both squares

$$\begin{array}{ccc} \mathbf{Coh}^{\mathsf{ext}}(\widehat{X}_Z) & \longrightarrow & \mathbf{Coh}^{\mathsf{ext}}(X) \\ \Big\downarrow{\scriptstyle \partial_0\times\partial_2} & & \Big\downarrow{\scriptstyle \partial_0\times\partial_2} \\ \mathbf{Coh}(\widehat{X}_Z)\times\mathbf{Coh}(\widehat{X}_Z) & \longrightarrow & \mathbf{Coh}(X)\times\mathbf{Coh}(X) \end{array}$$

and

$$\begin{array}{ccc} \mathbf{Coh}^{\mathsf{ext}}(\widehat{X}_Z) & \longrightarrow & \mathbf{Coh}^{\mathsf{ext}}(X) \\ \Big\downarrow{\scriptstyle \partial_1} & & \Big\downarrow{\scriptstyle \partial_1} \\ \mathbf{Coh}(\widehat{X}_Z) & \longrightarrow & \mathbf{Coh}(X) \end{array}$$

are pullback.

This theorem shows that ${}^{\mathsf{red}}\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)$ generalizes the global nilpotent cone. Our approach passes through first defining $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)$ as a functor of points, and only after proving that its reduced is a closed substack of $\mathbf{Coh}_{\mathsf{ps}}(X)$.

This theorem implies that the map

$$\partial_0\times\partial_2\colon \mathbf{Coh}^{\mathsf{ext}}(\widehat{X}_Z)\longrightarrow \mathbf{Coh}(\widehat{X}_Z)\times\mathbf{Coh}(\widehat{X}_Z)$$

is representable by lci derived Artin stacks (it is, in fact, a linear stack as in [PS23, Proposition 3.6]), and that the map

$$\partial_1\colon \mathbf{Coh}^{\mathsf{ext}}(\widehat{X}_Z)\longrightarrow \mathbf{Coh}(\widehat{X}_Z)$$

is representable by ind-proper algebraic spaces. Notice that both source and target of these stacks are indgeometric, so it is a priori unclear that these maps are representable by geometric stacks.

Finally, in order to have a more streamlined treatment of the Hall product, we introduce the class of *admissible indgeometric derived stacks*, which is essentially characterized by the property of having a geometric reduced stack (see Theorem 2.20). We then extend Khan's theory of motivic Borel-Moore homology to this class of stacks in §3. We can summarize the main result obtained there as follows:

Theorem C (Theorem 3.5)**.** There exists a lax symmetric monoidal functor

$$\mathsf{H}_*^{\mathsf{BM}}(-;\mathbb{Q})\colon \mathsf{Corr}^{\times}(\mathsf{indGeom}_k^{\mathsf{adm}})_{\mathsf{rep.lci,rpas}}\longrightarrow \mathsf{Pro}(\mathsf{Mod}_{\mathbb{Q}}^{\heartsuit})$$

that sends $\mathcal{X}$ to the pro-object

$$\mathsf{H}_*^{\mathsf{BM}}(\mathcal{X};\mathbb{Q}) := \text{“}\lim_{\mathcal{U}\Subset\mathcal{X}}\text{”}\, \mathsf{H}_*^{\mathsf{BM}}({}^{\mathsf{red}}\mathcal{U};\mathbb{Q})\ ,$$

where the limit is taken over the quasi-compact admissible open substacks of $\mathcal{X}$. This assignment is covariant in morphisms that are representable by ind-proper algebraic spaces and contravariant in morphisms that are representable by derived lci morphisms of geometric stacks.

The proof is essentially a routine extension of the six operation formalism, whose main steps are carried out axiomatically in §3. We also prove a much more general result (in the spirit of [Kha19]), dealing with any oriented motivic Borel-Moore homology theory.

Finally, we combine all the ingredients discussed so far to prove the following.

Theorem D (Theorem 5.7)**.** Let X be a smooth surface over a field k of characteristic zero and let $j\colon Z \hookrightarrow X$ be the inclusion of a closed subscheme. Assume that X admits a projective compactification $\overline{X}$ that contains Z as a closed subscheme. Then, there exists a unital associative topological algebra structure on

$$\mathbf{HA}_{\widehat{X}_Z} := \mathsf{H}_*^{\mathsf{BM}}(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z);\mathbb{Q})$$

with the multiplication $p_*q^!$, where the map p and q are those in the diagram

$$\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)\times\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z) \xleftarrow{\ q\ } \mathcal{S}_2\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z) \xrightarrow{\ p\ } \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)\ .$$

Moreover:

- if $Z' \hookrightarrow X'$ is a second closed immersion satisfying the above assumptions, an abstract isomorphism $\widehat{X}_Z \simeq \widehat{X'}_{Z'}$ of formal schemes induces an isomorphism of topological algebras
$$\mathbf{HA}_{\widehat{X}_Z} \simeq \mathbf{HA}_{\widehat{X'}_{Z'}}\ .$$
- if $i\colon Z' \hookrightarrow Z$ is a nested closed subscheme of X, then the direct image i_* gives rise to a continuous algebra morphism
$$i_*\colon \mathbf{HA}_{\widehat{X}_{Z'}} \longrightarrow \mathbf{HA}_{\widehat{X}_Z}\ .$$

Point D of the above theorem plays a crucial role in the explicit computations done in [DPS+25, §**??**], as it allows to reduce global computations to ‘local’ ones. Its proof follows from the fact that *by construction* $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)$ only depends on $\widehat{X}_Z$ as a formal scheme, and not on the full pair (X, Z).

If there is an algebraic torus T acting on X such that Z is T-invariant, the above theorem extends verbatim to the equivariant setting.

We state and prove Theorem D in the more general context of an arbitrary motivic formalism $\mathbf{D}$, and the construction of $\mathbf{HA}^{\mathbf{D}}_{X,Z}$ is functorial in $\mathbf{D}$: any transformation $\mathbf{D}\to\mathbf{D}'$ induces a continuous algebra morphism $\mathbf{HA}^{\mathbf{D}}_{X,Z}\to\mathbf{HA}^{\mathbf{D}'}_{X,Z}$. When $\mathbf{D}$ is taken to be sheaves on the $\mathbb{C}$-analytification, one recovers the Borel-Moore homology discussed above, but this framework allows to deal simultaneously with motivic Chow groups and G-theory. As in [PS23], our approach hinges on the notion of *2-Segal structures* considered by Dyckerhoff and Kapranov [DK19].

Notation. We set $\mathbb{N} := \mathbb{Z}_{\geqslant 0}$.

We follow the notation introduced in [PS23, §1.6][1]. In particular, we use the *implicitly derived convention*: given a morphism of derived stacks $f\colon X\to Y$, we let

$$f^*\colon \mathsf{QCoh}(Y)\to\mathsf{QCoh}(X)$$

[1]With the difference that we denote with Spc instead of $\mathcal{S}$ the ∞-category of spaces.

be the *derived* pullback functor, and we let

$$f_*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$$

be the *derived* pushforward. Similarly, all fiber products, Hom sheaves and spaces, and tensor products will be understood in the derived sense, unless otherwise stated.

For a smooth projective complex surface X, let $K_0(X)$ be its *Grothendieck group* and let $N(X)$ be its *numerical Grothendieck group*, where the latter is defined by:

$$N(X) \coloneqq K_0(X)/\equiv ,$$

where $F_1 \equiv F_2$ if $\mathsf{ch}(F_1) = \mathsf{ch}(F_2)$ for $F_1, F_2 \in K_0(X)$. Then, $N(X)$ is a finitely generated free abelian group. In addition, we denote by $N_1(X)$ the subgroup of numerical equivalence classes of divisors on X.

We denote by $\mathsf{NS}(X)$ the *Neron-Severi group* of X. For a coherent sheaf E on X whose support has dimension less than or equal to one, we denote by $\ell(E) \in \mathsf{NS}(X)$ the fundamental one cycle of E.

Let $\mathsf{Coh}_{\leqslant 1}(X) \subset \mathsf{Coh}(X)$ be the subcategory of sheaves $\mathcal{E}$ with $\dim \mathsf{Supp}(\mathcal{E}) \leq 1$. We define the subgroup $N_{\leqslant 1}(X) \subset N(X)$ to be

$$N_{\leqslant 1}(X) \coloneqq \mathsf{Im}(K_0(\mathsf{Coh}_{\leqslant 1}(X)) \longrightarrow N(X)) .$$

Note that we have an isomorphism $N_{\leqslant 1}(X) \simeq \mathsf{NS}(X) \oplus \mathbb{Z}$ sending E to the pair $(\ell(E), \chi(E))$. We shall identify an element $v \in N_{\leqslant 1}(X)$ with $(\beta, n) \in \mathsf{NS}(X) \oplus \mathbb{Z}$ by the above isomorphism.

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2. Admissible indgeometric stacks

In this section, we introduce the class of *admissible indgeometric stacks*, that are essentially characterized by the property that their reduced is a geometric stack, see Theorem 2.20. We study their main properties, in particular establishing the existence of canonical ind-presentations, see Theorem 2.23. This foundational work will be later needed to define Borel-Moore homology for these stacks, together with its natural functorialities. The main source of examples for us would be the derived stack of coherent sheaves on a projective scheme X set-theoretically supported on a closed subscheme Z, that will be discussed in detail in §4.

2.1. **Generalities on derived stacks.** We fix a base commutative noetherian ring k. We denote by Poly_k the category of polynomial k-algebras in finitely many variables and we write dCAlg_k for the sifted completion of Poly_k. We refer to dCAlg_k as the ∞-category of *derived commutative k-algebras*. We also set $\mathsf{dAff}_k := \mathsf{dCAlg}_k^{\mathsf{op}}$, and we refer to it as the ∞-category of affine derived schemes. We set

$$\mathsf{PreSt}_k := \mathsf{PSh}(\mathsf{dAff}_k) \ .$$

Following our general convention for this part, PSh denotes the ∞-category of presheaves with coefficients in Spc. We now identify a series of important properties of derived prestacks that we need through the main text.

We start with *laftness*. For an integer $n \geqslant 0$, we let ${}^{\leqslant n}\mathsf{dAff}_k$ be the full subcategory spanned by n-coconnective affine derived schemes, i.e., the objects of the form $\mathsf{Spec}(A)$ where A is a derived commutative ring satisfying

$$\pi_i(A) \simeq 0 \quad \text{for } i > n \ .$$

We give the following:

Definition 2.1. A morphism $F \to G$ of derived prestacks is said to be *locally almost of finite type* (*laft*) if for every integer $n \geqslant 0$ and every cofiltered diagram $S_\bullet \colon I \to {}^{\leqslant n}\mathsf{dAff}_k$ with limit S, the square

$$\begin{array}{ccc} \operatorname*{colim}_{i \in I^{\mathsf{op}}} F(S_i) & \longrightarrow & F(S) \\ \downarrow & & \downarrow \\ \operatorname*{colim}_{i \in I^{\mathsf{op}}} G(S_i) & \longrightarrow & G(S) \end{array}$$

is a pullback square. We say that a derived prestack F is *laft* if the structural morphism $F \to \mathsf{Spec}(k)$ is laft. ⊘

Next, we discuss *convergence*. We let ${}^{<\infty}\mathsf{dAff}_k$ be the full subcategory spanned by *eventually coconnective* affine derived schemes, i.e., the objects of the form $\mathsf{Spec}(A)$ where A is an animated ring satisfying

$$\pi_i(A) \simeq 0 \quad \text{for } i \gg 0 \ .$$

We denote by ${}^{<\infty}j \colon {}^{<\infty}\mathsf{dAff}_k \hookrightarrow \mathsf{dAff}_k$ be the natural inclusion. We set

$${}^{<\infty}\mathsf{PreSt}_k := \mathsf{PSh}({}^{<\infty}\mathsf{dAff}_k) \ .$$

Left Kan extension along ${}^{<\infty}j$ allows to see ${}^{<\infty}\mathsf{PreSt}_k$ as a full subcategory of PreSt_k. We introduce the following:

Definition 2.2. A morphism $F \to G$ of derived prestacks is said to be *convergent* if for every affine derived scheme $S \in \mathsf{dAff}_k$ the square

$$\begin{array}{ccc} F(S) & \longrightarrow & \lim_{n \geqslant 0} F(\mathrm{t}_{\leqslant n}(S)) \\ \downarrow & & \downarrow \\ G(S) & \longrightarrow & \lim_{n \geqslant 0} G(\mathrm{t}_{\leqslant n}(S)) \end{array}$$

is a pullback. When $G = \mathsf{Spec}(k)$, we say that F is *convergent*. We denote by ${}^{\mathsf{conv}}\mathsf{PreSt}_k$ the full subcategory of PreSt_k spanned by convergent prestacks. ⊘

Recollection 2.3 (Convergence)**.** A derived prestack F is convergent if and only if the canonical morphism

$$F \longrightarrow {}^{<\infty}j_* \, {}^{<\infty}j^*(F)$$

is an equivalence, where ${}^{<\infty}j_*$ denotes the right Kan extension along ${}^{<\infty}j$. We set

$$ {}^{\mathsf{conv}}(-) := {}^{<\infty}j_* {}^{<\infty}j^* \colon \mathsf{PreSt}_k \longrightarrow \mathsf{PreSt}_k \ , $$

and we refer to it as the *convergent completion functor*. It provides a left adjoint for the inclusion ${}^{<\infty}\mathsf{PreSt}_k \hookrightarrow \mathsf{PreSt}_k$. Concretely, for every derived prestack F and every affine derived scheme S, one has

$$ {}^{\mathsf{conv}}F(S) := \lim_{n \geqslant 0} F(\mathsf{t}_{\leqslant n}(S)) \ . $$

In particular, a derived prestack and its convergent completion always share the same classical truncation. Notice that by construction ${}^{\mathsf{conv}}(-)$ commutes with arbitrary limits in PreSt_k. △

Finally, we discuss *infinitesimal cohesiveness*. Fix $S = \mathsf{Spec}(A) \in \mathsf{dAff}_k$ and $\mathcal{F} \in \mathsf{QCoh}(S)_{\geqslant 0}$. We let $A \oplus \mathcal{F}$ denote the split square-zero extension of A by the connective A-module $\mathcal{F}$, and we set

$$ S[\mathcal{F}] := \mathsf{Spec}(A \oplus \mathcal{F}) \in \mathsf{dAff}_k \ . $$

A *derivation of A with coefficients in $\mathcal{F}$* is a section of the canonical projection $A \oplus \mathcal{F} \to A$. Whenever $\pi_0(\mathcal{F}) = 0$, a derivation d with coefficients in $\mathcal{F}$ gives rise to a pushout square of the form

$$ \begin{array}{ccc} S[\mathcal{F}] & \xrightarrow{\delta_d} & S \\ \downarrow{\scriptstyle \delta_0} & & \downarrow \\ S & \longrightarrow & S_d[\mathcal{F}] \end{array} \ , $$

where δ_d corresponds to d and δ_0 to the zero derivation. Here $S_d[\mathcal{F}]$ is again an affine derived scheme and it is referred to as the *square-zero extension of S by $\mathcal{F}$ associated to d*.

Definition 2.4. A morphism $F \to G$ of derived prestacks is said to be *infinitesimally cohesive* if for every $S \in \mathsf{dAff}_k$, every $\mathcal{F} \in \mathsf{QCoh}(S)_{\geqslant 1}$ and every derivation $d \colon S[\mathcal{F}] \to S$, the square

$$ \begin{array}{ccc} F(S_d[\mathcal{F}]) & \longrightarrow & F(S) \times_{F(S[\mathcal{F}])} F(S) \\ \downarrow & & \downarrow \\ G(S_d[\mathcal{F}]) & \longrightarrow & G(S) \times_{G(S[\mathcal{F}])} G(S) \end{array} $$

is a pullback. We say that a derived prestack F is *infinitesimally cohesive* if the structural morphism $F \to \mathsf{Spec}(k)$ is infinitesimally cohesive. ⊘

Finally, we recall the notion of associated reduced prestack.

Notation 2.5. Let ${}^{\mathsf{red}}\mathsf{dAff}_k$ be the full subcategory of dAff_k spanned by discrete and reduced affine schemes. Let

$$ {}^{\mathsf{red}}j \colon {}^{\mathsf{red}}\mathsf{dAff}_k \longrightarrow \mathsf{dAff}_k $$

be the canonical inclusion. Given $F \in \mathsf{dSt}$, we set

$$ {}^{\mathsf{red}}F := {}^{\mathsf{red}}j_! {}^{\mathsf{red}}j^*(F) \ , $$

where ${}^{\mathsf{red}}j_!$ denotes the left Kan extension along ${}^{\mathsf{red}}j$. We refer to ${}^{\mathsf{red}}F$ as the *reduced stack of F* (see [CPT+17, §2.1]). ⊘

Definition 2.6. A morphism $f \colon X \to Y$ between derived stacks is said to be a *nil-equivalence* if the induced morphism ${}^{\mathsf{red}}f \colon {}^{\mathsf{red}}X \to {}^{\mathsf{red}}Y$ is an equivalence. ⊘

We conclude this section of preliminaries with a lemma that will be repeatedly useful throughout the text. Recall first that a map of derived prestack $F \to G$ is said to be *formally étale* if it admits 0 as cotangent complex.

Lemma 2.7. *Let $f \colon F \to G$ be a formally étale morphism between laft derived prestacks. Assume that f is convergent and infinitesimally cohesive. Then, f is an equivalence if and only if it is a nil-equivalence.*

Proof. If f is an equivalence, the same goes for ${}^{\mathrm{red}}f$. As for the converse, it suffices to show that for every affine derived scheme $S = \mathrm{Spec}(A)$, the map f induces an equivalence on S-points. Since both F and G are laft, it is enough to treat the case where S is almost of finite presentation over k. Proceeding by induction on the Postnikov tower, we further reduce to treat the case where S is underived. In this case, the map $A = \pi_0(A) \to \pi_0(A)_{\mathrm{red}}$ can be factored as a finite composition of square-zero extensions. Proceeding by induction on the length of this presentation, the conclusion follows combining convergence, infinitesimal cohesiveness and [Lur17, Theorem 7.4.1.23]. □

2.2. **Indgeometric stacks and their basic properties.** Mimicking [GR17, §2.1], we introduce the following:

Definition 2.8. A functor $F\colon \mathrm{dAff}_k^{\mathrm{op}} \to \mathrm{Spc}$ is said to be an *n-indgeometric derived stack* if it satisfies the following two conditions:

(1) F is convergent;

(2) there exists a filtered diagram $X_\bullet\colon I \to \mathrm{PreSt}_k$ together with a map
$$\phi\colon X \coloneqq \operatorname*{colim}_{\alpha\in I} X_\alpha \longrightarrow F$$
such that:
  (a) for every $\alpha \in I$, X_α is an n-geometric derived stack;
  (b) for every morphism $\alpha \to \beta$ in I, the transition map $X_\alpha \to X_\beta$ is a closed immersion almost of finite type;
  (c) the restriction
$$^{<\infty}j^*\colon X|_{^{<\infty}\mathrm{dAff}_k^{\mathrm{op}}} \longrightarrow F|_{^{<\infty}\mathrm{dAff}_k^{\mathrm{op}}}$$
  is an equivalence in $^{<\infty}\mathrm{PreSt}_k$.

We refer to the diagram $\{X_\alpha\}_{\alpha\in I}$ as a *presentation for F*.

We say that F is *indgeometric derived stack* if it is an n-indgeometric derived stack for some $n \geqslant 0$, and we denote by $\mathrm{indGeom}_k$ the full subcategory of PreSt_k spanned by indgeometric stacks. ⊘

Warning 2.9. Note that in [CW23, Definition 4.1], the authors introduce a notion of ind-geometric stacks, which is more general than ours. In particular, the definition of *loc. cit.* is the natural generalization to stacks of the definition of ind-schemes in [GR14]. Some of the results proved in this section overlap with similar ones proved in [CW23, §4.2 & §4.3]. △

Remark 2.10. Let $\{X_\alpha\}_{\alpha\in I}$ be a presentation for a indgeometric stack F and let
$$X \coloneqq \operatorname*{colim}_{\alpha\in I} X_\alpha\ ,$$
the colimit being computed in PreSt_k. Then, the structural morphism $\phi\colon X \to F$ is not necessarily an equivalence, but it induces a canonical equivalence $\overline{\phi}\colon {}^{\mathrm{conv}}X \xrightarrow{\sim} F$. △

Warning 2.11. Notice that in the above definition the presentation $\{X_\alpha\}_{\alpha\in I}$ is not part of the structure defining F. Notice also that, despite the terminology, F is not defined as an ind-object in some category. Of course, if we *fix* some presentation $\{X_\alpha\}_{\alpha\in I}$ for F, then we can define an ind-object
$$\text{“}\operatorname*{colim}_{\alpha\in I}\text{”}\, X_\alpha \in \mathrm{Ind}(\mathrm{dGeom}_k)\ ,$$
which, a priori, *depends on the presentation* of F. Our first step is to show that under some mild conditions, it is possible to attach to F a *canonical* ind-object. △

Before starting working towards the goal sketched in Warning 2.11, let us collect a couple of basic properties of indgeometric derived stacks. We start by the following simple lemma, asserting that asking for convergence is a harmless assumption (compare with Remark 2.10):

Lemma 2.12. *Let $X_\bullet \colon I \to \mathsf{PreSt}_k$ be a filtered diagram and assume that:*

(a) there exists an integer n such that for every $i \in I$, X_i is an n-geometric derived stack;

(b) for every morphism $i \to j$ in I, the morphism $X_i \to X_j$ is a closed immersion almost of finite type.

Set

$$X \coloneqq \operatorname*{colim}_{i \in I} X_i \ ,$$

the colimit being computed in PreSt_k. Then ${}^{\mathsf{conv}}X$ is an n-indgeometric derived stack.

Proof. Set $F \coloneqq {}^{\mathsf{conv}}X$. Then F is convergent by assumption, and since ${}^{\mathsf{conv}}(-)$ is a left adjoint to the inclusion ${}^{\mathsf{conv}}\mathsf{PreSt}_k \hookrightarrow \mathsf{PreSt}_k$, there is a canonical map $X \to F$. Since ${}^{<\infty}j \colon {}^{<\infty}\mathsf{dAff}_k \hookrightarrow \mathsf{dAff}_k$ is fully faithful, the counit of the adjunction ${}^{<\infty}j^* \dashv {}^{<\infty}j_*$ is an equivalence. The conclusion follows. □

Another easy but handy fact is that one can ask for a more restrictive condition on the presentation of an indgeometric derived stack at no cost. Indeed, we have the following.

Lemma 2.13. *For a convergent derived stack $F \colon \mathsf{dAff}_k^{\mathsf{op}} \to \mathsf{Spc}$, the following conditions are equivalent:*

(1) there exists a presentation $\{X_\alpha\}_{\alpha \in I}$ for F satisfying Definition 2.8–(2);

(2) there exists a presentation $\{X_\alpha\}_{\alpha \in I}$ for F satisfying Definition 2.8–(2) and moreover for every $\alpha \in I$, $X_\alpha \in {}^{<\infty}\mathsf{PreSt}_k$.

Proof. Obviously (2) implies (1).

For the converse, fix a presentation $\{X_\alpha\}_{\alpha \in I}$ for F satisfying Definition 2.8–(2). Considering $\mathbb{N}$ as a poset in the natural way, we see that $I \times \mathbb{N}$ is obviously filtered. We define a functor

$$Y_{\bullet,*} \colon \longrightarrow \mathsf{PreSt}_k$$

by the rule $Y_{\alpha,n} \coloneqq \mathsf{t}_{\leqslant n}(X_\alpha)$. There are canonical maps

$$\operatorname*{colim}_{n \in \mathbb{N}} \mathsf{t}_{\leqslant n}(X_\alpha) \longrightarrow X_\alpha \ ,$$

which induce a canonical morphism

$$\operatorname*{colim}_{(\alpha,n) \in I \times \mathbb{N}} Y_{\alpha,n} \longrightarrow F \ .$$

Conditions (2a) and (2b) in Definition 2.8 are trivially satisfied. As for condition (2c), our assumption is that for every $S \in {}^{<\infty}\mathsf{dAff}_k$ the canonical morphism

$$\operatorname*{colim}_{\alpha \in I} \mathsf{Map}_{\mathsf{PreSt}_k}(S, X_\alpha) \longrightarrow \mathsf{Map}_{\mathsf{PreSt}_k}(S, F)$$

is an equivalence. Thus, to conclude it is enough to argue that the canonical morphism

$$\operatorname*{colim}_{n \in \mathbb{N}} \mathsf{Map}_{\mathsf{PreSt}_k}(S, \mathsf{t}_{\leqslant n}(X_\alpha)) \longrightarrow \mathsf{Map}_{\mathsf{PreSt}_k}(S, X_\alpha)$$

is an equivalence. Choose an integer m such that $S \in {}^{\leqslant m}\mathsf{dAff}_k$. Then, we have

$$\begin{aligned} \operatorname*{colim}_{n \in \mathbb{N}} \mathsf{Map}_{\mathsf{PreSt}_k}(S, \mathsf{t}_{\leqslant n}(X_\alpha)) &\simeq \operatorname*{colim}_{n \in \mathbb{N}} \mathsf{Map}_{\mathsf{PreSt}_k}(S, \mathsf{t}_{\leqslant m}(\mathsf{t}_{\leqslant n}(X_\alpha))) \simeq \mathsf{Map}_{\mathsf{PreSt}_k}(S, \mathsf{t}_{\leqslant m}(X_\alpha)) \\ &\simeq \mathsf{Map}_{\mathsf{PreSt}_k}(S, X_\alpha) \ , \end{aligned}$$

whence the conclusion. □

Next, let us observe that indgeometric stacks automatically satisfy flat hyperdescent, and hence define objects in dSt_k. To see this, we need the following mild generalization of [GR17, Chapter 2, Proposition 1.2.2].

Lemma 2.14. *Let F be an n-indgeometric stack. Then for every m-truncated affine derived scheme $S \in {}^{\leqslant m}\mathsf{dAff}_k$, $\mathsf{Map}(S, F)$ is $(n+m)$-truncated.*

Proof. Since filtered colimits in Spc commute with taking homotopy groups and since colimits in PreSt_k are computed objectwise, we can replace F by an n-geometric derived stack. In this case, the statement is well known, and follows for instance from [TV08a, Lemma 2.1.1.2]. ☐

Proposition 2.15. *Every indgeometric stack satisfies flat hyperdescent, and in particular defines an object in* dSt_k.

Proof. Let $S_\bullet \to S$ be a flat hypercover in dAff and let F be a indgeometric stack. Consider the commutative square

$$\begin{array}{ccc} \mathrm{Map}(S,F) & \longrightarrow & \lim_{[n]\in\mathbf{\Delta}} \mathrm{Map}(S_n,F) \\ \downarrow & & \downarrow \\ \lim_{m\geqslant 0} \mathrm{Map}(\mathrm{t}_{\leqslant m}(S),F) & \longrightarrow & \lim_{[n]\in\mathbf{\Delta}} \mathrm{Map}(\mathrm{t}_{\leqslant m}(S_n),F) \end{array} .$$

We have to prove that the top horizontal map is an equivalence. Since F is convergent, the vertical maps are equivalences. Therefore, it is enough to prove the same statement assuming in addition that S (and hence each S_n) is m-truncated for some $m \geqslant 0$.

Choose a presentation $\{X_\alpha\}_\alpha$ for F by ℓ-geometric stacks. Then, we are called to check that the canonical map

$$\operatorname*{colim}_{\alpha} \mathrm{Map}(S,X_\alpha) \longrightarrow \lim_{[n]\in\mathbf{\Delta}} \operatorname*{colim}_{\alpha} \mathrm{Map}(S_n,X_\alpha)$$

is an equivalence. However, the previous lemma guarantees that $\mathrm{Map}(S_n,X_\alpha)$ is $(m+\ell)$-truncated for every $[n]\in\mathbf{\Delta}$ and every integer $\ell\geqslant 0$. Therefore, the right hand side is the limit of a diagram with values in $\mathsf{Spc}_{\leqslant m+\ell}$, and therefore the canonical map

$$\lim_{[n]\in\mathbf{\Delta}} \operatorname*{colim}_{\alpha} \mathrm{Map}(S_n,X_\alpha) \longrightarrow \lim_{[n]\in\mathbf{\Delta}_{\leqslant m+\ell+2}} \operatorname*{colim}_{\alpha} \mathrm{Map}(S_n,X_\alpha)$$

is an equivalence. At this point, the conclusion follows from the fact that filtered colimits commute with finite limits. ☐

2.3. **Admissibility.** For our applications, we only need a special class of indgeometric stacks, that we call *admissible indgeometric stacks*. They are akin to the notion of inf-scheme (and not of ind-inf-scheme) introduced in [GR17, Definition 2.3.1.2].

Definition 2.16. Let F be an indgeometric derived stack. We say that:

(1) F is *ind-qcqs* if it admits a presentation $\{X_\alpha\}$ where each X_α is a quasi-compact and quasi-separated geometric derived stack;

(2) F is *qcqs* if it is ind-qcqs with a presentation $\{X_\alpha\}_{\alpha\in I}$ where all the transition maps are nil-equivalences;

(3) F is *admissible* if there exists a (possibly transfinite) sequence

$$\varnothing = U_0 \hookrightarrow U_1 \hookrightarrow \cdots U_\alpha \hookrightarrow U_{\alpha+1} \hookrightarrow \cdots$$

of open Zariski immersions between qcqs indgeometric stacks, whose colimit in PreSt_k is F. We refer to the collection $\{U_\alpha\}$ as an *admissible open exhaustion of* F.

We let $\mathsf{indGeom}_k^{\text{ind-qcqs}}$, $\mathsf{indGeom}_k^{\text{qcqs}}$, and $\mathsf{indGeom}_k^{\text{adm}}$ denote the full subcategories of $\mathsf{indGeom}_k$ spanned by ind-qcqs, qcqs, and admissible objects, respectively. ⊘

Remark 2.17. Let F be an indgeometric derived stack and let $\{X_\alpha\}_{\alpha\in I}$ be a presentation. If $f\colon J\to I$ is a cofinal map, then the restricted diagram $\{X_{f(\beta)}\}_{\beta\in J}$ is again a presentation for F. In particular, F is qcqs if and only if there exists a cofinal map $f\colon J\to I$ such that all the transition maps in $\{X_{f(\beta)}\}_{\beta\in J}$ are nil-equivalences. Notice that since I is filtered, for every $\alpha_0\in I$ the forgetful map $I_{\alpha_0/}\to I$ is cofinal. In other words, F is qcqs if and only if it admits a presentation where the transition maps become nil-equivalences for "sufficiently large indexes". △

A very important fact concerning the notion of admissibility is that it can be tested at the reduced level, in particular completely disregarding the derived structure. To prove this fact, we need the following two technical results.

Lemma 2.18.

(1) Let F be an n-indgeometric stack and let $S \in {}^{<\infty}\mathsf{dGeom}_k^{\mathsf{qcqs}}$ be an eventually coconnective qcqs geometric derived stack. Then, for every presentation $\{X_\alpha\}_\alpha$ of F, the canonical map

$$\operatorname*{colim}_\alpha \mathsf{Map}_{\mathsf{dSt}}(S, X_\alpha) \longrightarrow \mathsf{Map}_{\mathsf{dSt}}(S, F)$$

is an equivalence.

(2) Let $S \in \mathsf{indGeom}_k^{\mathsf{qcqs}}$ and let $F \in \mathsf{indGeom}_k^{\mathsf{adm}}$ be an admissible indgeometric derived stack. Then, for any quasi-compact open exhaustion $\{U_\alpha\}$ of F, the canonical map

$$\operatorname*{colim}_\alpha \mathsf{Map}_{\mathsf{dSt}_k}(S, U_\alpha) \longrightarrow \mathsf{Map}_{\mathsf{dSt}_k}(S, F) \tag{2.1}$$

is an equivalence.

Proof. We first prove (1). Since S is geometric and qcqs, we can find a smooth hypercover $S_\bullet$ of S with the property that for every $[a] \in \mathbf{\Delta}$, S_a is a *finite* disjoint union of derived affines. Furthermore, if $S \in {}^{<m}\mathsf{dGeom}_k^{\mathsf{qc}}$, then we can equally assume that $S_a \in {}^{<m}\mathsf{dAff}_k$. At this point, consider the following commutative diagram:

$$\begin{array}{ccc} \mathsf{Map}_{\mathsf{dSt}_k}(S, F) & \longrightarrow & \operatorname{colim}_\alpha \mathsf{Map}_{\mathsf{dSt}_k}(S, X_\alpha) \\ \downarrow & & \downarrow \\ \lim_{[a]\in\mathbf{\Delta}} \mathsf{Map}_{\mathsf{dSt}_k}(S_a, F) & \longrightarrow & \operatorname{colim}_\alpha \lim_{[a]\in\mathbf{\Delta}} \mathsf{Map}_{\mathsf{dSt}_k}(S_a, X_\alpha) \end{array} .$$

Proposition 2.15 guarantees that the vertical maps are equivalences. It is therefore enough to prove that the bottom horizontal arrow is an equivalence. Applying Lemma 2.14, we see that all the mapping spaces are $(n+m)$-truncated. We can therefore replace $\mathbf{\Delta}$ by $\mathbf{\Delta}_{\leqslant n+m+2}$, whence the conclusion.

We now prove (2). For every index α, the map $i_\alpha \colon U_\alpha \to F$ is representable by open Zariski immersions. It immediately follows that the canonical map

$$U_\alpha \longrightarrow U_\alpha \times_F U_\alpha$$

is an equivalence. In other words, the map i_α is (-1)-truncated. It follows that the induced map $\mathsf{Map}_{\mathsf{dSt}_k}(S, U_\alpha) \to \mathsf{Map}_{\mathsf{dSt}_k}(S, F)$ is (-1)-truncated as well in Spc. Since the colimit is filtered, it follows that the map (2.1) is also (-1)-truncated. To complete the proof, it is enough to check that it is surjective on connected components. Let therefore $S \to F$ be a morphism in dSt_k. Since i_α is representable by open Zariski immersions, we immediately see that the canonical map

$$\mathsf{Map}_{\mathsf{dSt}_k}(S, U_\alpha) \longrightarrow \mathsf{Map}_{\mathsf{dSt}_k}({}^{\mathsf{red}}S, U_\alpha) \times_{\mathsf{Map}_{\mathsf{dSt}_k}({}^{\mathsf{red}}S, F)} \mathsf{Map}_{\mathsf{dSt}_k}(S, F)$$

is an equivalence. In other words: a map $f \colon S \to F$ factors through U_α if and only if its restriction ${}^{\mathsf{red}}f \colon {}^{\mathsf{red}}S \to F$ factors through U_α. We can therefore replace S by ${}^{\mathsf{red}}S$; in other words, we can assume from the very beginning that $S \in \mathsf{dGeom}_k^{\mathsf{qc}}$. In this case, set $V_\alpha \coloneqq S \times_F U_\alpha$. This gives an increasing open exhaustion of S by the open V_α. Since S is quasi-compact, there must exist an index α such that $S = V_\alpha$, so that f must factor through U_α. The proof is thus complete. □

Lemma 2.19. *Let F be a derived stack and let $G \to {}^{\mathsf{red}}F$ be a morphism representable by open Zariski immersions. Then, there exists an essentially unique derived stack $\overline{G}$ and dashed arrows making the square*

$$\begin{array}{ccc} G & \dashrightarrow & \overline{G} \\ \downarrow & & \downarrow \\ {}^{\mathsf{red}}F & \longrightarrow & F \end{array}$$

a pullback. Moreover, the map $\overline{G} \to F$ is representable by open Zariski immersions.

Proof. Recall that the canonical map

$$({}^{\mathrm{red}}F)_{\mathsf{dR}} \longrightarrow F_{\mathsf{dR}}$$

is an equivalence.[2] We define $\overline{G}$ as the fiber product

$$\begin{array}{ccccc} G & \dashrightarrow & \overline{G} & \longrightarrow & G_{\mathsf{dR}} \\ \downarrow & & \downarrow & & \downarrow \\ {}^{\mathrm{red}}F & \longrightarrow & F & \longrightarrow & F_{\mathsf{dR}} \end{array} .$$

The canonical map $G \to G_{\mathsf{dR}}$ makes the outer rectangle commutative, so the existence of the dashed arrow follows immediately. Furthermore, unraveling the definitions we see that the outer rectangle is in fact a pullback, so the same holds for the square on the left. We are thus reduced to check that the map $G_{\mathsf{dR}} \to F_{\mathsf{dR}}$ is representable by open Zariski immersions. Fix $S \in \mathsf{dAff}_k$ and a morphism $x \colon S \to F_{\mathsf{dR}}$, corresponding to a morphism ${}^{\mathrm{red}}S \to F$. By assumption, the map $U := S \times_{{}^{\mathrm{red}}F} G \to S$ is an open Zariski immersion. It is thus enough to check that the square

$$\begin{array}{ccc} U & \longrightarrow & G_{\mathsf{dR}} \\ \downarrow & & \downarrow \\ S & \longrightarrow & F_{\mathsf{dR}} \end{array}$$

is a pullback. Notice that both vertical maps are (-1)-truncated. Then, the conclusion follows by observing that for every $T \in \mathsf{dAff}_k$, a map $T \to S$ factors through U if and only if ${}^{\mathrm{red}}T \to {}^{\mathrm{red}}S$ factors through ${}^{\mathrm{red}}U$, hence if and only if the composite ${}^{\mathrm{red}}T \to {}^{\mathrm{red}}F$ factors through G. □

Theorem 2.20. *Let $\mathcal{X}$ be an indgeometric derived stack.*

(1) $\mathcal{X}$ is a qcqs indgeometric derived stack if and only if ${}^{\mathrm{red}}\mathcal{X}$ is a qcqs geometric stack.

(2) $\mathcal{X}$ is an admissible indgeometric derived stack if and only if ${}^{\mathrm{red}}\mathcal{X}$ is a quasi-separated geometric stack.

Proof. We first establish the "only if" implications. Fix a presentation $\{X_\alpha\}_{\alpha \in I}$ for F. Then it follows that

$${}^{\mathrm{red}}F \simeq \operatorname*{colim}_{\alpha \in I} {}^{\mathrm{red}}X_\alpha ,$$

the colimit being computed in PreSt_k. If F is qcqs, the above colimit becomes eventually constant, so both statements follow automatically.

We now complete the proof of point (1). Assume that ${}^{\mathrm{red}}\mathcal{X}$ is a qcqs geometric derived stack. Pick any qcqs presentation $\{X_\alpha\}_{\alpha \in I}$ for $\mathcal{X}$. The canonical map

$$\operatorname*{colim}_{\alpha \in I} {}^{\mathrm{red}}X_\alpha \longrightarrow {}^{\mathrm{red}}\mathcal{X}$$

is then an isomorphism. Lemma 2.18 guarantees that ${}^{\mathrm{red}}\mathcal{X}$ is a retract of ${}^{\mathrm{red}}X_\alpha$ for some $\alpha \in I$. Thus, for every $\beta \in I_{\alpha/}$, ${}^{\mathrm{red}}\mathcal{X}$ is also a retract of ${}^{\mathrm{red}}X_\beta$. Moreover, since we required the transition maps to be closed embeddings, we see that they are (-1)-truncated morphisms in the category of *reduced* geometric derived stacks. Thus, the map ${}^{\mathrm{red}}X_\beta \to {}^{\mathrm{red}}\mathcal{X}$ is also (-1)-truncated, and therefore, having a retract, it is an equivalence. It follows that for every $\beta \in I_{\alpha/}$, the map $X_\beta \to \mathcal{X}$ is a nil-equivalence, and therefore that the transition maps in $\{X_\beta\}_{\beta \in I_{\alpha/}}$ are nil-equivalences. Thus, the conclusion follows from Remark 2.17.

[2] Here $(-)_{\mathsf{dR}}$ is the de Rham stack. Concretely, for any derived stack F and every affine scheme S, one has $F_{\mathsf{dR}}(S) := F({}^{\mathrm{red}}S)$.

Now complete the proof of point (2). Assume therefore that ${}^{\mathrm{red}}\mathcal{X}$ is a quasi-separated geometric derived stack. Then [PS23, Lemma A.1] allows to pick a quasi-compact open exhaustion $\{U_\alpha\}_{\alpha\in I}$ of ${}^{\mathrm{red}}\mathcal{X}$. Since ${}^{\mathrm{red}}\mathcal{X}$ is quasi-separated, every the same goes for every U_α. Applying Lemma 2.19, we see that there exists a uniquely defined derived stack V_α equipped with a Zariski open immersion $V_\alpha \to \mathcal{X}$ such that the square

$$\begin{array}{ccc} U_\alpha & \longrightarrow & V_\alpha \\ \downarrow & & \downarrow \\ {}^{\mathrm{red}}\mathcal{X} & \longrightarrow & \mathcal{X} \end{array}$$

a pullback. It follows that V_α is an indgeometric derived stack, and that ${}^{\mathrm{red}}V_\alpha \simeq U_\alpha$. The uniqueness of V_α immediately implies that if $U_\alpha \subseteq U_\beta$ then $V_\alpha \subseteq V_\beta$. Thus, the first half of the proof guarantees that V_α is a qcqs geometric derived stack. Since $\{V_\alpha\}_{\alpha\in I}$ obviously is an open exhaustion of $\mathcal{X}$, it follows that $\mathcal{X}$ is an admissible indgeometric derived stack. □

Corollary 2.21. *Let $\mathcal{X}$ be an admissible indgeometric derived stack and let $\mathcal{U} \to \mathcal{X}$ be a morphism representable by quasi-compact open Zariski immersions. Then, $\mathcal{U}$ is a qcqs indgeometric derived stack.*

Proof. The representability assumption implies that $\mathcal{U} \to \mathcal{X}$ is a convergent morphism. Since $\mathcal{X}$ is convergent, the same goes for $\mathcal{U}$. Besides, if $\{X_\alpha\}_{\alpha\in I}$ is a presentation for $\mathcal{X}$, then it follows formally that $\{\mathcal{U} \times_{\mathcal{X}} X_\alpha\}_{\alpha\in I}$ is a presentation for $\mathcal{U}$. Therefore, $\mathcal{U}$ is an indgeometric derived stack. Besides, there is a canonical morphism

$${}^{\mathrm{red}}\mathcal{U} \longrightarrow \mathcal{U} \times_{\mathcal{X}} {}^{\mathrm{red}}\mathcal{X} ,$$

and since $\mathcal{U} \to \mathcal{X}$ is representable by open Zariski immersions, we immediately see that this morphism is an equivalence. Since ${}^{\mathrm{red}}\mathcal{X}$ is a quasi-separated geometric stack and ${}^{\mathrm{red}}\mathcal{U}$ is a quasi-compact open inside ${}^{\mathrm{red}}\mathcal{X}$, we obtain that ${}^{\mathrm{red}}\mathcal{U}$ is qcqs geometric stack and the conclusion follows from Theorem 2.20–(1). □

Corollary 2.22. *Let $\mathcal{X}$ be a qcqs indgeometric derived stack and let $\{X_\alpha\}_{\alpha\in I}$ be any presentation. Define I' to be the full subcategory of I spanned by those indexes α for which the structural morphism $X_\alpha \to \mathcal{X}$ is a nil-equivalence. Then I' is a cofinal subcategory of I, and in particular $\{X_\alpha\}_{\alpha\in I'}$ is again a presentation for $\mathcal{X}$.*

Proof. Consider the structural morphism ${}^{\mathrm{red}}\mathcal{X} \to \mathcal{X}$. Since $\mathcal{X}$ is a qcqs indgeometric derived stack, Theorem 2.20 implies that ${}^{\mathrm{red}}\mathcal{X}$ is a qcqs geometric stack. Besides,

$${}^{\mathrm{red}}\mathcal{X} \simeq \operatorname*{colim}_{\alpha\in I} {}^{\mathrm{red}}X_\alpha .$$

Thus, Lemma 2.18 implies that ${}^{\mathrm{red}}\mathcal{X}$ is a retract of ${}^{\mathrm{red}}X_\alpha$ for some $\alpha \in I$. Since ${}^{\mathrm{red}}X_\alpha \to {}^{\mathrm{red}}\mathcal{X}$ is a monomorphism, it follows that ${}^{\mathrm{red}}\mathcal{X} \simeq {}^{\mathrm{red}}X_\alpha$. This shows that I' is non-empty. Replacing I by $I_{\beta/}$ for any $\beta \in I$, we conclude that I' is a cofinal subcategory of I. □

In order to define Borel-Moore homology of admissible indgeometric derived stacks in §3, we need to make the ind-structure on such objects explicit. Following [DPS22, §II.7], we consider the restricted Yoneda embeddings

$$\Phi_{\mathsf{ind\text{-}qcqs}} \colon \mathsf{indGeom}_k^{\mathsf{ind\text{-}qcqs}} \longrightarrow \mathsf{PSh}({}^{<\infty}\mathsf{dGeom}_k^{\mathsf{qcqs}}) ,$$

and

$$\Phi_{\mathsf{qcqs}} \colon \mathsf{indGeom}_k^{\mathsf{adm}} \longrightarrow \mathsf{PSh}(\mathsf{indGeom}_k^{\mathsf{qcqs}}) .$$

The choices of ${}^{<\infty}\mathsf{dGeom}_k^{\mathsf{qcqs}}$ and $\mathsf{indGeom}_k^{\mathsf{qcqs}}$ are justified by the following result.

Theorem 2.23.

(1) *The functor* $\Phi_{\mathsf{ind\text{-}qcqs}}$ *commutes with limits, is fully faithful and factors through* $\mathsf{Ind}({}^{<\infty}\mathsf{dGeom}_k^{\mathsf{qcqs}})$. *Furthermore, if* $\{X_\alpha\}_{\alpha\in I}$ *is a qcqs presentation for* $\mathcal{X} \in \mathsf{indGeom}_k^{\mathsf{ind\text{-}qcqs}}$, *then*

$$\Phi_{\mathsf{ind\text{-}qcqs}}(\mathcal{X}) \simeq \underset{(n,\alpha)\in\mathbb{N}\times I}{\text{“colim”}} \mathsf{t}_{\leqslant n} X_\alpha ,$$

where $\mathsf{t}_{\leqslant n}$ *denotes the n-th Postnikov truncation functor.*

(2) *The functor* Φ_{qcqs} *commutes with limits, is fully faithful and factors through* $\mathsf{Ind}(\mathsf{indGeom}_k^{\mathsf{qcqs}})$. *Furthermore, if* $\{U_\alpha\}_{\alpha\in I}$ *is a quasi-compact open exhaustion of F, then*

$$\Phi_{\mathsf{qcqs}}(\mathcal{X}) \simeq \underset{\alpha\in I}{\text{“colim”}}\, X_\alpha .$$

Proof. Choose an ind-qcqs presentation $\{X_\alpha\}_{\alpha\in I}$ for $\mathcal{X}$. Using Lemma 2.13, we see that

$$\{\mathsf{t}_{\leqslant n}(X_\alpha)\}_{(\alpha,n)\in I\times\mathbb{N}}$$

is again a presentation for $\mathcal{X}$, whose pieces are eventually coconnective. At this point, the conclusion follows combining [DPS22, Proposition II.7.6] and Lemma 2.18. □

Notation 2.24. The two parts of Theorem 2.23 provide us with a limit-preserving and fully faithful functor

$$(-)_{\mathsf{ind}} \colon \mathsf{indGeom}_k^{\mathsf{adm}} \to \mathsf{Ind}(\mathsf{indGeom}^{\mathsf{qcqs}}) \to \mathsf{Ind}(\mathsf{Ind}({}^{<\infty}\mathsf{dGeom}_k^{\mathsf{qcqs}})) ,$$

which we refer to as the *indization functor*. ⊘

Corollary 2.25. *The full subcategory* $\mathsf{indGeom}_k^{\mathsf{ind\text{-}qcqs}} \subset \mathsf{PreSt}_k$ *is closed under finite limits.*

Proof. It is enough to consider the case of fiber products. For this, apply first Corollary B.12 to $\mathsf{Ind}({}^{<\infty}\mathsf{dGeom}_k^{\mathsf{qcqs}})$ with P the collection of closed immersions, and then combine it with Theorem 2.23–(1). □

Corollary 2.26. *The category* $\mathsf{indGeom}_k^{\mathsf{adm}}$ *is closed under finite limits in* PreSt_k.

Proof. It suffices to show that $\mathsf{indGeom}_k^{\mathsf{adm}}$ is closed under fiber products. Consider therefore a fiber product $\mathcal{W} := \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ of admissible indgeometric stacks. Corollary 2.25 implies that $\mathcal{W}$ is again indgeometric. On the other hand,

$${}^{\mathsf{red}}\mathcal{W} \simeq {}^{\mathsf{red}}({}^{\mathsf{red}}\mathcal{X} \times_{{}^{\mathsf{red}}\mathcal{Y}} {}^{\mathsf{red}}\mathcal{Z}) .$$

Then the conclusion follows combining Theorem 2.20 and the fact that quasi-separated geometric derived stacks are closed under fiber products. □

We will need one more criterion to establish admissibility. To state it, we first prove:

Lemma 2.27. *Let* $f \colon \mathcal{X} \to \mathcal{Y}$ *be a morphism of derived indgeometric stacks. Assume that f is representable by smooth geometric stacks. Then the square*

$$\begin{array}{ccc} {}^{\mathsf{red}}\mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ {}^{\mathsf{red}}\mathcal{Y} & \longrightarrow & \mathcal{Y} \end{array}$$

is a pullback.

Proof. Choose a presentation $\{Y_\alpha\}_{\alpha\in I}$ for $\mathcal{Y}$ and set $X_\alpha := Y_\alpha \times_{\mathcal{Y}} \mathcal{X}$. Since f is representable by geometric stacks, it follows formally that $\{X_\alpha\}_{\alpha\in I}$ is a presentation for $\mathcal{X}$. Then Proposition 2.15 guarantees that

$$\mathcal{Y} \simeq \underset{\alpha\in I}{\operatorname{colim}}\, Y_\alpha \qquad \text{and} \qquad \mathcal{X} \simeq \underset{\alpha\in I}{\operatorname{colim}}\, X_\alpha$$

hold in dSt_k. In particular, since ${}^{\mathsf{red}}j \colon {}^{\mathsf{red}}\mathsf{dAff}_k \to \mathsf{dAff}_k$ is cocontinuous, it follows from [PY16, Lemma 2.20] that both ${}^{\mathsf{red}}j_!$ and ${}^{\mathsf{red}}j^*$ commute with colimits and hence that the canonical morphisms

$$\operatorname*{colim}_{\alpha \in I} {}^{\mathsf{red}}Y_\alpha \longrightarrow {}^{\mathsf{red}}\mathcal{Y} \quad \text{and} \quad \operatorname*{colim}_{\alpha \in I} {}^{\mathsf{red}}X_\alpha \longrightarrow {}^{\mathsf{red}}\mathcal{X}$$

are equivalences. Since these colimit are filtered, it follows that it suffices to argue that for each $\alpha \in I$ the square

$$\begin{array}{ccc} {}^{\mathsf{red}}X_\alpha & \longrightarrow & X_\alpha \\ \downarrow & & \downarrow{\scriptstyle f_\alpha} \\ {}^{\mathsf{red}}Y_\alpha & \longrightarrow & Y_\alpha \end{array}$$

is a pullback. Notice that after applying ${}^{\mathsf{red}}j^*$, the horizontal arrows become equivalences and therefore the square becomes a pullback. It suffices then to argue that the fiber product ${}^{\mathsf{red}}Y_\alpha \times_{Y_\alpha} X_\alpha$ is automatically reduced. Since f_α is smooth by assumption, this is automatic. □

Corollary 2.28. *Let*

$$\begin{array}{ccc} \mathcal{X}_U & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ U & \xrightarrow{u} & Y \end{array}$$

be a pullback square of derived stacks. Assume that Y is a geometric derived stack, that u is a smooth epimorphism and that $\mathcal{X}$ is indgeometric. Then $\mathcal{X}$ is admissible if and only if $\mathcal{X}_U$ is admissible.

Proof. Both U and $\mathcal{Y}$ are admissible, so if $\mathcal{X}$ is admissible Corollary 2.26 implies that $\mathcal{X}_U$ is admissible. For the vice-versa, we observe that since u is smooth the square

$$\begin{array}{ccc} {}^{\mathsf{red}}U & \longrightarrow & {}^{\mathsf{red}}Y \\ \downarrow & & \downarrow \\ U & \longrightarrow & Y \end{array}$$

is a pullback square. Let

$$\mathcal{X}' := {}^{\mathsf{red}}U \times_{{}^{\mathsf{red}}Y} {}^{\mathsf{red}}\mathcal{X} .$$

The canonical morphism $\mathcal{X}' \to \mathcal{X}_U$ induces an equivalence ${}^{\mathsf{red}}\mathcal{X}' \simeq {}^{\mathsf{red}}\mathcal{X}_U$. Thus, if $\mathcal{X}_U$ is admissible, ${}^{\mathsf{red}}\mathcal{X}_U$ is a quasi-separated by Theorem 2.20. By assumption, the canonical morphism $\mathcal{X}' \to {}^{\mathsf{red}}\mathcal{X}$ is representable by smooth derived stacks. In particular, Lemma 2.27 guarantees that $\mathcal{X}'$ is itself reduced, and therefore it is a quasi-separated geometric stack. Since ${}^{\mathsf{red}}U \to {}^{\mathsf{red}}Y$ is again a smooth epimorphism, we deduce from [TV08b, Corollary 1.3.4.5] that ${}^{\mathsf{red}}\mathcal{X}$ is itself a quasi-separated geometric stack, so the conclusion follows from Theorem 2.20. □

3. Borel-Moore homology for admissible indgeometric stacks

In this section we define Borel-Moore homology for admissible indgeometric stacks, and establish its basic functorialities. We use in an essential way the motivic theory developed by A. Khan and its collaborators (see e.g. [Kha19, Kha21]). The extension from geometric to admissible indgeometric is a formal procedure, essentially made possible thanks to the theory developed in [DPS22, §II.1]. We refer to *loc. cit.* for all the relevant notation.

3.1. **Borel-Moore homology for admissible indgeometric derived stacks.** Fix a *motivic formalism* $\mathbf{D}^*$ (in the sense of [DPS22, Definition II.1.5]) and its associated six-functors formalism

$$^{\mathrm{pro}}\mathbf{D}^*_! \colon \mathsf{Corr}\big(\mathsf{dGeom}_k^{\mathsf{qs}}\big)_{\mathsf{lft,all}} \longrightarrow \mathsf{CAlg}(\mathsf{Pro}(\mathsf{Cat}_\infty)) \ .$$

Let $S \in \mathsf{dGeom}_k^{\mathsf{qcqs}}$ be a qcqs geometric derived stack. For an oriented $\mathcal{A} \in \mathsf{CAlg}(\mathbf{D}(S))$ and an abelian subgroup $\Gamma \subset \mathsf{Pic}(\mathbf{D}(S))$ closed under Thom twists. In [DPS22, §II.1.4], we defined for a quasi-separated geometric derived stack locally of finite type $\mathcal{X}$ over S its *relative Borel-Moore homology groups with coefficients in* $\mathcal{A}$:

$$\mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{X}/S;\mathcal{A}) \ ,$$

as a Γ-graded *topological* abelian group. This is an extension of Khan's framework [Kha19], and the topology keeps track of the non-quasi-compactness of $\mathcal{X}$.

Remark 3.1 (Underlying topology). Let us take $S = \mathsf{Spec}(k)$. It follows combining [DPS22, Remarks II.1.44 and II.1.49] that $\mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{X}/S;\mathcal{A})$ is explicitly given by the formula

$$\mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{X}/S;\mathcal{A}) \simeq \bigoplus_{\alpha\in\pi_0(\mathcal{X})} \lim_{\mathcal{U}\subset_{\mathsf{qc}}\mathcal{X}_\alpha} \mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{U}/S;\mathcal{A}) \ ,$$

where:

(1) the direct sum is taken over the set of connected components of $\mathcal{X}$, and if $\alpha \in \pi_0(\mathcal{X})$, we denote by $\mathcal{X}_\alpha$ the corresponding connected component;

(2) the limit is taken over the set of quasi-compact open substacks of $\mathcal{X}_\alpha$;

(3) each group $\mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{U}/S;\mathcal{A})$ can be expressed in terms of the six-operations, and essentially coincide with Khan's motivic Borel-Moore homology (except for the fact that we are summing together all twists by elements in Γ, see [DPS22, Remark II.1.24] for more on this point);

(4) both the inverse limit and the coproduct are topologized in the natural way, endowing the building pieces $\mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{U}/S;\mathcal{A})$ with the discrete topology.

Technically speaking, the results obtained in [DPS22] are strictly stronger, in that we do not work with topological abelian groups, but with free categorical (co)completions. However, in applications only this underlying topological structure will be taken into account.
△

Contrary to [Kha19], the above homology groups are not obtained directly out of the six-functors formalism, as there is a certain *renormalization* procedure that needs to be carried out and that allows to have larger functoriality properties (see [DPS22, Remarks II.1.50 and II.1.54]).

Definition 3.2. A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is said to be *locally rpas* if for every connected component $\mathcal{X}_0 \subset \mathcal{X}$ the induced map $\mathcal{X}_0 \to \mathcal{Y}$ is representable by proper algebraic spaces; and that is said to be *finitely connected* if for every connected component $\mathcal{Y}_0 \subset \mathcal{Y}$ the preimage $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_0$ has finitely many connected components. ⊘

With this terminology, we establish in [DPS22, §II.1.4] the following functoriality properties for $\mathsf{H}_*^{\mathbf{D},\Gamma}(-/S;\mathcal{A})$:

- for derived lci, quasi-compact and finitely connected morphisms $f\colon \mathcal{X} \to \mathcal{Y}$, a *Gysin pullback*

$$f^! \colon \mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{Y}/S;\mathcal{A}) \longrightarrow \mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{X}/S;\mathcal{A}) \ ,$$

  which is continuous for the natural topology on both sides;

- for locally rpas morphisms $f\colon \mathcal{X} \to \mathcal{Y}$, a proper pushforward

$$f_* \colon \mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{X}/S;\mathcal{A}) \longrightarrow \mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{Y}/S;\mathcal{A}) \ .$$

Together with the exterior products (essentially induced by the algebra structure on $\mathcal{A}$), these properties produce a lax-monoidal functor

$$\mathsf{H}_0^{\mathbf{D},\Gamma}(-/S;\mathcal{A})\colon \mathsf{Corr}^{\times}(\mathsf{dGeom}_S^{\mathsf{qs}})_{\mathsf{qc.lci}\,\cap\,\mathsf{fconn},\,\mathsf{lrpas}} \longrightarrow \mathsf{Pro}^{\sqcup}(\mathsf{Mod}_R^{\heartsuit})\ ,$$

see [DPS22, Theorem II.1.53].

We now extend this framework to admissible indgeometric derived stacks. In light of Theorem 2.20, we have a well defined functor

$$^{\mathsf{red}}(-)\colon \mathsf{indGeom}_S^{\mathsf{adm}} \longrightarrow \mathsf{Geom}_S^{\mathsf{qs}}\ .$$

We extend the terminology above:

Definition 3.3. A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of admissible indgeometric derived stacks is *finitely connected* (resp. *locally rpas*) if the underlying morphism $^{\mathsf{red}}f\colon {}^{\mathsf{red}}\mathcal{X} \to {}^{\mathsf{red}}\mathcal{Y}$ has the same property. On the other hand, we say that f is *quasi-compact* (resp. *derived lci*) if it is representable by quasi-compact (resp. derived lci) geometric derived stacks.
⊘

With this terminology, the proper pushforward and the exterior products carry over from [DPS22, Theorem II.1.53]. Concerning the Gysin pullback, we have the following.

Remark 3.4 (Gysin pullback). Fix a qcqs geometric derived stack S and a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of admissible indgeometric derived stacks over S. Assume that f is quasi-compact, derived lci and finitely connected. In this situation, the morphism $f'\colon \mathcal{X} \times_{\mathcal{Y}} {}^{\mathsf{red}}\mathcal{Y} \to {}^{\mathsf{red}}\mathcal{Y}$ has the same properties, and it is nil-equivalent to f. Besides, $^{\mathsf{red}}\mathcal{Y}$ is a quasi-separated geometric derived stack by Theorem 2.20, and the same goes for

$$^{\mathsf{red}}\Big(\mathcal{X} \times_{\mathcal{Y}} {}^{\mathsf{red}}\mathcal{Y}\Big) \simeq {}^{\mathsf{red}}\mathcal{X}\ .$$

Thus, the Gysin pullback of [DPS22, Theorem II.1.53] yields a well defined map

$$f^!\colon \mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{Y}/S;\mathcal{A}) \longrightarrow \mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{X}/S;\mathcal{A})\ .$$

△

We can summarize this discussion stating the following:

Theorem 3.5. *Let $S \in \mathsf{dGeom}_k^{\mathsf{qcqs}}$, $\mathcal{A} \in \mathsf{CAlg}(\mathbf{D}(S))$ and let $\Gamma \subseteq \mathsf{Pic}(\mathbf{D}^*(S))$ be an abelian subgroup. Assume that $\mathcal{A}$ is oriented and that Γ is closed under Thom twists. Then the construction*

$$\mathsf{H}_0^{\mathbf{D},\Gamma}(-/S;\mathcal{A})\colon \mathsf{Corr}^{\times}(\mathsf{indGeom}_S^{\mathsf{adm}})_{\mathsf{qc.lci}\,\cap\,\mathsf{fconn},\,\mathsf{lrpas}} \longrightarrow \mathsf{Pro}^{\sqcup}(\mathsf{Mod}_R^{\heartsuit})$$

that sends $\mathcal{X} \to S$ to the topological R-module $\mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{X}/S;\mathcal{A})$ (disregarding the extra Γ-grading) and a correspondence

$$\begin{array}{ccc} & \mathcal{Z} & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle p} \\ \mathcal{X} & & \mathcal{Y} \end{array}$$

where f is representable by finitely connected and quasi-compact lci geometric derived stacks and p is locally rpas to the composite

$$p_* \circ f^!\colon \mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{X}/S;\mathcal{A}) \longrightarrow \mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{Y}/S;\mathcal{A})$$

defines a lax-monoidal functor. Moreover, if $(\mathbf{D}',\mathcal{A}',\Gamma')$ is a second motivic formalism with a choice of an oriented ring of coefficients $\mathcal{A}'$ and an abelian subgroup Γ' closed under Thom twist, then a morphism

$$(s,\phi)\colon (\mathbf{D},\mathcal{A},\Gamma) \longrightarrow (\mathbf{D}',\mathcal{A}',\Gamma')$$

induces a lax symmetric monoidal transformation

$$\mathsf{H}_0^{\mathbf{D},\Gamma}(-/S;\mathcal{A}) \longrightarrow \mathsf{H}_0^{\mathbf{D}',\Gamma'}(-/S;\mathcal{A}')\ .$$

Example 3.6. Let $S \in \mathsf{dGeom}_k^{\mathsf{qcqs}}$ and $\mathcal{X} \in \mathsf{indGeom}_S^{\mathsf{adm}}$.

(1) when $\mathbf{D}^* := \mathbf{DM}^*_{\mathbb{Q}}$ is the *rational Voevodsky's formalism* (cf. [DPS22, Example II.1.8]) and $\mathcal{A} := \mathsf{H}\mathbb{Q}$ is the motivic Eilenberg-MacLane $\mathbb{E}_\infty$-ring spectrum, and $\Gamma := \mathbb{Z}\langle 1\rangle$, we write

$$\mathsf{H}^{\mathsf{mot}}_\bullet(\mathcal{X}/S;0) := \mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{X}/S;\mathsf{H}\mathbb{Q}) .$$

This recovers (motivically defined) Chow groups of $\mathcal{X}$.

(2) When $\mathbf{D}^* := \mathbf{DM}^*_{\mathbb{Q}}$ is the rational Voevodsky's formalism and $\mathcal{A} := \mathsf{KGL}^{\mathsf{et}}$ is the étale hypersheafification of the algebraic K-theory spectrum, and $\Gamma := \mathbb{Z}\langle 1\rangle$, we write

$$\mathbf{G}(\mathcal{X}/S) := \mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{X}/S;\mathsf{KGL}^{\mathsf{et}}) .$$

This recovers the algebraic G-theory of $\mathcal{X}$.

(3) When $\mathbf{D}^* := {}^{\mathsf{top}}\mathbf{D}$ is the *topological formalism* (cf. [DPS22, Example II.1.11]), $\mathcal{A} := \mathbb{Q}$, and $\Gamma := \mathbb{Z}\langle 1/2\rangle$ (see [DPS22, Remark II.1.24]), we simply write

$$\mathsf{H}^{\mathsf{BM}}_\bullet(\mathcal{X}/S) := \mathsf{H}_0^{\mathsf{BM},\Gamma}(\mathcal{X}/S;\mathbb{Q}) .$$

We will also be interested in taking $\Gamma := \mathbb{Z}\langle 1\rangle$, in which case we write

$$\mathsf{H}^{\mathsf{BM}}_{\mathsf{even}}(\mathcal{X}/S) := \mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{X}/S;\mathbb{Q}) .$$

The natural inclusion $\mathbb{Z}\langle 1\rangle \subset \mathbb{Z}\langle 1/2\rangle$ induces a continuous morphism of algebras

$$\mathsf{H}^{\mathsf{BM}}_{\mathsf{even}}(\mathcal{X}/S) \longrightarrow \mathsf{H}^{\mathsf{BM}}_\bullet(\mathcal{X}/S) .$$

Also, the natural transformation $\mathbf{DM}_{\mathbb{Q}} \to {}^{\mathsf{top}}\mathbf{D}_{\mathbb{Q}}$ (see [DPS22, Remark II.1.13]) induces a continuous map

$$\mathsf{H}^{\mathsf{mot}}_\bullet(\mathcal{X}/S) \longrightarrow \mathsf{H}^{\mathsf{BM}}_{\mathsf{even}}(\mathcal{X}/S) ,$$

given by the cycle class map.

△

Remark 3.7. Let $S \in \mathsf{dGeom}_k^{\mathsf{qcqs}}$ and $\mathcal{X} \in \mathsf{indGeom}_S^{\mathsf{adm}}$. Assume that there are actions by a torus T on $\mathcal{X}$. We consider S equipped with the trivial T-action. By construction

$$S \times_{[S/T]} [\mathcal{X}/T] \simeq \mathcal{X} .$$

Therefore, Corollary 2.28 implies that each $[\mathcal{X}/T]$ is admissible. Thus, for fixed $\mathcal{A} \in \mathsf{CAlg}(\mathbf{D}(S))$ and a fixed abelian group $\Gamma \subseteq \mathsf{Pic}(\mathbf{D}^*(S))$, we set

$$\mathsf{H}_0^{\mathbf{D},\Gamma,T}(\mathcal{X};\mathcal{A}) := \mathsf{H}_0^{\mathsf{BM},\pi^*\Gamma}([\mathcal{X}/T]/[S/T];\pi^*\mathcal{A}) ,$$

where $\pi\colon [S/T] \to S$ is the coarse moduli space map. In particular, we denote by $\mathsf{H}^T_\bullet(\mathcal{X})$, $\mathbf{G}^T(\mathcal{X})$, $\mathsf{H}^{\mathsf{mot},T}_\bullet(\mathcal{X})$, $\mathsf{H}^T_{\mathsf{even}}(\mathcal{X})$ the equivariant versions of the examples discussed in Example 3.6. △

We fix an abelian group $(\Lambda, +)$. We define the category of Λ-*graded derived stacks* as:

$$\Lambda\text{-}\mathsf{dSt}_k := \mathsf{Fun}(\Lambda, \mathsf{dSt}_k) ,$$

and we consider it with the symmetric monoidal structure induced by Λ via *Day's convolution* (see [DPS22, Recollection I.6.3]). This symmetric monoidal structure propagates to the ∞-category of correspondences $\mathsf{Corr}(\Lambda\text{-}\mathsf{dSt}_k)$.

Definition 3.8. Let P be a property of derived stacks (resp. of morphisms of derived stacks). A Λ-graded derived stack F (resp. a morphism $F \to G$) in $\mathsf{Fun}(\Lambda, \mathsf{dSt}_k)$ is *said to have the property* P if for every $\mathbf{v} \in \Lambda$ the derived stack $F(\mathbf{v})$ (resp. the morphism $F(\mathbf{v}) \to G(\mathbf{v})$) has the property P. ⊘

We denote by $\Lambda\text{-}\mathsf{indGeom}_k^{\mathsf{adm}}$ the full subcategory of $\Lambda\text{-}\mathsf{dSt}_k$ spanned by admissible indgeometric stacks. Concretely,

$$\Lambda\text{-}\mathsf{indGeom}_k^{\mathsf{adm}} := \mathsf{Fun}(\Lambda, \mathsf{indGeom}_k^{\mathsf{adm}}) .$$

In the same way, we have a well defined symmetric monoidal ∞-category of correspondences

$$\mathsf{Corr}^{\times}(\Lambda\text{-}\mathsf{indGeom}_k^{\mathsf{adm}})_{\mathsf{qc.lci}\,\cap\,\mathsf{fconn},\,\mathsf{lrpas}} .$$

Fix now a coefficient ring R of characteristic zero. Once again, Day's convolution endows the category

$$\Lambda\text{-}\mathsf{Pro}^{\sqcup}(\mathsf{Mod}_R^{\heartsuit}) := \mathsf{Fun}(\Lambda, \mathsf{Pro}^{\sqcup}(\mathsf{Mod}_R^{\heartsuit}))$$

with a symmetric monoidal structure. Combining the formal properties of Day's convolution (see [DPS22, Recollection I.6.3]) with Theorem 3.5, we obtain:

Theorem 3.9. *Let $\mathbf{D}^*$ be a motivic formalism. Let $S \in \mathsf{dGeom}^{\mathsf{qcqs}}$, $\mathcal{A} \in \mathsf{CAlg}(\mathbf{D}(S))$ and let $\Gamma \subseteq \mathsf{Pic}(\mathbf{D}^*(S))$ be an abelian subgroup. Assume that $\mathcal{A}$ is oriented and that Γ is closed under Thom twists. Then, the construction*

$$\mathsf{H}_0^{\mathbf{D},\Gamma}(-/S;\mathcal{A})\colon \mathsf{Corr}^{\times}(\Lambda\text{-}\mathsf{indGeom}_k^{\mathsf{adm}})_{\mathsf{qc.lci}\,\cap\,\mathsf{fconn},\,\mathsf{lrpas}} \longrightarrow \Lambda\text{-}\mathsf{Pro}^{\sqcup}(\mathsf{Mod}_R^{\heartsuit})$$

that sends $\mathcal{X} \to S$ to

$$\mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{X}/S;\mathcal{A}) := \bigoplus_{\mathbf{v}\in\Lambda} \mathsf{H}_0^{\mathbf{D},\Gamma}(\mathcal{X}(\mathbf{v})/S;\mathcal{A})$$

and whose functoriality is given as in Theorem 3.5 defines a lax symmetric monoidal functor.

Remark 3.10. All the constructions described in Example 3.6 and Remark 3.7 carry over to the Λ-graded setting. △

4. Derived moduli of coherent sheaves on formal schemes

In this section we study families of nilpotent sheaves on formal schemes. All throughout this section, we fix a field k of characteristic zero.

4.1. Formal completions in derived geometry.

We start by discussing *formal completions*. Our starting point is the following:

Definition 4.1 ([CPT⁺17, Definition 2.1.3])**.** Let X be a quasi-compact and quasi-separated derived scheme locally almost of finite presentation and let $j\colon Z \to X$ be a closed immersion. The *formal completion* $\widehat{X}_Z$ of X along Z is the derived stack fitting in the following pullback diagram:

$$\begin{array}{ccc} \widehat{X}_Z & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle \lambda_X} \\ Z_{\mathsf{dR}} & \xrightarrow{j_{\mathsf{dR}}} & X_{\mathsf{dR}} \end{array} .$$

⊘

Since X is noetherian, $\widehat{X}_Z$ is a derived indscheme (and more precisely an ind-inf-scheme), see [GR17, Chapter 2, Example 3.1.3]. An explicit ind-presentation is provided in [GR14, Proposition 6.5.5], but since we will have to use it, we proceed to give a brief review.

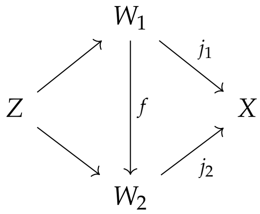
As a starting point, observe that ${}^{\mathsf{red}}Z \to Z$ induces an equivalence $({}^{\mathsf{red}}Z)_{\mathsf{dr}} \to Z_{\mathsf{dr}}$. For this reason, we can assume without loss of generality that Z is underived and reduced to begin with.

Definition 4.2. A *laft thickening of Z inside X* is a factorization of $j\colon Z \to X$ as

$$Z \xrightarrow{j_{Z,W}} W \xrightarrow{j_W} X \quad ,$$

where $j_{Z,W}$ is a nil-isomorphism[3], W is a laft derived scheme, and j_W is a closed immersion. We denote by $\mathcal{T}_{Z/\!/X}$ the full subcategory of $\mathsf{dSt}_{Z/\!/X}$ spanned by laft thickenings of Z inside X. ⊘

Remark 4.3. Consider a morphism



in $\mathcal{T}_{Z/\!/X}$. Since j_1 and j_2 are closed immersion, it immediately follows that the same goes for f. △

Notice that if W is a laft thickening of Z inside X, the canonical morphism $Z_{\mathsf{dr}} \to W_{\mathsf{dr}}$ is an equivalence. It follows that the canonical morphism

$$\widehat{X}_Z \longrightarrow \widehat{X}_W$$

is an equivalence and that we have a canonical morphism $W \to \widehat{X}_Z$. In turn, this induces a canonical comparison map

$$\operatorname*{colim}_{W \in \mathcal{T}_{Z/\!/X}} W \longrightarrow \widehat{X}_Z \,, \tag{4.1}$$

the colimit being computed in dSt_k. We have the following.

Proposition 4.4 ([GR14, Proposition 6.5.5])**.** *In the above setting:*

(1) the ∞-category $\mathcal{T}_{Z/\!/X}$ is filtered;

(2) the canonical morphism (4.1) is an equivalence. Moreover, the colimit can be computed in derived prestacks.

Notice that a priori one needs all derived thickenings to compute $\widehat{X}_Z$. Nevertheless, when X is underived then the same goes for $\widehat{X}_Z$, indeed we have the following result.

Proposition 4.5. *Let $j\colon Z \to X$ be a closed immersion of classical schemes almost of finite presentation over k. Then, $\widehat{X}_Z$ is underived and it coincides with the classical formal completion of X along Z.*

Proof. The question is easily seen to be local on X, and in the affine case it follows from [Lur18, Corollary 7.3.6.9]. See also [HLP23, Proposition 2.1.4]. □

Remark 4.6. In [GR17, Chapter 9, §5], the authors construct a derived version of the infinitesimal neighborhood of order n of Z inside X. We denote this construction by $Z^{(n)}$. When X is underived, its truncation $\mathrm{t}_0(Z^{(n)})$ coincides with the usual n-th order thickening of Z inside X. If X is underived and j is a regular closed immersion, $Z^{(n)}$ is underived and it coincides with the usual infinitesimal neighborhood of order n, $\mathsf{Spec}_X(\mathcal{O}_X/\mathcal{J}_Z^{n+1})$, where $\mathcal{J}_Z$ is the ideal of definition of Z. Finally, the natural functor

$$\mathbb{N} \longrightarrow \mathcal{T}_{Z/\!/X}$$

sending n to $Z^{(n)}$ is cofinal and therefore there is a canonical equivalence

$$\operatorname*{colim}_n Z^{(n)} \simeq \widehat{X}_Z \,.$$

Notice that when X is underived, Proposition 4.5 implies that the canonical comparison map

$$\operatorname*{colim}_n \mathrm{t}_0(Z^{(n)}) \longrightarrow \operatorname*{colim}_n Z^{(n)}$$

[3] i.e., ${}^{\mathsf{red}}j_{Z,W}$ is an isomorphism.

is an equivalence, even when the individual maps $\mathrm{t}_0(Z^{(n)}) \to Z^{(n)}$ are not. △

For later use, we introduce a simple recognition criterion for formal completions.

Lemma 4.7. *Let $f\colon F \to G$ be a formally étale morphism between laft, convergent and infinitesimally cohesive derived stacks. Then, f exhibits F as the formal completion of G along ${}^{\mathsf{red}}F \to G$.*

Proof. First of all, observe that the canonical map

$$({}^{\mathsf{red}}F)_{\mathsf{dR}} \longrightarrow F_{\mathsf{dR}}$$

is an equivalence. It is therefore sufficient to argue that the square

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ \downarrow & & \downarrow \\ F_{\mathsf{dR}} & \longrightarrow & G_{\mathsf{dR}} \end{array}$$

is a pullback. Since both F_{dR} and G_{dR} are formally étale over $\mathsf{Spec}(k)$, the bottom horizontal map is formally étale as well. Since by assumption f is formally étale, the same goes for the induced morphism

$$F \longrightarrow \widehat{G}_F \ .$$

Besides, since F and G are laft, convergent and infinitesimally cohesive, the same goes for $\widehat{G}_F$ and therefore for the above morphism. In particular, Lemma 2.7 shows that it suffices to check that the induced morphism ${}^{\mathsf{red}}F \to {}^{\mathsf{red}}\widehat{G}_F$ is an equivalence. For this, it suffices to argue that the above square is a pullback when evaluated on $S \in {}^{\mathsf{red}}\mathsf{dAff}_k$. Since for such a choice of S the vertical arrows in the above square become equivalences, the conclusion follows. □

We now introduce a global counterpart of Definition 4.1.

Definition 4.8. A *derived formal scheme* is a derived stack $\mathcal{X} \in \mathsf{dSt}_k$ such that there exists an effective epimorphism

$$\coprod_i \mathcal{U}_i \longrightarrow \mathcal{X}$$

where each $\mathcal{U}_i \to \mathcal{X}$ is representable by open Zariski immersions and each $\mathcal{U}_i$ is a formal completion in the sense of Definition 4.1. We write fdSch_k for the full subcategory of dSt_k spanned by derived formal schemes. ⊘

Remark 4.9. If $\mathcal{X}$ is a derived formal scheme, then ${}^{\mathsf{red}}\mathcal{X}$ is an ordinary scheme. △

Definition 4.10. We say that $\mathcal{X}$ is *quasi-compact* if ${}^{\mathsf{red}}\mathcal{X}$ is quasi-compact.

We write $\mathsf{fdSch}_k^{\mathsf{qc}}$ for the full subcategory of $\mathsf{fdSch}_k^{\mathsf{qc}}$ spanned by quasi-compact derived formal schemes. ⊘

Example 4.11. Let $S \in \mathsf{dSch}_k$ be a derived scheme. Then, the induced morphism

$$(\mathrm{t}_0(S))_{\mathsf{dR}} \longrightarrow S_{\mathsf{dR}}$$

is an equivalence, and in particular S coincides with the formal completion of S along $\mathrm{t}_0(S)$. More generally, if Z is a closed subscheme of X and $S \in \mathsf{dSch}_k$ is a derived scheme, then $\widehat{X}_Z \times S$ coincides with the formal completion of $X \times S$ along $Z \times \mathrm{t}_0(S_0)$. It follows that if $\mathcal{X}$ is a derived formal scheme, the same goes for $\mathcal{X} \times S$ for every derived scheme S. △

Despite being naturally indgeometric stacks, derived formal schemes have representable diagonal. This particular (well-known) property will play an important role in what follows, so we include a short proof.

Lemma 4.12. *Let X be a quasi-compact and quasi-separated derived scheme locally almost of finite presentation and let $j\colon Z \hookrightarrow X$ be a closed immersion. Then, the natural morphism*

$$\widehat{X}_Z \longrightarrow X$$

is (-1)-truncated.

Proof. Since (-1)-truncated morphisms are stable under pullbacks, it is enough to verify that $Z_{\mathsf{dr}} \to X_{\mathsf{dr}}$ is (-1)-truncated. Unraveling the definitions, we see that we have to check that for every derived affine scheme S locally almost of finite type the fibers of the morphism

$$\mathsf{Map}_{\mathsf{dSt}}({}^{\mathsf{red}}S, Z) \longrightarrow \mathsf{Map}_{\mathsf{dSt}}({}^{\mathsf{red}}S, X)$$

are empty or contractible. This is true because $Z \to X$ is a closed immersion. □

Proposition 4.13. *Let $\mathcal{X}$ be a derived formal scheme. Then, the diagonal*

$$\Delta_{\mathcal{X}}\colon \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X}$$

is representable by derived schemes.

Proof. The question is Zariski-local on $\mathcal{X}$, so we can assume from the very beginning that $\mathcal{X} \simeq \widehat{X}_Z$ for some quasi-compact and quasi-separated derived scheme locally almost of finite presentation and some closed immersion $j\colon Z \hookrightarrow X$. Let $S = \mathsf{Spec}(A)$ be an affine derived scheme and let $x\colon S \to \widehat{X}_Z \times \widehat{X}_Z$. It follows from Proposition 4.4 that we can find a laft thickening W of Z inside X and a factorization of x as

$$S \xrightarrow{x'} W \times W \longrightarrow \widehat{X}_Z \times \widehat{X}_Z \quad .$$

Since

$$(W \times W) \times_{\widehat{X}_Z \times \widehat{X}_Z} \widehat{X}_Z \simeq W \times_{\widehat{X}_Z} W\,,$$

it suffices to prove that this is a derived scheme. Using Lemma 4.12 we see that the canonical map

$$W \times_{\widehat{X}_Z} W \longrightarrow W \times_X W$$

is an equivalence, whence the conclusion. □

The above Proposition immediately implies the following.

Corollary 4.14. *Let $\mathcal{X}$ be a derived formal scheme and let S be a derived scheme. Then, any morphism $S \to \mathcal{X}$ is representable by derived schemes.*

4.2. **Nilpotent quasi-coherent sheaves.** In this section, we shall define *nilpotent* quasi-coherent sheaves on a formal scheme. When dealing with a formal completion $\widehat{X}_Z$, nilpotent quasi-coherent sheaves are simply the colimit

$$\mathsf{QCoh}^{\mathsf{nil}}(\widehat{X}_Z) \simeq \operatorname*{colim}_{W \in \mathfrak{T}_{Z/\!/X}} \mathsf{QCoh}(W)\,,$$

where the transition maps are given by the pushforwards. This could be taken as definition, but some extra care is needed to ensure a good functoriality and to extend it to more general derived formal schemes. For this reason, we take a slightly longer route to the definition of $\mathsf{QCoh}^{\mathsf{nil}}$.

We start fixing some terminology.

Definition 4.15. We say that a derived formal scheme $\mathcal{X}$ is *affine* if there exists an equivalence $\mathcal{X} \simeq \widehat{X}_Z$ with X being affine.

We write fdAff_k for the full subcategory of fdSch_k spanned by derived formal affine schemes. We also denote $(\mathsf{fdSch}_k)_{\mathsf{sch}}$ (resp. $(\mathsf{fdAff}_k)_{\mathsf{sch}}$) for the non-full subcategory of fdSch_k (resp. fdAff_k) having all objects and whose morphisms are the *schematic* ones (i.e., those representable by derived schemes). ⊘

Notice that Proposition 4.13 implies that whenever S is a derived scheme and $\mathcal{X}$ is a derived formal scheme, *any* map $S \to \mathcal{X}$ is schematic.

Construction 4.16. We define a Grothendieck topology $\tau^{\mathsf{qc}}_{\mathsf{r\text{-}Zar}}$ on fdAff_k declaring that covers are sets of schematic affine open Zariski immersions $\{\mathcal{U}_i \to \mathcal{X}\}_{i\in I}$ with I being *finite* and such that for every $S \in \mathsf{dAff}_k$ and every $S \to \mathcal{X}$ the induced family

$$\{\mathcal{U}_i \times_{\mathcal{X}} S \longrightarrow S\}_{i\in I}$$

is an affine open Zariski cover of S. We refer to $\tau^{\mathsf{qc}}_{\mathsf{r\text{-}Zar}}$ as the *quasi-compact Zariski topology* on fdAff_k, and we observe that it restricts to a Grothendieck topology on $(\mathsf{fdAff}_k)_{\mathsf{sch}}$. ⊘

The restricted Yoneda embedding yields fully faithful and left exact functors

$$\mathsf{fdSch}_k \hookrightarrow \mathsf{Sh}\big((\mathsf{fdAff}_k)_{\mathsf{sch}}, \tau^{\mathsf{qc}}_{\mathsf{r\text{-}Zar}}\big) \quad \text{and} \quad (\mathsf{fdAff}_k)_{\mathsf{sch}} \hookrightarrow \mathsf{Sh}\big((\mathsf{fdAff}_k)_{\mathsf{sch}}, \tau^{\mathsf{qc}}_{\mathsf{r\text{-}Zar}}\big) .$$

On the other hand, following [DPS22, §II.7], we take $\mathcal{T} = \mathsf{dSt}_k$, P to be the property of being an affine derived scheme and Q the property of being a closed immersion. Combining Proposition 4.4 and [DPS22, Proposition II.7.6], we see that the restricted Yoneda yields a fully faithful functor

$$\mathsf{fdAff}_k \hookrightarrow \mathsf{Ind}(\mathsf{dAff}_k) . \tag{4.2}$$

The definition of the quasi-compact Zariski topology makes sense *verbatim* within $\mathsf{Ind}(\mathsf{dAff}_k)$, and the above morphism becomes a continuous and cocontinuous morphism of sites (see [PY16, Definitions 2.12 & 2.17]). Similarly, write $\mathsf{Ind}(\mathsf{dAff}_k)_{\mathsf{sch}}$ for the non-full subcategory of $\mathsf{Ind}(\mathsf{dAff}_k)$ having all objects and whose morphisms are representable ones. Then, the above embedding restricts to a fully faithful functor

$$(\mathsf{fdAff}_k)_{\mathsf{sch}} \hookrightarrow (\mathsf{Ind}(\mathsf{dAff}_k))_{\mathsf{sch}} , \tag{4.3}$$

which is again a continuous and cocontinuous morphism of sites.

In particular, there is a well defined restriction functor

$$\mathsf{Sh}(\mathsf{Ind}(\mathsf{dAff}_k), \tau_{\mathsf{r\text{-}Zar}}) \longrightarrow \mathsf{Sh}(\mathsf{fdAff}_k, \tau_{\mathsf{r\text{-}Zar}})$$

as well as its schematic counterpart

$$\mathsf{Sh}(\mathsf{Ind}(\mathsf{dAff}_k)_{\mathsf{sch}}, \tau^{\mathsf{qc}}_{\mathsf{r\text{-}Zar}}) \longrightarrow \mathsf{Sh}((\mathsf{fdAff}_k)_{\mathsf{sch}}, \tau^{\mathsf{qc}}_{\mathsf{r\text{-}Zar}}) .$$

It follows from [PY16, Lemmas 2.13 & 2.18] that both admit a left and a right adjoint. Besides, the right adjoint coincides exactly with right Kan extension respectively along the embeddings (4.2) and (4.3). In particular, it is fully faithful and therefore the same goes for the left adjoint. These left adjoints yields fully faithful and left exact embeddings

$$\mathsf{fdSch}_k \hookrightarrow \mathsf{Sh}\big(\mathsf{Ind}(\mathsf{dAff}_k), \tau^{\mathsf{qc}}_{\mathsf{r\text{-}Zar}}\big) \qquad \text{and} \qquad (\mathsf{fdSch}_k)^{\mathsf{qc}}_{\mathsf{sch}} \hookrightarrow \mathsf{Sh}\big((\mathsf{Ind}(\mathsf{dAff}_k))_{\mathsf{sch}}, \tau^{\mathsf{qc}}_{\mathsf{r\text{-}Zar}}\big) .$$

We exploit this embedding to functorially attach categorical invariants to derived formal schemes, as we are going to explain now.

We now begin the main construction. Let

$$\mathbf{Q}^* \colon \mathsf{dAff}_k^{\mathsf{op}} \longrightarrow \mathsf{Cat}_\infty$$

be a functor with values in (possibly large but not necessarily presentable) ∞-categories satisfying the following:

Assumption 1. For every pullback square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \big\downarrow{\scriptstyle g'} & & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

in dAff_k, the induced diagram

$$\begin{array}{ccc} \mathbf{Q}(Y) & \xrightarrow{f^*} & \mathbf{Q}(X) \\ \big\downarrow{\scriptstyle g^*} & & \big\downarrow{\scriptstyle f'^*} \\ \mathbf{Q}(Y') & \xrightarrow{g'^*} & \mathbf{Q}(X') \end{array}$$

is horizontally right adjointable. ⊘

In particular, for every $f\colon S \to T$, the induced functor $f^*\colon \mathbf{Q}(T) \to \mathbf{Q}(S)$ has a right adjoint, that we denote f_*. This determines a functor

$$\mathbf{Q}_*\colon \mathsf{dAff}_k \longrightarrow \mathsf{Cat}_\infty ,$$

and we denote by

$$^{\mathsf{ind}}\mathbf{Q}_*\colon \mathsf{Ind}(\mathsf{dAff}_k) \longrightarrow \mathsf{Cat}_\infty$$

its canonical ind-extension. We denote in the same way the restriction

$$^{\mathsf{ind}}\mathbf{Q}_*\colon \mathsf{fdAff}_k \longrightarrow \mathsf{Cat}_\infty .$$

Warning 4.17. It is extremely important for our purposes that the above ind-extension is computed in Cat_∞ and not in Pr^{R}. In the main example of interest, $\mathbf{Q}^* = \mathsf{QCoh}^*$ so it would be possible to compute the ind-extension inside Pr^{R}. However, it does not yield the correct result. △

Assumption 1 guarantees that $\mathbf{Q}^*$ induces a well defined functor

$$^{\mathsf{ind}}\mathbf{Q}^*\colon \mathsf{Ind}(\mathsf{dAff}_k)^{\mathsf{op}}_{\mathsf{sch}} \longrightarrow \mathsf{Cat}_\infty .$$

We have the following result.

Proposition 4.18. *Assume that $\mathbf{Q}^*$ satisfies descent for the Zariski topology on dAff_k. Then, $^{\mathsf{ind}}\mathbf{Q}^*$ satisfies descent for the quasi-compact Zariski topology $\tau^{\mathsf{qc}}_{\mathsf{r\text{-}Zar}}$ on $\mathsf{Ind}(\mathsf{dAff}_k)$. In particular, $^{\mathsf{ind}}\mathbf{Q}^*$ canonically extends to a functor*

$$^{\mathsf{ind}}\mathbf{Q}^*\colon \mathsf{Sh}\big((\mathsf{fdAff}_k)_{\mathsf{sch}}, \tau^{\mathsf{qc}}_{\mathsf{r\text{-}Zar}}\big)^{\mathsf{op}} \longrightarrow \mathsf{Cat}_\infty .$$

By restriction, we obtain a well defined functor

$$^{\mathsf{ind}}\mathbf{Q}^*\colon (\mathsf{fdSch}^{\mathsf{qc}}_k)^{\mathsf{op}}_{\mathsf{sch}} \longrightarrow \mathsf{Cat}_\infty .$$

Proof. Because of the finiteness constraint on $\tau^{\mathsf{qc}}_{\mathsf{r\text{-}Zar}}$-covers, it is enough to prove that if $\{\mathcal{U}, \mathcal{V}\}$ is a representable open Zariski cover of $\mathcal{X}$ and

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{V} \\ \big\downarrow & & \big\downarrow{\scriptstyle j} \\ \mathcal{U} & \xrightarrow{i} & \mathcal{X} \end{array}$$

is a pullback square in $\mathsf{Ind}(\mathsf{dAff}_k)$, then the induced map

$$^{\mathsf{ind}}\mathbf{Q}^*(\mathcal{X}) \longrightarrow {}^{\mathsf{ind}}\mathbf{Q}^*(\mathcal{U}) \times_{^{\mathsf{ind}}\mathbf{Q}^*(\mathcal{W})} {}^{\mathsf{ind}}\mathbf{Q}^*(\mathcal{V})$$

is an equivalence. Fix a presentation

$$\mathcal{X} \simeq \text{“}\operatorname*{colim}_\alpha\text{”}\, X_\alpha .$$

Set

$$U_\alpha \coloneqq \mathcal{U} \times_{\mathcal{X}} X_\alpha , \qquad V_\alpha \coloneqq \mathcal{V} \times_{\mathcal{X}} X_\alpha , \qquad W_\alpha \coloneqq \mathcal{W} \times_{\mathcal{X}} X_\alpha ,$$

so that

$$\mathcal{U} \simeq \text{“}\operatorname*{colim}_\alpha\text{”}\, U_\alpha , \qquad \mathcal{V} \simeq \text{“}\operatorname*{colim}_\alpha\text{”}\, V_\alpha , \qquad \mathcal{W} \simeq \text{“}\operatorname*{colim}_\alpha\text{”}\, W_\alpha .$$

Unraveling the definitions, we find that

$$^{\mathrm{ind}}\mathbf{Q}(\mathcal{X}) \simeq \operatorname*{colim}_{\alpha} \mathbf{Q}_*(X_\alpha) \ ,$$

and similarly for $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ in place of $\mathcal{X}$. Notice that the pullback functoriality along i and j is guaranteed by Assumption 1. At this point, the conclusion simply follows from the fact that filtered colimits in Cat_∞ commute with finite limits. □

Our main example of interest is

$$\mathbf{Q}^* := \mathsf{QCoh}^* \colon \mathsf{dAff}_k^{\mathrm{op}} \longrightarrow \mathsf{Cat}_\infty \ .$$

Then Assumption 1 is satisfied (see e.g. [Lur18, Proposition 2.5.4.5]), and the functor satisfies (flat, hence) Zariski descent. Thus, Proposition 4.18 shows that this functor extends to $^{\mathrm{ind}}\mathbf{Q}^*$.

Definition 4.19. With respect to the above choice of $\mathbf{Q}^*$, we set

$$\mathsf{QCoh}^{\mathsf{nil}} := {}^{\mathrm{ind}}\mathbf{Q}^* \colon (\mathsf{fdSch}^{\mathsf{qc}})^{\mathrm{op}}_{\mathsf{sch}} \longrightarrow \mathsf{Cat}_\infty \ ,$$

and we refer to $\mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X})$ as the ∞-category of *nilpotent quasi-coherent sheaves on* $\mathcal{X}$. ⊘

The following list of examples covers all important cases for us.

Example 4.20.

(1) When X is a quasi-compact derived scheme, there is a canonical identification

$$\mathsf{QCoh}^{\mathsf{nil}}(X) \simeq \mathsf{QCoh}(X) \ .$$

(2) Consider now the case of a formal derived scheme $\mathcal{X} \simeq \widehat{X}_Z$. By construction, when X is affine we have an equivalence

$$\mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X}) \simeq \operatorname*{colim}_{W \in \mathfrak{T}_{Z/\!/X}} \mathsf{QCoh}(W) \ ,$$

the colimit being taken with respect to pushforwards. Since filtered colimits in Cat_∞ commute with finite limits and since $\mathsf{QCoh}^{\mathsf{nil}}$ satisfies descent for $\tau^{\mathsf{qc}}_{\mathsf{r\text{-}Zar}}$ (see Proposition 4.18), that the above description remains valid for X quasi-compact.

(3) Combining the above two points, it follows that for every affine derived scheme S, every quasi-compact derived scheme X and any closed derived subscheme $Z \hookrightarrow X$, there are canonical equivalences

$$\mathsf{QCoh}^{\mathsf{nil}}(\widehat{X}_Z \times S) \simeq \operatorname*{colim}_{W \in \mathfrak{T}_{Z/\!/X}} \mathsf{QCoh}(W \times S) \simeq \operatorname*{colim}_{W' \in \mathfrak{T}_{Z \times S/\!/X \times S}} \mathsf{QCoh}(W') \ .$$

Indeed, when X is affine this follows from the fact that both

$$\operatorname*{colim}_{W \in \mathfrak{T}_{Z/\!/X}} W \times S \qquad \text{and} \qquad \operatorname*{colim}_{W' \in \mathfrak{T}_{Z \times S/\!/X \times S}} W'$$

provide ind-presentations for $\widehat{X}_Z \times S$, and the general case follows by Zariski descent.

△

Notation 4.21. Let $\mathcal{X} \simeq \widehat{X}_Z$, with X a quasi-compact derived scheme. Given a thickening W of Z inside X and $\mathcal{F} \in \mathsf{QCoh}(W)$, we write $[\mathcal{F}]$ for the element of $\mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X})$ determined by $\mathcal{F}$. ⊘

We now single out a special subcategory of $\mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X})$:

Definition 4.22 (Nilpotent almost perfect sheaves)**.** Let $\mathcal{X}$ be a quasi-compact derived formal scheme and let $\mathfrak{F} \in \mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X})$ be a nilpotent quasi-coherent sheaf.

- Assume that $\mathcal{X} \simeq \widehat{X}_Z$ is an affine derived formal scheme. We say that $\mathfrak{F}$ is *almost perfect* if it is of form $[\mathcal{F}]$ for some thickening W of Z inside X and some $\mathcal{F} \in \mathsf{APerf}(W)$.

- In the general case, we say that $\mathfrak{F} \in \mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X})$ is *almost perfect* if it is almost perfect in the above sense on an affine Zariski cover.

We let $\mathsf{APerf}^{\mathsf{nil}}(\mathcal{X})$ denote the full subcategory of $\mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X})$ spanned by nilpotent almost perfect sheaves. ⊘

Remark 4.23. Assume that $\mathcal{X} \simeq \widehat{X}_Z$. In this case,

$$\mathsf{APerf}^{\mathsf{nil}}(\widehat{X}_Z) \simeq \operatorname*{colim}_{W \in \mathfrak{T}_{Z/\!/X}} \mathsf{APerf}(W) .$$

Notice that the transition functors are given by pushforward, which preserve almost perfect sheaves because the transition maps in $\mathfrak{T}_{Z/\!/X}$ are closed immersions. △

Warning 4.24. Let $\mathcal{X}$ be a derived formal scheme. Then $\mathcal{X}$ is in particular a derived stack, so $\mathsf{APerf}(\mathcal{X})$ is well defined, but it typically differs from $\mathsf{APerf}^{\mathsf{nil}}(\mathcal{X})$. Consider for instance the following example: take $X = \mathbb{A}^1_k$, $Z = \mathsf{Spec}(k)$ embedded as 0 inside X and $\mathcal{X} := \widehat{X}_Z$. Then

$$\mathsf{APerf}(\mathcal{X}) \simeq \lim_n \mathsf{APerf}(k[T]/(T^n)) \simeq \mathsf{APerf}(k[\![T]\!]) ,$$

while

$$\mathsf{APerf}^{\mathsf{nil}}(\mathcal{X}) \simeq \operatorname*{colim}_n \mathsf{APerf}(k[T]/(T^n)) .$$

Here the limit is computed with respect to the pullback, while the colimit is computed with respect to the pushforward. The ring of formal power series $k[\![T]\!]$ defines an object in $\mathsf{APerf}(\mathcal{X})$ but not in $\mathsf{APerf}^{\mathsf{nil}}(\mathcal{X})$. △

Variant 4.25. Let S be a derived stack and let fdSch_S be the full subcategory of $\mathsf{dSt}_{/S}$ spanned by morphisms representable by formal derived schemes. Let $(\mathsf{fdSch}_S)_{\mathsf{sch}}$ be the (non full) subcategory spanned by schematic morphisms. Then Proposition 4.18 allows to define

$$\mathsf{QCoh}^{\mathsf{nil}} \colon (\mathsf{fdSch}_S)^{\mathsf{op}}_{\mathsf{sch}} \longrightarrow \mathsf{Cat}_\infty .$$

Informally,

$$\mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X}) \simeq \lim_{T \in \mathsf{dAff}_{/S}} \mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X} \times_S T) .$$

Indeed, if $T' \to T$ is a morphism in $\mathsf{dAff}_{/S}$, then the induced morphism $\mathcal{X} \times_S T' \to \mathcal{X} \times_S T$ is schematic, and therefore Proposition 4.18 allows to define the above limit. This applies in particular to the following situation: let G be an algebraic group acting on a quasi-compact scheme X and let $Z \hookrightarrow X$ be a closed subscheme which is G-invariant. Then the maps

$$[X/G] \longrightarrow \mathrm{B}G \qquad \text{and} \qquad [Z/G] \simeq Z \times \mathrm{B}G \to \mathrm{B}G$$

are schematic. The formal completion of $[X/G]$ along $[Z/G]$ coincides with $[\widehat{X}_Z/G]$, and the structural map to $\mathrm{B}G$ is representable by formal derived schemes. In particular, $\mathsf{QCoh}^{\mathsf{nil}}([\widehat{X}_Z/G])$ is well defined. ⊘

4.3. **Derived moduli of nilpotent sheaves.** We now adapt the definition of the derived stack of coherent sheaves in [PS23, §2.1] to the formal setting.

When $\mathcal{X}$ is a quasi-compact derived formal scheme and $S \in \mathsf{dAff}_k$ is an affine derived scheme, we refer to objects in $\mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X} \times S)$ as *S-families of nilpotent quasi-coherent sheaves on $\mathcal{X}$*. This is justified by the contravariant functoriality of $\mathsf{QCoh}^{\mathsf{nil}}$ in schematic morphisms: whenever we have closed point $\mathsf{Spec}(K) \to S$, the induced map $i \colon \mathcal{X}_K := \mathcal{X} \times_S \mathsf{Spec}(K) \to \mathcal{X} \times S$ is representable and we therefore obtain a pullback

$$i^* \colon \mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X} \times S) \longrightarrow \mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X}_K) .$$

Definition 4.26 (Flat nilpotent families)**.** Let $\mathcal{X}$ be a quasi-compact derived formal scheme and let $S \in \mathsf{dAff}_k$ be an affine derived scheme. Let $\mathfrak{F} \in \mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X} \times S)$ be an S-family of nilpotent quasi-coherent sheaves.

- Assume that $\mathcal{X} \simeq \widehat{X}_Z$ is an affine derived formal scheme. We say that $\mathfrak{F}$ is *S-flat* if it is of form $[\mathcal{F}]$ for some thickening W of Z inside X and some $\mathcal{F} \in \mathsf{QCoh}(W \times S)$ which is S-flat in the sense of [PS23, Definition 2.1].
- In the general case, we say that $\mathfrak{F} \in \mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X})$ is *S-flat* if it is almost perfect in the above sense on an affine Zariski cover.

We let $\mathsf{QCoh}^{\mathsf{nil}}_S(\mathcal{X} \times S)$ denote the full subcategory of $\mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X} \times S)$ spanned by S-flat nilpotent quasi-coherent sheaves. ⊘

Definition 4.27 (Families of nilpotent coherent sheaves)**.** Let $\mathcal{X}$ be a quasi-compact derived formal scheme and let $S \in \mathsf{dAff}_k$ be an affine derived scheme. We say that $\mathfrak{F} \in \mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X} \times S)$ is an *S-family of nilpotent coherent sheaves on* $\mathcal{X}$ if it is both almost perfect and S-flat.

We write $\mathsf{Coh}^{\mathsf{nil}}_S(\mathcal{X} \times S)$ for the full subcategory of $\mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X} \times S)$ spanned by S-families of coherent sheaves on $\mathcal{X}$. ⊘

For the following definition, recall from Remark 4.9 that if $\mathcal{X}$ is a quasi-compact derived formal scheme, then ${}^{\mathsf{red}}\mathcal{X}$ is a quasi-compact scheme. In particular, Corollary 4.14 guarantees that the canonical map ${}^{\mathsf{red}}i_{\mathcal{X}} \colon {}^{\mathsf{red}}\mathcal{X} \to \mathcal{X}$ is schematic. It follows that for every $S \in \mathsf{dAff}_k$, there is a well defined functor

$$ {}^{\mathsf{red}}i^*_{\mathcal{X},S} \colon \mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X} \times S) \longrightarrow \mathsf{QCoh}^{\mathsf{nil}}({}^{\mathsf{red}}\mathcal{X} \times S) \simeq \mathsf{QCoh}({}^{\mathsf{red}}\mathcal{X} \times S)\ . $$

We can therefore introduce:

Definition 4.28 (Properly supported nilpotent families)**.** Let $\mathcal{X}$ be a quasi-compact derived formal scheme and let $S \in \mathsf{dAff}_k$ be an affine derived scheme. We say that $\mathfrak{F} \in \mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X} \times S)$ is *S-properly supported* (or *properly supported relative to* S) if ${}^{\mathsf{red}}i^*_{\mathcal{X},S}(\mathfrak{F})$ is properly supported in the sense of [PS23, Definition 2.27].

We write $\mathsf{QCoh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X} \times S)$ for the full subcategory of $\mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X} \times S)$ spanned by S-properly supported families and we set

$$ \mathsf{Coh}^{\mathsf{nil}}_{S,\mathsf{ps}}(\mathcal{X} \times S) \coloneqq \mathsf{Coh}^{\mathsf{nil}}_S(\mathcal{X} \times S) \cap \mathsf{QCoh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X} \times S)\ . $$

⊘

Remark 4.29. When ${}^{\mathsf{red}}\mathcal{X}$ is proper, then $\mathsf{QCoh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X} \times S) = \mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X} \times S)$, and similarly for families of nilpotent coherent sheaves. △

We can upgraded Remark 4.23 thanks to the following lemma.

Lemma 4.30. *Let $f \colon Y_1 \to Y_2$ be a finite morphism between quasi-compact derived schemes. Then, an object $\mathcal{F} \in \mathsf{APerf}(Y_1 \times S)$ belongs to $\mathsf{Coh}_{S,\mathsf{ps}}(Y_1 \times S)$ if and only if $f_{S,*}(\mathcal{F})$ belongs to $\mathsf{Coh}_{S,\mathsf{ps}}(Y_2 \times S)$. Thus, the functor $f_{S,*} \colon \mathsf{QCoh}(Y_1 \times S) \to \mathsf{QCoh}(Y_2 \times S)$ induces a well defined functor*

$$ \mathsf{Coh}_{S,\mathsf{ps}}(Y_1 \times S) \longrightarrow \mathsf{Coh}_{S,\mathsf{ps}}(Y_2 \times S)\ . $$

Proof. Since f is finite, in particular proper, it takes $\mathsf{APerf}(Y_1 \times S)$ to $\mathsf{APerf}(Y_2 \times S)$, see [Lur18, Theorem 5.6.0.2]. Besides, the image of a subset of $Y_1 \times S$ proper over S via f_S will still be proper over S. Therefore, $f_{S,*}$ preserves the properly supported condition. It also reflects it, because since f is proper, the preimage via f of a proper subset of $Y_2 \times S$ will be proper in $Y_1 \times S$. We are left to show that it preserves and reflects S-flatness. Let $\mathcal{F} \in \mathsf{APerf}(Y_1 \times S)$ be S-flat and let $\mathcal{G} \in \mathsf{QCoh}^{\heartsuit}(Y_2 \times S)$. Write

$$ q_S \colon Y_1 \times S \longrightarrow S \qquad \text{and} \qquad p_S \colon Y_2 \times S \longrightarrow S $$

for the canonical projections. Then, the projection formula yields

$$ p_S^*(\mathcal{G}) \otimes_{\mathcal{O}_{X_S}} f_{S,*}(\mathcal{F}) \simeq f_{S,*}(f^* p_S^*(\mathcal{G}) \otimes_{\mathcal{O}_{Y_S}} \mathcal{F}) \simeq f_*(q_S^*(\mathcal{G}) \otimes_{\mathcal{O}_{Y_S}} \mathcal{F})\ . $$

Recall that f being finite guarantees that it is affine as well. Then, the conclusion follows from the fact that $f_{S,*}$ is t-exact and conservative. □

Corollary 4.31. *Let X be a quasi-compact derived scheme and let $Z \hookrightarrow X$ be a closed subscheme. Then, for every affine derived scheme $S \in \mathsf{dAff}_k$, we have*

$$\mathsf{Coh}^{\mathsf{nil}}_{S,\mathsf{ps}}(\widehat{X}_Z \times S) \simeq \operatorname*{colim}_{W \in \mathfrak{T}_{Z/\!/X}} \mathsf{Coh}_{S,\mathsf{ps}}(W \times S) ,$$

the colimit being computed with respect to pushforward.

Proof. The colimit is well defined thanks to Lemma 4.30. By construction there is a canonical map

$$\operatorname*{colim}_{W \in \mathfrak{T}_{Z/\!/X}} \mathsf{Coh}_{S,\mathsf{ps}}(W \times S) \longrightarrow \mathsf{Coh}^{\mathsf{nil}}_{S,\mathsf{ps}}(\widehat{X}_Z \times S)$$

which is fully faithful. Essential surjectivity follows unwinding Definition 4.27. □

The contravariant functoriality of $\mathsf{QCoh}^{\mathsf{nil}}$ in schematic morphisms and [PS23, Lemma 2.4] immediately imply the following.

Lemma 4.32. *Let $\mathcal{X}$ be a quasi-compact derived formal scheme and let $f\colon T \to S$ be a morphism of derived affine schemes. Then, the functor*

$$f^*_X \colon \mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X} \times S) \longrightarrow \mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X} \times T)$$

takes S-flat (resp. S-proper) families to T-flat (resp. T-proper) ones. In particular, it induces a well defined functor

$$f^*_X \colon \mathsf{Coh}^{\mathsf{nil}}_{S,\mathsf{ps}}(\mathcal{X} \times S) \longrightarrow \mathsf{Coh}^{\mathsf{nil}}_{T,\mathsf{ps}}(\mathcal{X} \times T) .$$

Construction 4.33. Fix a quasi-compact derived formal scheme $\mathcal{X}$. Using the above lemma, we introduce the following derived prestack

$$\widetilde{\mathbf{Coh}}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X}) \colon \mathsf{dAff}^{\mathsf{op}} \longrightarrow \mathsf{Spc}$$

by the rule

$$\widetilde{\mathbf{Coh}}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X})(S) \coloneqq \mathsf{Coh}^{\mathsf{nil}}_{S,\mathsf{ps}}(\mathcal{X} \times S)^{\simeq} ,$$

where $(-)^{\simeq}$ denotes the maximal ∞-groupoid. Finally, we let

$$\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X}) \coloneqq {}^{\mathsf{conv}}\widetilde{\mathbf{Coh}}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X})$$

be its convergent completion. We write $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X})$ for its truncation. ⊘

Example 4.34. Let $\mathcal{X} \simeq \widehat{X}_Z$ for X a quasi-compact derived scheme and let $S \in \mathsf{dAff}_k$ be an affine derived scheme. Then combining Recollection 2.3 and Corollary 4.31, we find

$$\begin{aligned} \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X})(S) &\simeq \lim_n \operatorname*{colim}_{W \in \mathfrak{T}_{Z/\!/X}} \mathbf{Coh}_{\mathsf{ps}}(W)(\mathsf{t}_{\leqslant n}(S)) \\ &\simeq \lim_n \operatorname*{colim}_{W \in \mathfrak{T}_{Z/\!/X}} \mathsf{Coh}_{S,\mathsf{ps}}(W \times \mathsf{t}_{\leqslant n}(S))^{\simeq} . \end{aligned}$$

Notice that when computing the truncation $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X})$, the exterior limit disappears. △

4.4. **Indgeometricity.** Our first goal is to show that for formal derived schemes of the form $\widehat{X}_Z$, the derived prestack $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$ is indgeometric under some mild assumptions on X (see Theorem 4.39 below). Before giving the precise statement, we need to introduce the following:

Definition 4.35. Let X be a derived scheme. We say that X *satisfies the perfect resolution property* if for every $\mathcal{F} \in \mathsf{APerf}^{\heartsuit}(X)$ there exists $\mathcal{E} \in \mathsf{Perf}(X)_{\geqslant 0}$ and a morphism

$$\mathcal{E} \longrightarrow \mathcal{F}$$

inducing a surjection on π_0. We say that X *universally satisfies the perfect resolution property* if for every derived affine scheme S, $X \times S$ satisfies the perfect resolution property. ⊘

Example 4.36. If X is underived and it has the resolution property (i.e., for every classical coherent sheaf $\mathcal{F}$ there exists an epimorphism $\mathcal{E} \to \mathcal{F}$, with $\mathcal{E}$ being a vector bundle), then it also satisfies the perfect resolution property. In particular, if X is quasi-projective, then it universally satisfies the perfect resolution property. △

Lemma 4.37. *Let $j\colon X \hookrightarrow \overline{X}$ be a closed immersion of derived schemes. If $\overline{X}$ (universally) satisfies the perfect resolution property, then the same goes for X.*

Proof. We treat the universal case; the other follows by the same proof below taking $S = \mathsf{Spec}(k)$.

Fix an affine derived scheme S and let $\mathcal{F} \in \mathsf{APerf}^{\heartsuit}(X \times S)$. Then $j_S\colon X \times S \to \overline{X} \times S$ is again a closed immersion and $\overline{X} \times S$ satisfies the perfect resolution property. Let $\mathcal{F} \in \mathsf{APerf}^{\heartsuit}(X \times S)$. Then $j_{S,*}(\mathcal{F}) \in \mathsf{APerf}^{\heartsuit}(\overline{X} \times S)$. It follows that there exists $\mathcal{E} \in \mathsf{Perf}(\overline{X} \times S)_{\geqslant 0}$ and a morphism

$$\mathcal{E} \longrightarrow j_{S,*}(\mathcal{F})$$

being surjective on π_0. Since j_S is a closed immersion, it follows that the composite

$$j_S^*(\mathcal{E}) \longrightarrow j_S^*(j_{S,*}(\mathcal{F})) \longrightarrow \mathcal{F}$$

is also surjective on π_0. □

Corollary 4.38. *Smooth schemes universally satisfy the perfect resolution property.*

Proof. Let Y be a smooth scheme. Note that every object in $\mathsf{APerf}^{\heartsuit}(Y)$ is perfect. Thus, Y satisfies the perfect resolution property.

Let S be a derived affine scheme and let X be a smooth scheme. Since S is affine, there exists a closed immersion in $\mathbb{A}^n_k$ for some $n \geqslant 0$. Thus, the induced morphism $X \times S \to X \times \mathbb{A}^n_k$ is a closed immersion. Since $Y \coloneqq X \times \mathbb{A}^n_k$ is again smooth, Y satisfies the perfect resolution property. Therefore, the conclusion follows from Lemma 4.37. □

We can now formulate the main result of this section.

Theorem 4.39. *Let X be a quasi-compact derived scheme and let Z be a closed subscheme of X. Assume that X universally satisfies the perfect resolution property. Then, the derived stack $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$ is indgeometric.*

Remark 4.40. The main reason we have to restrict to $\widehat{X}_Z$ instead of considering a more general derived formal scheme $\mathcal{X}$ is that we do not know whether the canonical map

$$\operatorname*{colim}_{W \in \mathcal{T}_{\mathsf{red}\,\mathcal{X}/\!/\mathcal{X}}} W \longrightarrow \mathcal{X}$$

is an equivalence. In general, [GR17, Theorem 4.1.3] implies that

$$\mathcal{X} \simeq \operatorname*{colim}_{W \in \mathcal{T}^{\mathsf{nil\text{-}cl}}_{\mathsf{red}\,\mathcal{X}/\!/\mathcal{X}}} W\,, \tag{4.4}$$

where $\mathcal{T}^{\mathsf{nil\text{-}cl}}_{\mathsf{red}\,\mathcal{X}/\!/\mathcal{X}}$ is the full subcategory spanned by nil-closed thickenings. In other words, we are dropping the assumption that the structural morphism $W \to \mathcal{X}$ is a closed immersion. In turn, the transition morphisms in $\mathcal{T}^{\mathsf{nil\text{-}cl}}_{\mathsf{red}\,\mathcal{X}/\!/\mathcal{X}}$ are no longer forced to be closed immersions, and thus Proposition 4.45 below can no longer be used to construct a presentation for $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X})$. It is worth remarking that [GR17, Theorem 4.1.3] cover objects that are much more general than quasi-compact derived formal schemes, so it is quite possible that (4.4) is an equivalence in general. △

Before giving the proof of Theorem 4.39, we need some technical preliminaries. The key statement is Proposition 4.45 below. We start with two general constructions.

Construction 4.41. Let $f\colon Y \to Y$ be a closed immersion between quasi-compact and quasi-separated schemes locally almost of finite presentation over k. Notice that given a morphism $T \to S$ of derived affine schemes, the square

$$\begin{array}{ccc} Y \times T & \longrightarrow & Y \times S \\ \big\downarrow f_T & & \big\downarrow f_S \\ X \times T & \longrightarrow & X \times S \end{array}$$

is a derived pullback. In particular, derived base-change (see e.g. [Lur18, Proposition 2.5.4.5]) and Lemma 4.30 imply that f induces a well-defined morphism

$$\mathbf{f}_* \colon \mathbf{Coh}_{\mathsf{ps}}(Y) \longrightarrow \mathbf{Coh}_{\mathsf{ps}}(X) \ .$$

Our goal is to establish that, under mild assumptions on X, $\mathbf{f}_*$ is itself a closed immersion. ⊘

Construction 4.42. Let $p\colon X \to S$ be a flat morphism of derived schemes, with S being affine. Given $f\colon T \to S$, we write $X_T := T \times_S X$ and we let $f_X\colon X_T \to X$ be the naturally induced morphism. Let $\phi\colon \mathcal{G} \to \mathcal{F}$ be a morphism in $\mathsf{APerf}(X)$. We define the functor of points

$$\mathsf{Z}_\phi \colon \mathsf{dAff}_{/S}^{\mathsf{op}} \longrightarrow \mathsf{Spc}$$

sending $f\colon T \to S$ to

$$\mathsf{Map}_{\mathsf{Map}_{\mathsf{QCoh}(X_T)}(f_X^*(\mathcal{G}), f_X^*(\mathcal{F}))}(f_X^*(\phi), 0) \ ,$$

i.e., the space of nullhomotopies of $f_X^*(\phi)\colon f_X^*(\mathcal{G}) \to f_X^*(\mathcal{F})$ in $\mathsf{QCoh}(X_T)$. Notice that there is a canonical natural transformation $i\colon \mathsf{Z}_\phi \to S$. ⊘

Remark 4.43. In the setting of Construction 4.42, assume that both $\mathcal{F}$ and $\mathcal{G}$ are connective and that $\mathcal{F}$ is S-flat. For every *underived* affine scheme T, $f_X^*(\mathcal{F})$ is T-flat and hence it belongs to $\mathsf{QCoh}^\heartsuit(X_T)$. Since $f_X^*(\mathcal{G})$ is connective, it follows that

$$\mathsf{Map}_{\mathsf{QCoh}(X_T)}\big(f_X^*(\mathcal{G}), f_X^*(\mathcal{F})\big) \simeq \mathsf{Map}_{\mathsf{QCoh}^\heartsuit(X_T)}\big(\pi_0(f_X^*(\mathcal{G})), f_X^*(\mathcal{F})\big) \ .$$

Under this equivalence, we see that $\mathsf{Map}_{\mathsf{QCoh}^\heartsuit(X_T)}\big(\pi_0(f_X^*(\mathcal{G})), f_X^*(\mathcal{F})\big)$ is a *set*, and $\mathsf{Z}_\phi(T)$ can be explicitly characterized as follows:

$$\mathsf{Z}_\phi(T) = \begin{cases} * & \text{if } \pi_0(f_X^*(\phi)) = 0 \ , \\ \emptyset & \text{otherwise} \ . \end{cases}$$

Said differently, $\mathsf{t}_0(\mathsf{Z}_\phi)$ is identified with the subfunctor of $\mathsf{Map}_{\mathsf{dSt}_k}(-, S)$ that sends an underived affine scheme T to the collection of morphisms $f\colon T \to S$ such that $\pi_0(f_X^*(\phi)) = 0$. △

The following lemma is a generalization of [FGI+05, Theorem 5.8 and Remark 5.9] to the context of derived geometry.

Lemma 4.44. *In the setting of Construction 4.42, assume that $\mathcal{G}$ is connective and that $\mathcal{F}$ is S-flat. Then:*

(1) *let $\alpha\colon \mathcal{E} \to \mathcal{G}$ be a morphism with $\mathcal{E}$ connective and let*

$$\psi\colon \mathcal{E} \longrightarrow \mathcal{F}$$

be the morphism obtained by composition with ϕ. If α induces an epimorphism on π_0, then the induced natural transformation

$$\mathsf{Z}_\phi \longrightarrow \mathsf{Z}_\psi$$

is an equivalence after passing to the truncations.

(2) *Assume furthermore that $\mathcal{G}$ is perfect and that $\mathcal{F}$ is S-properly supported. Then, Z_ϕ is a derived scheme and the morphism $i\colon \mathsf{Z}_\phi \to S$ is a closed immersion.*

Proof. Statement (1) follows directly from the definitions and from Remark 4.43.

Let us prove Statement (2). Consider $\mathcal{H}om_X(\mathcal{G}, \mathcal{F}) \simeq \mathcal{G}^\vee \otimes_{\mathcal{O}_X} \mathcal{F}$. Notice that for every $f\colon T \to S$ and every $M \in \mathsf{QCoh}^\heartsuit(T)$, the complex

$$M \otimes_{\mathcal{O}_T} p_{T,*}\mathcal{H}om_{X_T}(f_X^*(\mathcal{G}), f_X^*(\mathcal{F})) \simeq p_{T,*}\big(\mathcal{H}om_{X_T}(f_X^*(\mathcal{G}), p_T^*(M) \otimes f_X^*(\mathcal{F}))\big)$$

is in negative homological degrees (where $p_T\colon X_T := X \times_S T \to T$ and $f_X\colon X_T \to X$ are the naturally induced morphisms). Thus,

$$\mathcal{H} := p_*\mathcal{H}om_X(\mathcal{G}, \mathcal{F})$$

has tor-amplitude $\leqslant 0$ on S. Since $\mathcal{F}$ is S-properly supported, we have $\mathcal{H} \in \mathsf{APerf}(S)$ and therefore $\mathcal{H}$ is perfect. Its dual $\mathcal{H}^\vee$ is connective, so it follows that

$$V := \mathsf{Spec}_S(\mathsf{Sym}_{\mathcal{O}_S}(\mathcal{H}^\vee))$$

is a derived scheme. For every $f\colon T = \mathsf{Spec}(A) \to S$, we have

$$\begin{aligned} \mathsf{Map}_{/S}(T, V) &\simeq \mathsf{Map}_{\mathsf{QCoh}(T)}(f^*(\mathcal{H}^\vee), A) \\ &\simeq \mathsf{Map}_{\mathsf{QCoh}(T)}(A, f^*(\mathcal{H})) \\ &\simeq \mathsf{Map}_{\mathsf{QCoh}(T)}(A, p_{T,*}(\mathcal{H}om_{X_T}(f_X^*(\mathcal{G}), f_X^*(\mathcal{F})))) \ . \end{aligned}$$

It follows that the morphism $\phi\colon \mathcal{G} \to \mathcal{F}$ is classified by a morphism $s_\phi\colon S \to V$. Write $s_0\colon S \to V$ for the zero section, which is clearly a closed immersion. Unraveling the definitions, we see that

$$\begin{array}{ccc} Z_\phi & \xrightarrow{\ i\ } & S \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle s_\phi} \\ S & \xrightarrow{\ s_0\ } & V \end{array}$$

is a derived pullback square, whence the conclusion. □

Proposition 4.45. *Assume that $f\colon Y \to X$ is a closed immersion and assume that X universally satisfies the perfect resolution property. Then*

$$\mathbf{f}_*\colon \mathbf{Coh}_{\mathsf{ps}}(Y) \longrightarrow \mathbf{Coh}_{\mathsf{ps}}(X)$$

is a closed immersion as well.

Proof. This is a statement that only depends on the truncation

$$\mathbf{f}_*\colon \mathfrak{Coh}_{\mathsf{ps}}(Y) \to \mathfrak{Coh}_{\mathsf{ps}}(X) \ .$$

Let S be an underived affine scheme and let $x\colon S \to \mathfrak{Coh}_{\mathsf{ps}}(X)$ be a point classifying a family of S-properly supported, S-flat coherent sheaves $\mathcal{F} \in \mathsf{APerf}(X \times S)$. Since S is underived, S-flatness implies that $\mathcal{F} \in \mathsf{QCoh}^\heartsuit(X \times S)$. Set

$$\mathcal{I}_S := \mathsf{fib}\big(\mathcal{O}_{X\times S} \to f_{S,*}(\mathcal{O}_{Y\times S})\big) \ .$$

Then x belongs to the image of f_* if and only if the map

$$\phi\colon \pi_0(\mathcal{I} \otimes \mathcal{F}) \longrightarrow \mathcal{F}$$

is zero. It follows that

$$S \times_{\mathfrak{Coh}_{\mathsf{ps}}(X)} \mathfrak{Coh}_{\mathsf{ps}}(Y) \simeq Z_\phi \ .$$

Since X universally satisfies the perfect resolution property, we can find a connective perfect complex $\mathcal{E}$ and a morphism

$$\mathcal{E} \longrightarrow \pi_0(\mathcal{I} \otimes \mathcal{F})$$

which induces an epimorphism on π_0. At this point, the conclusion follows from Lemma 4.44. □

Proof of Theorem 4.39. The derived stack $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$ is convergent by definition. It is therefore enough to construct a presentation. Notice that Example 4.34 implies that for every $S \in {}^{<\infty}\mathsf{dAff}_k$ there is an equivalence

$$\operatorname*{colim}_{W \in \mathcal{T}_{Z/\!/X}} \mathbf{Coh}_{\mathsf{ps}}(W)(S) \xrightarrow{\sim} \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)(S) .$$

Besides, it follows from [PS23, §2.3.1] or [HLP23, Theorem 5.2.2] that $\mathbf{Coh}_{\mathsf{ps}}(W)$ is a derived 1-geometric stack, locally almost of finite presentation (see also [Sta25, Tag 0DLX] for the analogous statement on truncations). This shows that condition (2c) in Definition 2.8 is satisfied. We are therefore left to check condition (2b); in other words, we have to check that for a morphism $W \to W'$ in $\mathcal{T}_{Z/\!/X}$ the induced morphism

$$\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(W) \longrightarrow \mathbf{Coh}_{\mathsf{ps}}(W')$$

is a closed immersion almost of finite type. Since both source and target are locally almost of finite presentation, the above morphism is automatically almost of finite type. Since X universally satisfies the perfect resolution property, Lemma 4.37 guarantees that the same goes for W'. At this point, the conclusion follows from Proposition 4.45. □

Theorem 4.39 and Proposition 2.15 yield the following.

Corollary 4.46. *Under the assumptions of Theorem 4.39, the derived prestack $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$ is a derived stack, i.e., it satisfies étale hyperdescent.*

4.5. **Nilpotent versus set-theoretic support.** We keep fixing a field of characteristic zero k. We also fix a quasi-compact and quasi-separated derived k-scheme X together with a closed immersion $j\colon Z \to X$. The goal of this section is to provide an alternative description of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$ in terms of $\mathbf{Coh}_{\mathsf{ps}}(X)$. This alternative description is particularly well suited for explicit computations, but it is not intrinsic in the formal completion $\widehat{X}_Z$ and relies on the extra knowledge of the ambient space X.

Denote by $i\colon U \to X$ the inclusion of the open complementary of Z.

Definition 4.47. Let $S \in \mathsf{dAff}_k$ be an affine derived scheme. The ∞-category of *S-families of quasi-coherent sheaves on X set-theoretically supported on Z* is the kernel

$$\mathsf{QCoh}_Z(X \times S) := \ker\left(i_S^*\colon \mathsf{QCoh}(X \times S) \to \mathsf{QCoh}(U \times S)\right) .$$

Moreover, given $\mathcal{F} \in \mathsf{QCoh}_Z(X \times S)$ we say that $\mathcal{F}$ is *almost perfect* (resp. *S-flat*, *S-properly supported*) if it is almost perfect (resp. S-flat, S-properly supported) as an object in $\mathsf{QCoh}(X \times S)$. ⊘

It is straightforward to check that the above definition cuts a derived substack

$$\mathbf{Coh}_{Z,\mathsf{ps}}(X) \hookrightarrow \mathbf{Coh}_{\mathsf{ps}}(X) ,$$

that parametrizes S-flat families of S-properly supported almost perfect sheaves on X set-theoretically supported on Z.

Notation 4.48. The universal property of colimits produces a well defined map

$$\hat{\jmath}_*\colon \mathsf{QCoh}^{\mathsf{nil}}(\widehat{X}_Z \times S) \longrightarrow \mathsf{QCoh}_Z(X \times S) , \tag{4.5}$$

that sends $[\mathcal{F}]$ to $j_{W,*}(\mathcal{F})$ (see Notation 4.21 for the notation $[\mathcal{F}]$). ⊘

The functor $\hat{\jmath}_*$ in Formula (4.5) induces a well defined morphism at the level of derived stacks

$$\hat{\jmath}_*\colon \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z) \longrightarrow \mathbf{Coh}_{Z,\mathsf{ps}}(X) .$$

We can now state the main result of this section.

Theorem 4.49. *The morphism $\hat{\jmath}_*$ defined above is an equivalence of derived stacks.*

The proof of this theorem will be given in §4.5.2, after introducing pro-quasicoherent sheaves, which are essential to provide an equivalent description for $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$.

4.5.1. *Digression: pro-quasicoherent sheaves.* Pro-quasicoherent sheaves are a technical tool introduced in [GR14, GR17] that plays an important role in the study of formal geometry (for instance in the proof of the indschematic analogue of Lurie's representability theorem, see Theorem 2.1.7.7 in *loc. cit.*). Throughout the section k denotes a fixed animated commutative ring of characteristic zero.

For a derived affine scheme $S \in \mathsf{dAff}_k$, we set

$$\mathsf{ProQCoh}(S) := \mathsf{Pro}(\mathsf{QCoh}(S)) .$$

Given a morphism $f \colon T \to S$ in dAff_k, we have an induced functor

$$\mathsf{Pro}(f^*) \colon \mathsf{ProQCoh}(S) \longrightarrow \mathsf{ProQCoh}(T) .$$

This gives rise to a functor

$$\mathsf{ProQCoh} \colon \mathsf{dAff}_k^{\mathsf{op}} \longrightarrow \mathsf{Cat}_\infty ,$$

which we right Kan extend to

$$\mathsf{ProQCoh} \colon \mathsf{PreSt}_k^{\mathsf{op}} \longrightarrow \mathsf{Cat}_\infty .$$

In other words, for a derived stack X we set

$$\mathsf{ProQCoh}(X) \simeq \lim_{S \in \mathsf{dAff}_{/X}} \mathsf{ProQCoh}(S) .$$

Given a morphism $f \colon X \to Y$ of derived prestacks, we write

$$^{\mathsf{pro}}f^* \colon \mathsf{ProQCoh}(Y) \longrightarrow \mathsf{ProQCoh}(X)$$

for the induced functor.

Observe that

$$\mathsf{Pro}(\mathsf{QCoh}(X))^{\mathsf{op}} \simeq \mathsf{Ind}(\mathsf{QCoh}(X)^{\mathsf{op}}) .$$

Since $(-)^{\mathsf{op}}$ is a self-equivalence of Cat_∞, it follows that, in a larger Grothendieck universe, $\mathsf{ProQCoh}(X)^{\mathsf{op}}$ is presentable for every derived prestack X. Thus, $\mathsf{ProQCoh}(X)$ is both complete and cocomplete. Using the completeness of $\mathsf{ProQCoh}(X)$, we see that the canonical functor

$$\mathsf{QCoh}(X) \longrightarrow \mathsf{ProQCoh}(X)$$

can be extended to

$$\mathsf{Pro}(\mathsf{QCoh}(X)) \longrightarrow \mathsf{ProQCoh}(X) .$$

In general, this is not an equivalence. However, we have the following result.

Proposition 4.50.

(1) *The functor* $\mathsf{ProQCoh} \colon \mathsf{dAff}_k^{\mathsf{op}} \to \mathsf{Cat}_\infty$ *satisfies Zariski descent.*

(2) *For every quasi-compact and quasi-separated derived k-scheme X, the canonical map*

$$\mathsf{Pro}(\mathsf{QCoh}(X)) \longrightarrow \mathsf{ProQCoh}(X)$$

is an equivalence.

Proof. The first point follows from [GR14, Lemma 4.2.6] (see also [Hel20, Proposition 3.2.9] for a more thorough argument in a slightly modified setting).

To prove (2), we first observe that (1) implies that $\mathsf{ProQCoh}$ descends to a colimit-preserving functor

$$\mathsf{Sh}(\mathsf{dAff}_k, \tau_{\mathsf{Zar}})^{\mathsf{op}} \longrightarrow \mathsf{Cat}_\infty .$$

In particular, if $\{U_i\}$ is an affine Zariski open cover of X and $U_\bullet$ denotes its Čech nerve, then the canonical map

$$\mathsf{ProQCoh}(X) \longrightarrow \lim_{[n] \in \mathbf{\Delta}} \mathsf{ProQCoh}(U_n)$$

is an equivalence. It suffices therefore to prove that the canonical map

$$\mathsf{Pro}(\mathsf{QCoh}(X)) \longrightarrow \lim_{[n]\in\mathbf{\Delta}} \mathsf{ProQCoh}(U_n)$$

is an equivalence. Since X is quasi-compact and quasi-separated, we can proceed by induction on the number of affines needed to cover X. At this point, the inductive hypothesis reduces us to check that the canonical map

$$\mathsf{Pro}(\mathsf{QCoh}(X)) \longrightarrow \lim_{[n]\in\mathbf{\Delta}} \mathsf{Pro}(\mathsf{QCoh}(U_n))$$

is an equivalence, which also follows from [GR14, Lemma 4.2.6]. □

Corollary 4.51. *Let $f\colon X \to Y$ be a morphism of quasi-compact and quasi-separated derived k-schemes. Then, the functor*

$$^{\mathsf{pro}}f^*\colon \mathsf{ProQCoh}(Y) \longrightarrow \mathsf{ProQCoh}(X)$$

is canonically identified to $\mathsf{Pro}(f^*)$ and in particular it admits $\mathsf{Pro}(f_*)$ as a right adjoint. Furthermore, $^{\mathsf{pro}}f^*$ commutes with arbitrary limits.

Proof. We use the identifications $\mathsf{ProQCoh}(X) \simeq \mathsf{Pro}(\mathsf{QCoh}(X))$ and $\mathsf{ProQCoh}(Y) \simeq \mathsf{Pro}(\mathsf{QCoh}(Y))$ supplied by Proposition 4.50. Let $\text{“}\lim_\alpha\text{''}\, \mathcal{F}_\alpha \in \mathsf{Pro}(\mathsf{QCoh}(Y))$ and $\text{“}\lim_\beta\text{''}\, \mathcal{G}_\beta \in \mathsf{Pro}(\mathsf{QCoh}(X))$. By definition we have

$$\mathsf{Pro}(f^*)\Big(\text{“}\lim_\alpha\text{''}\, \mathcal{F}_\alpha\Big) \simeq \text{“}\lim_\alpha\text{''}\, f^*(\mathcal{F}_\alpha) \quad \text{and} \quad \mathsf{Pro}(f_*)\Big(\text{“}\lim_\beta\text{''}\, \mathcal{G}_\beta\Big) \simeq \text{“}\lim_\beta\text{''}\, f_*(\mathcal{G}_\beta)\ .$$

Thus:

$$\begin{aligned}
\mathsf{Map}_{\mathsf{Pro}(\mathsf{QCoh}(X))}\Big(\mathsf{Pro}(f^*)\Big(\text{“}\lim_\alpha\text{''}\, \mathcal{F}_\alpha\Big), \text{“}\lim_\beta\text{''}\mathcal{G}_\beta\Big) &\simeq \mathsf{Map}_{\mathsf{Pro}(\mathsf{QCoh}(X))}\Big(\text{“}\lim_\alpha\text{''}\, f^*(\mathcal{F}_\alpha), \text{“}\lim_\beta\text{''}\, \mathcal{G}_\beta\Big) \\
&\simeq \lim_\beta \operatorname{colim}_\alpha \mathsf{Map}_{\mathsf{Pro}(\mathsf{QCoh}(X))}\Big(f^*(\mathcal{F}_\alpha), \mathcal{G}_\beta\Big) \\
&\simeq \lim_\beta \operatorname{colim}_\alpha \mathsf{Map}_{\mathsf{Pro}(\mathsf{QCoh}(X))}\Big(\mathcal{F}_\alpha, f_*(\mathcal{G}_\beta)\Big) \\
&\simeq \mathsf{Map}_{\mathsf{Pro}(\mathsf{QCoh}(X))}\Big(\text{“}\lim_\alpha\text{''}\, \mathcal{F}_\alpha, \text{“}\lim_\beta\text{''}\, f_*(\mathcal{G}_\beta)\Big) \\
&\simeq \mathsf{Map}_{\mathsf{Pro}(\mathsf{QCoh}(X))}\Big(\text{“}\lim_\alpha\text{''}\, \mathcal{F}_\alpha, \mathsf{Pro}(f_*)\Big(\text{“}\lim_\beta\text{''}\, \mathcal{G}_\beta\Big)\Big)\ ,
\end{aligned}$$

which shows that the adjunction $\mathsf{Pro}(f^*) \dashv \mathsf{Pro}(f_*)$ holds. Furthermore, $\mathsf{Pro}(f^*)$ commutes with filtered limits and with finite colimits. Since $\mathsf{Pro}(\mathsf{QCoh}(Y))$ and $\mathsf{Pro}(\mathsf{QCoh}(X))$ are stable, $\mathsf{Pro}(f^*)$ commutes with finite limits as well. Therefore, $\mathsf{Pro}(f^*)$ commutes with arbitrary limits. □

Corollary 4.52. *Let*

$$\begin{array}{ccc}
X' & \xrightarrow{g} & X \\
\big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle p} \\
Y' & \xrightarrow{f} & Y
\end{array}$$

be a pullback square of derived prestacks. Assume that p is representable by quasi-compact and quasi-separated derived schemes. Then, the induced square

$$\begin{array}{ccc}
\mathsf{ProQCoh}(Y) & \xrightarrow{^{\mathsf{pro}}f^*} & \mathsf{ProQCoh}(Y') \\
\big\downarrow{\scriptstyle ^{\mathsf{pro}}p^*} & & \big\downarrow{\scriptstyle ^{\mathsf{pro}}q^*} \\
\mathsf{ProQCoh}(X) & \xrightarrow{^{\mathsf{pro}}g^*} & \mathsf{ProQCoh}(X')
\end{array}$$

is vertically right adjointable.

Proof. Since Beck-Chevalley transformations are stable under composition of squares, we can assume without loss of generality that Y' is an affine derived scheme. In this case, observe that the statement holds when Y is a quasi-compact and quasi-separated derived scheme, as a consequence of the identifications ${}^{\mathrm{pro}}p^* \simeq \mathrm{Pro}(p^*)$ and ${}^{\mathrm{pro}}p_* \simeq \mathrm{Pro}(p_*)$ supplied by Corollary 4.51.

Let $S \in \mathsf{dAff}_k$ be an affine derived scheme and let $S \to Y$ be any morphism. By assumption, $X_S \coloneqq X \times_Y S$ is a quasi-compact and quasi-separated derived scheme. Write $p_S \colon X_S \to S$ for the canonical projection. Notice that the canonical morphism

$$\operatorname*{colim}_{S \in (\mathsf{dAff}_k)_{/Y}} X_S \longrightarrow X$$

is an equivalence in PreSt_k. Thus, the conclusion follows from the case already dealt with. □

4.5.2. *Comparison between the two definitions.* We fix a quasi-compact and quasi-separated derived scheme X together with a closed immersion $j \colon Z \to X$. In this section, we shall prove Theorem 4.49. It will be deduced from the following more precise categorical statement.

Theorem 4.53. *Let $S \in \mathsf{dAff}_k$ be a derived affine scheme. Then:*

1. *The functor*

$$\hat{\jmath}_* \colon \mathsf{QCoh}^{\mathsf{nil}}(\widehat{X}_Z \times S) \longrightarrow \mathsf{QCoh}_Z(X \times S)$$

is fully faithful.

2. *Let $\mathfrak{F} \in \mathsf{APerf}^{\mathsf{nil}}(\widehat{X}_Z \times S)$. Then $\mathfrak{F}$ is S-flat (resp. S-properly supported) if and only if $\hat{\jmath}_*(\mathfrak{F})$ is.*
3. *The functor $\hat{\jmath}_*$ preserves almost perfect objects. Conversely, if $\mathcal{F} \in \mathsf{QCoh}_Z(X \times S)$ is almost perfect and bounded (that is, $\pi_i(\mathcal{F}) \simeq 0$ for $i \gg 0$), then $\mathcal{F}$ lies in the essential image of $\hat{\jmath}_*$.*

The proof of Theorem 4.53 will occupy the rest of this section. Assuming it, let us give a proof of Theorem 4.49.

Proof of Theorem 4.49. We first notice that the derived stack $\mathbf{Coh}_{Z,\mathsf{ps}}(X)$ fits in the following pullback square:

$$\begin{array}{ccc} \mathbf{Coh}_{Z,\mathsf{ps}}(X) & \longrightarrow & \mathbf{Coh}_{\mathsf{ps}}(X) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{\ 0\ } & \mathbf{Coh}(U)\ , \end{array} \tag{4.6}$$

where the morphism $0 \colon \mathrm{Spec}(k) \to \mathbf{Coh}(U)$ selects the zero sheaf. It immediately follows that $\mathbf{Coh}_{Z,\mathsf{ps}}(X)$ is convergent. Since $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$ is convergent as well, it is enough to prove that for every $S \in {}^{<\infty}\mathsf{dAff}_k$, the morphism $\hat{\jmath}_*$ induces an equivalence

$$\hat{\jmath}_* \colon \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)(S) \longrightarrow \mathbf{Coh}_{Z,\mathsf{ps}}(X)(S)\ .$$

For this, it is enough to prove that this morphism is fully faithful and essentially surjective. Full faithfulness follows from Theorem 4.53–(1). On the other hand, essential surjectivity follows from points (2) and (3) of the same theorem, paired with the observation that since S is eventually coconnective, any almost perfect and S-flat sheaf $\mathcal{F} \in \mathsf{QCoh}_Z(X \times S)$ satisfies $\pi_i(\mathcal{F}) \simeq 0$ for $i \gg 0$. □

Before diving in the proof of Theorem 4.53, let us discuss one important consequence of Theorem 4.49. Composing the top horizontal arrow in the diagram (4.6) with $\hat{\jmath}_*$ we obtain a canonical morphism

$$\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z) \longrightarrow \mathbf{Coh}_{\mathsf{ps}}(X)\ . \tag{4.7}$$

We have the following.

Corollary 4.54. *The morphism (4.7) is formally étale and it exhibits* $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)$ *as the formal completion of* $\mathbf{Coh}_{\mathsf{ps}}(X)$ *along* ${}^{\mathsf{red}}\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)$. *In other words, the square*

$$\begin{array}{ccc} \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z) & \longrightarrow & \mathbf{Coh}_{\mathsf{ps}}(X) \\ \downarrow & & \downarrow \\ \left({}^{\mathsf{red}}\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)\right)_{\mathsf{dR}} & \longrightarrow & \mathbf{Coh}_{\mathsf{ps}}(X)_{\mathsf{dR}} \end{array}$$

is a pullback.

Proof. We start by the first statement. Since $\hat{\jmath}_*$ is an equivalence by Theorem 4.49, it is enough to prove that the top horizontal morphism in the square (4.6) is formally étale. Since that square is a pullback, it suffices to show that the bottom horizontal arrow is formally étale, which is indeed true as a consequence of the Nakayama lemma . This prove the first statement. As for the second one, in virtue of Lemma 4.7 and what we just proved, it suffices to argue that both $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)$ and $\mathbf{Coh}(X)$ are convergent and laft. For the latter, both properties are well known; on the other hand the former is convergent by definition. Concerning it being laft, observe that the pullback description (4.6) implies that $\mathbf{Coh}_{Z,\mathsf{ps}}(X)$ is the pullback of three laft stack, and that it is therefore laft itself. At this point, the conclusion follows from Theorem 4.49. □

Remark 4.55. It is natural to wonder whether ${}^{\mathsf{red}}\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)$ is actually a closed substack of $\mathbf{Coh}_{\mathsf{ps}}(X)$. This is is not entirely obvious, but it is indeed true, at least under some mild additional assumptions on X. See Corollary 4.66.
△

Before giving the proof of Theorem 4.49, we need some preliminaries that heavily rely on the machinery of pro-quasi-coherent sheaves.

Notation 4.56. Let W_0 be a thickening of Z inside X. To keep the notations manageable, write

$$j_{W_0}\colon W_0 \times S \longrightarrow X \times S\ , \quad \hat{\jmath}_{W_0}\colon W_0 \times S \longrightarrow \widehat{X}_Z \times S\ , \quad \hat{\jmath}\colon \widehat{X}_Z \times S \longrightarrow X \times S$$

for the canonical morphisms. Similarly, given a further thickening $W \in \mathcal{T}_{W_0 /\!/ X}$, we write

$$j_{W_0,W}\colon W_0 \times S \longrightarrow W \times S$$

for the canonical morphism. ⊘

Construction 4.57. It follows from Corollary 4.14 that $\hat{\jmath}_{W_0}\colon W_0 \times S \to \widehat{X}_Z \times S$ is representable by derived schemes. In particular, Corollary 4.52 (applied with $Y' = Y$) implies that the pro-restriction functor

$${}^{\mathsf{pro}}\hat{\jmath}_{W_0}^*\colon \mathsf{ProQCoh}(\widehat{X}_Z \times S) \longrightarrow \mathsf{ProQCoh}(W_0 \times S)$$

admits a right adjoint ${}^{\mathsf{pro}}\hat{\jmath}_{W_0,*}$. Thus, the commutative triangle

$$\begin{array}{ccc} \mathsf{ProQCoh}(X \times S) & \xrightarrow{\ {}^{\mathsf{pro}}\hat{\jmath}^*\ } & \mathsf{ProQCoh}(\widehat{X}_Z \times S) \\ & {\scriptstyle {}^{\mathsf{pro}}j_{W_0}^*}\searrow \quad \swarrow{\scriptstyle {}^{\mathsf{pro}}\hat{\jmath}_{W_0}^*} & \\ & \mathsf{ProQCoh}(W_0 \times S) & \end{array}$$

gives rise to a Beck-Chevalley transformation

$$\mathsf{BC}\colon {}^{\mathsf{pro}}\hat{\jmath}^*\, {}^{\mathsf{pro}}j_{W_0,*} \longrightarrow {}^{\mathsf{pro}}\hat{\jmath}_{W_0,*}\, {}^{\mathsf{pro}}\hat{\jmath}_{W_0}^*\, {}^{\mathsf{pro}}\hat{\jmath}^*\, {}^{\mathsf{pro}}j_{W_0,*} \simeq {}^{\mathsf{pro}}\hat{\jmath}_{W_0,*}\, {}^{\mathsf{pro}}j_{W_0}^*\, {}^{\mathsf{pro}}j_{W_0,*} \longrightarrow {}^{\mathsf{pro}}\hat{\jmath}_{W_0,*}\ ,$$

and therefore to a transformation

$${}^{\mathsf{pro}}j_{W_0}^*\, {}^{\mathsf{pro}}j_{W_0,*} \simeq {}^{\mathsf{pro}}\hat{\jmath}_{W_0}^*\, {}^{\mathsf{pro}}\hat{\jmath}^*\, {}^{\mathsf{pro}}j_{W_0,*} \xrightarrow{\mathsf{BC}} {}^{\mathsf{pro}}\hat{\jmath}_{W_0}^*\, {}^{\mathsf{pro}}\hat{\jmath}_{W_0,*}\ . \tag{4.8}$$

On the other hand, since $j_{W_0}\colon W_0 \times S \to X \times S$ is a map between quasi-compact and quasi-separated derived schemes, Corollary 4.51 implies that for every $\mathcal{F} \in \mathsf{QCoh}(W_0 \times S)$ there is a canonical identification

$$j_{W_0}^* j_{W_0,*}(\mathcal{F}) \simeq {}^{\mathsf{pro}}j_{W_0}^* \, {}^{\mathsf{pro}}j_{W_0,*}(\mathcal{F}) ,$$

so (4.8) restricts to a well defined transformation

$$j_{W_0}^* j_{W_0,*} \longrightarrow {}^{\mathsf{pro}}\hat{\jmath}_{W_0}^* \, {}^{\mathsf{pro}}\hat{\jmath}_{W_0,*} \tag{4.9}$$

between functors from $\mathsf{QCoh}(W_0 \times S)$ to $\mathsf{ProQCoh}(W_0 \times S)$. ⊘

The following is the key technical input needed for the proof of Theorem 4.53–(1).

Lemma 4.58. *In the setting of Construction 4.57, for every $\mathcal{F} \in \mathsf{QCoh}(W_0 \times S)$:*

(1) the natural transformation (4.9)

$$j_{W_0}^* j_{W_0,*}(\mathcal{F}) \longrightarrow {}^{\mathsf{pro}}\hat{\jmath}_{W_0}^* \, {}^{\mathsf{pro}}\hat{\jmath}_{W_0,*}(\mathcal{F})$$

is an equivalence;

(2) there is a canonical identification

$${}^{\mathsf{pro}}\hat{\jmath}_{W_0}^* \, {}^{\mathsf{pro}}\hat{\jmath}_{W_0,*}(\mathcal{F}) \simeq \underset{W \in \mathfrak{T}_{W_0 /\!/ X}^{\mathrm{op}}}{\text{“lim”}} j_{W_0,W}^* j_{W_0,W,*}(\mathcal{F})$$

in $\mathsf{ProQCoh}(W_0 \times S)$.

In particular, the pro-object on the right hand side is equivalent to a constant one.

Proof. We first prove (1). It follows from Lemma 4.12 that the canonical morphism

$$W_0 \times_{\widehat{X}_Z} W_0 \longrightarrow W_0 \times_X W_0$$

is an equivalence. In other words, writing W_0' for the above derived scheme, both the squares

$$\begin{array}{ccc} W_0' \times S & \xrightarrow{p} & W_0 \times S \\ \downarrow q & & \downarrow \hat{\jmath}_{W_0} \\ W_0 \times S & \xrightarrow{\hat{\jmath}_{W_0}} & \widehat{X}_Z \times S \end{array} \quad \text{and} \quad \begin{array}{ccc} W_0' \times S & \xrightarrow{p} & W_0 \times S \\ \downarrow q & & \downarrow j_{W_0} \\ W_0 \times S & \xrightarrow{j_{W_0}} & X \times S \end{array}$$

are pullback. Derived base-change [Lur18, Proposition 2.5.4.5] applies to the right square, yielding a canonical identification

$$j_{W_0}^* j_{W_0}^* \simeq p_* q^* .$$

On the other hand, Corollary 4.14 shows that $\hat{\jmath}_{W_0}$ is representable by quasi-compact and quasi-separated derived schemes and therefore Corollary 4.52 supplies a canonical identification

$${}^{\mathsf{pro}}\hat{\jmath}_{W_0}^* \, {}^{\mathsf{pro}}\hat{\jmath}_{W_0,*} \simeq {}^{\mathsf{pro}}p_* \, {}^{\mathsf{pro}}q^* .$$

At this point, the conclusion follows from Corollary 4.51, that supplies a canonical identification

$${}^{\mathsf{pro}}p_* \, {}^{\mathsf{pro}}q^*(\mathcal{F}) \simeq p_* q^*(\mathcal{F})$$

whenever $\mathcal{F} \in \mathsf{QCoh}(W_0 \times S)$.

We now prove (2). Using Proposition 4.4, we can write

$$\widehat{X}_Z \simeq \underset{W \in \mathfrak{T}_{W_0 /\!/ X}}{\mathrm{colim}} W ,$$

the colimit being computed in PreSt_k. It follows from the fact that PreSt_k is an ∞-topos that there is an equivalence

$$(W_0 \times_{\widehat{X}_Z} W_0) \times S \simeq \underset{W \in \mathfrak{T}_{W_0 /\!/ X}}{\mathrm{colim}} (W_0 \times_W W_0) \times S$$

in PreSt_k. Notice that the functor $\mathsf{ProQCoh}\colon \mathsf{PreSt}_k^{\mathsf{op}} \to \mathsf{Cat}_\infty$ preserves limits by construction; thus, we have

$$\mathsf{ProQCoh}((W_0 \times_{\widehat{X}_Z} W_0) \times S) \simeq \lim_{W \in \mathfrak{T}^{\mathsf{op}}_{W_0 /\!/ X}} \mathsf{ProQCoh}((W_0 \times_W W_0) \times S) .$$

Consider now the pullback square

$$\begin{array}{ccc} (W_0 \times_W W_0) \times S & \xrightarrow{p_W} & W_0 \times S \\ \big\downarrow q_W & & \big\downarrow j_{W_0,W} \\ W_0 \times S & \xrightarrow{j_{W_0,W}} & W \times S \end{array} .$$

Then combining [PY16, §8.2] and Corollary 4.51, we obtain that for every $\mathcal{F} \in \mathsf{QCoh}(W_0 \times S)$ there is a canonical identification

$${}^{\mathsf{pro}}p_* \, {}^{\mathsf{pro}}q^*(\mathcal{F}) \simeq \lim_{W \in \mathfrak{T}^{\mathsf{op}}_{W_0 /\!/ X}} p_{W,*} q_W^*(\mathcal{F}) ,$$

the limit being computed in $\mathsf{ProQCoh}(W_0 \times S)$. Recall from Proposition 4.50–(2) that there is a canonical equivalence

$$\mathsf{ProQCoh}(W_0 \times S) \simeq \mathsf{Pro}(\mathsf{QCoh}(W_0 \times S)) .$$

We can therefore write

$${}^{\mathsf{pro}}p_* \, {}^{\mathsf{pro}}q^*(\mathcal{F}) \simeq \underset{W \in \mathfrak{T}^{\mathsf{op}}_{W_0 /\!/ X}}{\text{“}\lim\text{”}} p_{W,*} q_W^*(\mathcal{F}) .$$

At this point, using derived base-change once more we obtain canonical identifications

$$p_{W,*} q_W^*(\mathcal{F}) \simeq j^*_{W_0,W} j_{W_0,W,*}(\mathcal{F}) ,$$

and the conclusion follows. □

We are now ready to prove the theorem.

Proof of Theorem 4.53. We start proving the full faithfulness. Let $\mathfrak{F}, \mathfrak{G} \in \mathsf{QCoh}^{\mathsf{nil}}(\widehat{X}_Z \times S)$. Without loss of generality, we can assume that $\mathfrak{F} \simeq [\mathcal{F}]$ and $\mathfrak{G} \simeq [\mathcal{G}]$ for almost perfect complexes $\mathcal{F}$ and $\mathcal{G}$ defined on the same thickening W_0. We have to prove that the canonical comparison map

$$\mathsf{Map}_{\mathsf{QCoh}^{\mathsf{nil}}(\widehat{X}_Z \times S)}([\mathcal{F}], [\mathcal{G}]) \longrightarrow \mathsf{Map}_{\mathsf{QCoh}(X \times S)}(j_{W_0,*}(\mathcal{F}), j_{W_0,*}(\mathcal{G})) \tag{4.10}$$

is an equivalence. For every intermediate thickening $W \in \mathfrak{T}_{W_0 /\!/ X}$, write

$$j_{W_0,W} \colon W_0 \times S \longrightarrow W \times S , \qquad j_W \colon W \times S \longrightarrow X \times S , \qquad \hat{\jmath}_W \colon W \times S \longrightarrow \widehat{X}_Z \times S$$

for the canonical morphisms. Then, the left-hand-side of (4.10) can be written as

$$\begin{aligned} \operatorname*{colim}_{W \in \mathfrak{T}_{W_0 /\!/ X}} \mathsf{Map}_{\mathsf{QCoh}(W \times S)}(j_{W_0,W,*}(\mathcal{F}), j_{W_0,W,*}(\mathcal{G})) &\simeq \operatorname*{colim}_{W \in \mathfrak{T}_{W_0 /\!/ X}} \mathsf{Map}_{\mathsf{QCoh}(W_0 \times S)}(j^*_{W_0,W} j_{W_0,W,*}(\mathcal{F}), \mathcal{G}) \\ &\simeq \mathsf{Map}_{\mathsf{Pro}(\mathsf{QCoh}(W_0 \times S))}\Big(\underset{W \in \mathfrak{T}_{W_0 /\!/ X}}{\text{“}\lim\text{”}} j^*_{W_0,W} j_{W_0,W,*}(\mathcal{F}), \mathcal{G}\Big) . \end{aligned}$$

On the other hand, one has

$$\mathsf{Map}_{\mathsf{APerf}(X \times S)}(j_{W_0,*}(\mathcal{F}), j_{W_0,*}(\mathcal{G})) \simeq \mathsf{Map}_{\mathsf{QCoh}(W_0 \times S)}(j^*_{W_0} j_{W_0,*}(\mathcal{F}), \mathcal{G}) ,$$

so we reduce ourselves to check that the naturally induced morphism

$$j^*_{W_0} j_{W_0,*}(\mathcal{F}) \longrightarrow \underset{W \in \mathfrak{T}_{W_0 /\!/ X}}{\text{“}\lim\text{”}} j^*_{W_0,W} j_{W_0,W,*}(\mathcal{F})$$

is an equivalence in

$$\mathsf{ProQCoh}(W_0 \times S) \simeq \mathsf{Pro}(\mathsf{QCoh}(W_0 \times S)) .$$

This is exactly what has been checked in points (1) and (2) of Lemma 4.58, so the conclusion follows.

After unraveling the definitions, point (2) follows directly from Lemma 4.30, which also implies the direct implication of point (3).

We are left to check the converse implication. We proceed by induction on the number m of non-vanishing homotopy groups of $\mathcal{F}$. When $m = 0$, up to a shift we can assume that $\mathcal{F} \in \mathsf{APerf}^{\heartsuit}(X \times S) \simeq \mathsf{APerf}^{\heartsuit}(\mathrm{t}_0(X \times S))$. Using Example 4.20–(3), we see that it is enough to find a thickening W_0 of $Z \times S$ inside $X \times S$ and $\mathcal{F}_0 \in \mathsf{APerf}^{\heartsuit}(W)$ together with an identification $\mathcal{F} \simeq j_{W_0,*}(\mathcal{F}_0)$. Since $\mathcal{F}|_{X \smallsetminus Z} \simeq 0$ and $\mathcal{F}$ is coherent, it is well known that this is possible. Let us now handle the inductive step. Let $\mathcal{F} \in \mathsf{APerf}_Z(X \times S)$ be bounded. Up to a shift, we can assume that $\mathcal{F}$ is connective and that m is the biggest integer such that $\pi_{m+1}(\mathcal{F}) \neq 0$. Consider the fiber sequence

$$\pi_{m+1}(\mathcal{F})[m+1] \longrightarrow \mathcal{F} \longrightarrow \tau_{\leqslant m}(\mathcal{F}) \ .$$

Since the restriction $i^* \colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(X \smallsetminus Z)$ is t-exact, using the inductive hypothesis we find two thickenings W_1 and W_2 of $Z \times S$ inside $X \times S$ together with almost perfect sheaves $\mathcal{F}_1 \in \mathsf{APerf}(W_1)$ and $\mathcal{F}_2 \in \mathsf{APerf}(W_2)$ and with identifications

$$j_{W_1,*}(\mathcal{F}_1) \simeq \pi_{m+1}(\mathcal{F})[m+1] \ , \qquad j_{W_2,*}(\mathcal{F}_2) \simeq \tau_{\leqslant m}(\mathcal{F}) \ .$$

Recall from Proposition 4.4–(1) that the category of thickenings $\mathcal{T}_{Z \times S /\!/ X \times S}$ is filtered. We can therefore assume without loss of generality that $W_1 = W_2$. We will write W_0 instead of W_1 or W_2. Observe now that the extension $\mathcal{F}$ determines a canonical element in

$$\pi_0 \mathsf{Map}_{\mathsf{QCoh}(X \times S)}\big(\tau_{\leqslant m}(\mathcal{F}), \pi_{m+1}(\mathcal{F})[m+2]\big) \simeq \pi_0 \mathsf{Map}_{\mathsf{QCoh}(X \times S)}\big(j_{W_0,*}(\mathcal{F}_1), j_{W_0,*}(\mathcal{F}_2)\big) \ .$$

On the other hand, as in the first half of this proof, Theorem 4.53–(1) supplies a canonical identification

$$\pi_0 \mathsf{Map}_{\mathsf{QCoh}(X \times S)}\big(j_{W_0,*}(\mathcal{F}_1), j_{W_0,*}(\mathcal{F}_2)\big) \simeq \operatorname*{colim}_{W \in \mathcal{T}_{W_0 /\!/ X \times S}} \pi_0 \mathsf{Map}_{\mathsf{QCoh}(W)}\big(j_{W_0,W,*}(\mathcal{F}_1), j_{W_0,W,*}(\mathcal{F}_2)\big) \ ,$$

where we used the fact that π_0 commutes with filtered colimits in Spc. Thus, there exists a thickening W of W_0 such that the extension determined by $\mathcal{F}$ is defined at the W-level. The conclusion follows. □

4.6. **Admissibility.** Let X be a quasi-compact scheme and let $Z \hookrightarrow X$ be a closed immersion. We saw in Theorem 4.39 that mild assumptions on X imply that $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$ is indgeometric. The goal of this section is to prove that under slightly stronger assumptions, $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$ is admissible (see Definition 2.16). We will achieve this by constructing an explicit admissible open exhaustion of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$ given by *Harder-Narasimhan strata*.

To begin with, fix a thickening $W \in \mathcal{T}_{Z /\!/ X}$. Construction 4.41 yields a well-defined morphism

$$\mathbf{j}_{W,*} \colon \mathbf{Coh}_{\mathsf{ps}}(W) \longrightarrow \mathbf{Coh}_{\mathsf{ps}}(X) \ ,$$

and the universal property of colimits and of the convergent completion (see Recollection 2.3) produce a canonical morphism

$$\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z) \longrightarrow \mathbf{Coh}_{\mathsf{ps}}(X) \ .$$

We will use it to induce the desired admissible open exhaustion of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$, out of the geometry of $\mathbf{Coh}_{\mathsf{ps}}(X)$.

To begin our construction, we start assuming that X is a projective scheme over a field k. This condition will be relaxed later on, see Theorem 4.65.

We denote by H an ample divisor on X. Given any coherent sheaf $\mathcal{E}$ on X, we define as usual the *Hilbert polynomial* $P_H(\mathcal{E}, t)$ *of* $\mathcal{E}$ as

$$P_H(\mathcal{E}, t) \coloneqq \chi(E \otimes \mathcal{O}_X(tH)) \in \mathbb{Q}[m] \ .$$

By [HL10, Lemma 1.2.1], $P_H(\mathcal{E}, t)$ can be uniquely written in the form

$$P_H(\mathcal{E}, t) = \sum_{i=0}^{\dim(\mathcal{E})} c_i(\mathcal{E}) \frac{t^i}{i!} .$$

Following [Sim94, §1], when $\dim(\mathcal{E}) \geq 1$, the *reduced Hilbert polynomial of* $\mathcal{E}$ is

$$p_H(\mathcal{E}, t) \coloneqq \frac{P_H(\mathcal{E}, t)}{c_{\dim(\mathcal{E})}(\mathcal{E})} .$$

Recall moreover that there is a natural ordering of polynomials given by the lexicographic order of their coefficients. Explicitly, $P(\leq)P'$ if and only if $P(m)(\leq)P'(m)$ for $m \gg 0$.[4] We can now introduce a notion of semistability.

Definition 4.59. A coherent sheaf $\mathcal{E}$ of dimension $m \geq 1$ on X is (semi)stable if it is pure and if for all proper non-trivial subsheaves $\mathcal{E}' \subsetneq \mathcal{E}$ one has $p_H(\mathcal{E}')(\leq)p_H(\mathcal{E})$. ⊘

If $\mathcal{E}$ is a coherent sheaf of dimension $m \geq 1$, it admits a unique filtration in (the *Harder-Narasimhan filtration*)

$$0 \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_\ell \coloneqq \mathcal{E}$$

where $\mathcal{E}_0 \subset \mathcal{E}$ is the maximal $(m-1)$-dimensional subsheaf of $\mathcal{E}$ and the factors $\mathcal{E}_i/\mathcal{E}_{i-1}$ are semistable and their reduced Hilbert polynomial satisfy

$$p_H(\mathcal{E}_1/\mathcal{E}_0) > \cdots > p_H(\mathcal{E}/\mathcal{E}_{\ell-1}) .$$

Employing standard terminology, the *minimal Hilbert polynomial of* $\mathcal{E}$ is defined by

$$p_{H\text{-}\min}(\mathcal{E}) \coloneqq p_H(\mathcal{E}/\mathcal{E}_{\ell-1}) .$$

If $\mathcal{E}_0 = 0$, *the maximal Hilbert polynomial of* $\mathcal{E}$ is defined by

$$p_{H\text{-}\max}(\mathcal{E}) \coloneqq p_H(\mathcal{E}_1) .$$

For any fixed polynomial $P(t) \in \mathbb{Q}[t]$, we let $\mathbf{Coh}_{\mathsf{ps}}(X; P(t))$ be the open and closed substack of $\mathbf{Coh}_{\mathsf{ps}}(X)$ parametrizing flat and properly supported families of coherent sheaves having $P(t)$ as their Hilbert polynomial. We also set

$$\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; P(t)) \coloneqq \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z) \times_{\mathbf{Coh}_{\mathsf{ps}}(X)} \mathbf{Coh}_{\mathsf{ps}}(X; P(t)) .$$

Similarly, for every thickening $W \in \mathfrak{T}_{Z /\!/ X}$, we set

$$\mathbf{Coh}_{\mathsf{ps}}(W; P(t)) \coloneqq \mathbf{Coh}_{\mathsf{ps}}(W) \times_{\mathbf{Coh}_{\mathsf{ps}}(X)} \mathbf{Coh}_{\mathsf{ps}}(X; P(t)) .$$

It follows that

$$\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z) = \bigsqcup_{P(t) \in \mathbb{Q}[t]} \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; P(t)) \quad \text{and} \quad \mathbf{Coh}_{\mathsf{ps}}(W) = \bigsqcup_{P(t) \in \mathbb{Q}[t]} \mathbf{Coh}_{\mathsf{ps}}(W; P(t)) .$$

Furthermore, there is a canonical morphism in PreSt_k

$$\operatorname*{colim}_{W \in \mathfrak{T}_{Z /\!/ X}} \mathbf{Coh}_{\mathsf{ps}}(W; P(t)) \longrightarrow \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; P(t))$$

which exhibits $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; P(t))$ as the convergent completion of the colimit.

This analysis reduces us to construct admissible open covers for each $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; P(t))$. This will be achieved thanks to the properties of Harder-Narasimhan filtrations. We therefore fix a polynomial $P(t)$ of degree m, as well as an auxiliary monic polynomial of the same degree $\alpha(t) = \sum_{i=0}^{m} \alpha_i t^i / i! \in \mathbb{Q}[t]$. When $m \geqslant 1$, we define

$$\mathfrak{U}_\alpha(X; P(t)) \subset \mathbf{Coh}_{\mathsf{ps}}(X; P(t))$$

[4] Here, we are using the convention in [HL10, Notation 1.2.5].

as the open substack parametrizing S-flat and properly supported families of coherent sheaves $\mathcal{E}$ on X for which on every geometric point $s \in S$ the inequality

$$p_{H\text{-}\min}(\mathcal{E}_s, t) > \alpha(t)$$

holds. When instead $m = 0$, i.e., $P(t)$ is constant, we conventionally set

$$\mathfrak{U}_\alpha(X; P(t)) \coloneqq \mathbf{Coh}_{\mathsf{ps}}(X; P(t)) ,$$

for any (constant) α. In either case, $\mathfrak{U}_\alpha(X; P(t))$ is a quasi-compact and quasi-separated open substack of $\mathbf{Coh}_{\mathsf{ps}}(X; P(t))$. For a fixed thickening $W \in \mathfrak{T}_{Z /\!/ X}$ we set

$$\mathfrak{U}_\alpha(W; P(t)) \coloneqq \mathbf{Coh}_{\mathsf{ps}}(W; P(t)) \times_{\mathbf{Coh}_{\mathsf{ps}}(X;P(t))} \mathfrak{U}_\alpha(X; P(t)) .$$

Similarly, we introduce

$$\mathfrak{U}_\alpha(\widehat{X}_Z; P(t)) \coloneqq \mathbf{Coh}_{\mathsf{ps}}(\widehat{X}_Z; P(t)) \times_{\mathbf{Coh}_{\mathsf{ps}}(X;P(t))} \mathfrak{U}_\alpha(X; P(t)) .$$

We have:

Lemma 4.60. *For every choice of polynomials $P(t), \alpha(t) \in \mathbb{Q}[t]$ as above, $\mathfrak{U}_\alpha(\widehat{X}_Z; P(t))$ is an ind-qcqs indgeometric derived stack.*

Proof. There is a canonical map in PreSt_k

$$\operatorname*{colim}_{W \in \mathfrak{T}_{Z/\!/X}} \mathfrak{U}_\alpha(W; P(t)) \longrightarrow \mathfrak{U}_\alpha(\widehat{X}_Z; P(t)) ,$$

which exhibits $\mathfrak{U}_\alpha(\widehat{X}_Z; P(t))$ as the convergent completion of the colimit. Besides, for every morphism $j_{W,W'} \colon W \to W'$ in $\mathfrak{T}_{Z/\!/X}$, there is an induced square

$$\begin{array}{ccc} \mathfrak{U}_\alpha(W; P(t)) & \longrightarrow & \mathfrak{U}_\alpha(W'; P(t)) \\ \downarrow & & \downarrow \\ \mathbf{Coh}_{\mathsf{ps}}(W; P(t)) & \xrightarrow{\mathbf{j}_{W,W',*}} & \mathbf{Coh}_{\mathsf{ps}}(W'; P(t)) , \end{array}$$

which furthermore is a pullback. Since X is projective, Lemma 4.37 and Corollary 4.38 imply that W' universally satisfies the perfect resolution property. Therefore, Proposition 4.45 implies that $\mathbf{j}_{W,W',*}$ is a closed immersion. Thus, the same goes for the top horizontal map in the above square. Finally, using Proposition 4.45 once more, we see that the natural morphism

$$\mathbf{j}_{W,*} \colon \mathbf{Coh}_{\mathsf{ps}}(W; P(t)) \longrightarrow \mathbf{Coh}_{\mathsf{ps}}(X; P(t))$$

is a closed immersion. In particular, it is quasi-compact and it thus follows that each $\mathfrak{U}_\alpha(W; P(t))$ is quasi-compact and quasi-separated. □

For any $k \in \mathbb{N}$, $k \geq 1$, let $Z^{(k)}$ denote Gaitsgory-Rozenblyum's k-thickening of Z along X (cf. Remark 4.6). We shall show the following.

Proposition 4.61. *Fix polynomials $P(t), \alpha(t) \in \mathbb{Q}[t]$ of the same degree m such that $\alpha(t)$ is monic. Then, there exists $k \in \mathbb{N}$, with $k \geq 1$, depending on $P(t)$ and $\alpha(t)$, such that the canonical map*

$$\mathfrak{U}_\alpha(\mathrm{t}_0(Z^{(k_1)}); P(t)) \longrightarrow \mathfrak{U}_\alpha(\mathrm{t}_0(Z^{(k_2)}); P(t))$$

is a nil-equivalence for all $k \leq k_1 \leq k_2$. In particular, $\mathfrak{U}_\alpha(\widehat{X}_Z; P(t))$ is a qcqs indgeometric derived stack.

We need some preliminary results. The following proposition will be proved in Appendix A.

Proposition 4.62. *Let $\mathcal{F}$ be a pure coherent sheaf on X of dimension $m \geq 1$, set-theoretically supported on Z. Then, there exists a filtration*

$$0 =: \mathcal{F}_{\ell+1} \subset \mathcal{F}_\ell \subset \mathcal{F}_{\ell-1} \subset \cdots \subset \mathcal{F}_0 \coloneqq \mathcal{F} ,$$

for $\ell \geq 1$, so that each subquotient is a pure m-dimensional sheaf with scheme-theoretic support contained in Z.

Lemma 4.63. *Fix a positive integer c. Then, any coherent sheaf $\mathcal{F}$ on X of dimension m, set-theoretically supported on Z, with fixed Hilbert polynomial having c as leading coefficient, which is also pure if $m \geq 1$, is scheme-theoretically supported on $\mathrm{t}_0(Z^{(c)})$.*

Proof. Let $\mathcal{F}$ be a coherent sheaf on X of dimension m, set-theoretically supported on Z, with fixed Hilbert polynomial $P(t)$.

Let us first consider the case $m = 0$. In this case, if $c = 1$, i.e., if $\mathcal{F}$ has length one, it is scheme-theoretically supported on Z. When $c \geq 2$, the sheaf $\mathcal{F}$ admits a Jordan-Hölder filtration where all subquotients are zero-dimensional sheaves of length one. Thus, all subquotients are scheme-theoretically supported on Z. Hence, the scheme-theoretic support of $\mathcal{E}$ is contained in $\mathrm{t}_0(Z^{(c)})$ as a closed subscheme.[5]

Now, let us assume that $m \geq 1$. By Proposition 4.62, there exists a filtration

$$0 =: \mathcal{F}_{\ell+1} \subset \mathcal{F}_\ell \subset \mathcal{F}_{\ell-1} \subset \cdots \subset \mathcal{F}_0 \coloneqq \mathcal{F} ,$$

for $\ell \geq 1$, so that each subquotient is a pure m-dimensional sheaf with scheme-theoretic support contained in Z. In particular, the scheme-theoretic support of $\mathcal{F}$ is contained in $\mathrm{t}_0(Z^{(\ell)})$. Since $1 \leq \ell \leq c$, we have that the scheme-theoretic support of $\mathcal{F}$ is also contained in $\mathrm{t}_0(Z^{(c)})$ as a closed subscheme. □

Proposition 4.64. *Fix polynomials $P(t) = \sum_{i=0}^m c_i\, t^i/i! \in \mathbb{Q}[t]$ and $\alpha(t) = \sum_{i=0}^m \alpha_i t^i/i! \in \mathbb{Q}[t]$ of the same degree $m \geq 1$ such that $\alpha(t)$ is monic. Then, there exists an integer $k \geq 1$, depending on $P(t)$ and $\alpha(t)$, so that any coherent sheaf $\mathcal{E}$ on X, set-theoretically supported on Z, with Hilbert polynomial $P(t)$, such that $p_{H\text{-}\min}(\mathcal{E}) > \alpha$ is scheme-theoretically supported on $\mathrm{t}_0(Z^{(k)})$.*

Proof. Fix a coherent sheaf $\mathcal{E}$ on X, set-theoretically supported on Z, with Hilbert polynomial $P(t)$. Consider the torsion filtration of $\mathcal{E}$:

$$0 \subseteq \mathcal{T}_0 \subseteq \cdots \subseteq \mathcal{T}_{m-1} \subseteq \mathcal{E} ,$$

where $\mathcal{T}_i$ is the maximal i-dimensional subsheaf of $\mathcal{E}$ for $0 \leq i \leq m-1$.

If $\mathcal{E}$ is pure, i.e., $\mathcal{T}_{m-1} = 0$, the claim follows from Lemma 4.63. Otherwise, set $\mathcal{F}_m \coloneqq \mathcal{E}/\mathcal{T}_{m-1}$. Then, $\mathcal{F}_m$ is pure m-dimensional and set-theoretically supported on Z, with $c_m(\mathcal{F}_m) = c_m$. By Lemma 4.63, $\mathcal{F}_m$ is scheme-theoretically supported on $\mathrm{t}_0(Z^{(c_m)})$.

Since $p_{H\text{-}\min}(\mathcal{E}) > \alpha$, we get $p_H(\mathcal{F}_m) \geq \alpha$. In particular, we have

$$P_H(\mathcal{F}_m) \geq c_m \cdot \alpha ,$$

which implies

$$P_H(\mathcal{T}_{m-1}) = P - P_H(\mathcal{F}_m) \leq P - c_m \cdot \alpha .$$

If $\mathcal{T}_{m-1}$ is pure, i.e., if $\mathcal{T}_{m-2} = 0$, by Lemma 4.63, we get that $\mathcal{T}_{m-1}$ is supported on $Z^{(d_{m,m-1})}$, where $k_{m,m-1} \coloneqq \lfloor c_{m-1} - c_m \cdot \alpha_{m-1} \rfloor$. Thus, the scheme-theoretic support of $\mathcal{E}$ is contained in $\mathrm{t}_0(Z^{(k)})$ as a closed subscheme, where $k \coloneqq c_m + k_{m,m-1}$.

Otherwise, consider the short exact sequence

$$0 \longrightarrow \mathcal{T}_{m-1}/\mathcal{T}_{m-2} \longrightarrow \mathcal{E}/\mathcal{T}_{m-2} \longrightarrow \mathcal{F}_m \longrightarrow 0 .$$

We have

$$\begin{aligned} c_{m-1}(\mathcal{T}_{m-1}/\mathcal{T}_{m-2}) = c_{m-1}(\mathcal{E}/\mathcal{T}_{m-2}) - c_{m-1}(\mathcal{F}_m) &\leq c_{m-1} - c_m \cdot \alpha_{m-1} \\ &\leq c_{m-1} - c_m \cdot \alpha_{m-1} . \end{aligned}$$

Now, $\mathcal{T}_{m-1}/\mathcal{T}_{m-2}$ is a pure $(m-1)$-dimensional sheaf. Hence, by Lemma 4.63, $\mathcal{T}_{m-1}/\mathcal{T}_{m-2}$ is scheme-theoretically supported on $\mathrm{t}_0(Z^{(k_{m,m-1})})$.

[5]Here and it what follows, we make use of the following fact: if a coherent sheaf $\mathcal{F}$ fits into a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$, its scheme-theoretic support is contained in the union of the scheme-theoretic supports of $\mathcal{F}_1$ and $\mathcal{F}_2$.

Furthermore, also $p_H(\mathcal{E}/\mathcal{T}_{m-2}) \geq \alpha$. Thus,

$$c_{m-2}(\mathcal{T}_{m-2}) = c_{m-2} - c_{m-2}(\mathcal{E}/\mathcal{T}_{m-2}) \leq c_{m-2} - c_m \cdot \alpha_{m-2} .$$

If $\mathcal{T}_{m-2}(\mathcal{E})$ is pure, i.e., if $\mathcal{T}_{m-3}(\mathcal{E}) = 0$, by Lemma 4.63, we get that $\mathcal{T}_{m-2}(\mathcal{E})$ is supported on $Z^{(k_{m,m-2})}$, where $k_{m,m-2} := \lfloor c_{m-2} - c_m \cdot \alpha_{m-2} \rfloor$. Thus, the scheme-theoretic support of $\mathcal{E}$ is contained in $\mathrm{t}_0(Z^{(k)})$ as a closed subscheme, where $k := k_m + k_{m,m-1} + k_{m,m-2}$.

By iterating this argument, we get the claim. □

Now, we are ready to prove the proposition.

Proof of Proposition 4.61. Let $k \geq 1$ be the integer, depending on $P(t)$ and $\alpha(t)$, whose existence is guaranteed by Proposition 4.64. Consider $k \leq k_1 \leq k_2$.

Assume that $m \geq 1$. Let $i_{k_1,k_2} \colon \mathrm{t}_0(Z^{(k_1)}) \to \mathrm{t}_0(Z^{(k_2)})$ be the canonical closed embedding. It is enough to prove that

$$\mathbf{i}_{k_1,k_2} \colon \mathfrak{U}_\alpha(\mathrm{t}_0(Z^{(k_1)}); P(t))(S) \longrightarrow \mathfrak{U}_\alpha(\mathrm{t}_0(Z^{(k_2)}); P(t))(S)$$

is an equivalence for any reduced scheme S. Let $\mathcal{E}$ be a S-flat family of coherent sheaves on $\mathrm{t}_0(Z^{(k_2)})$ with fiber-wise Hilbert polynomial $P(t)$. Let

$$\eta \colon \mathcal{E} \longrightarrow (\mathrm{id}_S \times i_{k_1,k_2})_*(\mathrm{id}_S \times i_{k_1,k_2})^*\mathcal{E}$$

be the canonical surjective morphism. It is enough to prove that η is an isomorphism.

Set $\mathcal{F} := (\mathrm{id}_S \times i_{k_1,k_2})_*(\mathrm{id}_S \times i_{k_1,k_2})^*\mathcal{E}$. By Proposition 4.64, the morphism η_s is an isomorphism for any $s \in S$. Hence, the Hilbert polynomial of $\mathcal{F}_s$ coincides with the Hilbert polynomial of $\mathcal{E}_s$ for any $s \in S$. By [HL10, Proposition 2.1.2], the Hilbert polynomial of $\mathcal{F}_s$ is locally constant as a function of $s \in S$. Since S is reduced, *loc.cit.* also implies that $\mathcal{F}$ is S-flat. Now, by [HL10, Lemma 2.1.4], the kernel of η_s if $\ker(\eta)_s$ for any $s \in S$. Thus, $\ker(\eta)$ has fiber-wise zero Hilbert polynomial, hence $\ker(\eta) = 0$.

When $m = 0$, the claim follows from a similar argument using Lemma 4.63.

Now, the first claim yields that $\mathfrak{U}_\alpha(\widehat{X}_Z; P(t))$ is a qcqs indgeometric derived stack thanks to the description of $\widehat{X}_Z$ in Remark 4.6. □

Theorem 4.65. *Let X be a quasi-projective scheme over a field k and let Z be a closed subscheme of X such that there exists a projective scheme Y, containing X as an open subscheme, such that Z is also closed in Y. Then, the indgeometric derived stack $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; P(t))$ is admissible for any polynomial $P(t) \in \mathbb{Q}[t]$.*

Proof. Since $\widehat{X}_Z \simeq \widehat{Y}_Z$, we can assume from the beginning that X is projective.

Fix a sequence $\alpha_0(t) > \alpha_1(t) > \cdots$ of monic polynomials in $\mathbb{Q}[t]$. Then, the open substacks

$$\mathfrak{U}_k(\widehat{X}_Z; P(t)) := \mathfrak{U}_{\alpha_k}(\widehat{X}_Z; P(t))$$

form an exhaustion of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; P(t))$, which is admissible thanks to Proposition 4.61 (cf. Remark 2.17). □

We conclude this section with the following corollary.

Corollary 4.66. *Let X be a quasi-projective scheme over a field k and let Z be a closed subscheme of X such that there exists a projective scheme Y, containing X as an open subscheme, such that Z is also closed in Y. Then, the morphism (4.7)*

$${}^{\mathsf{red}}\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z) \longrightarrow \mathbf{Coh}_{\mathsf{ps}}(X)$$

is a closed immersion.

Proof. It is enough to prove that for every morphism $\mathcal{U} \to \mathbf{Coh}_{\mathsf{ps}}(X)$ representable by quasi-compact open immersions, setting

$$\mathcal{V} := \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z) \times_{\mathbf{Coh}_{\mathsf{ps}}(X)} \mathcal{U} ,$$

the induced morphism ${}^{\mathsf{red}}\mathcal{V} \to \mathcal{U}$ is a closed immersion. Since $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)$ is admissible by Theorem 4.65. Thus, Corollary 2.21 implies that $\mathcal{V}$ is a qcqs indgeometric derived stack. In particular, Corollary 2.22 guarantees that we can find a thickening $W \in \mathcal{T}_{Z/\!/X}$ such the morphism

$$\mathcal{V} \times_{\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)} \mathbf{Coh}_{\mathsf{ps}}(W) \longrightarrow \mathcal{V}$$

is a nil-equivalence. In particular, the map ${}^{\mathsf{red}}\mathcal{V} \to \mathcal{U}$ is canonically identified with the map

$${}^{\mathsf{red}}\mathbf{Coh}_{\mathsf{ps}}(W) \times_{\mathbf{Coh}_{\mathsf{ps}}(X)} \mathcal{U} \longrightarrow \mathcal{U} .$$

It is therefore enough to argue that the map $\mathbf{Coh}_{\mathsf{ps}}(W) \to \mathbf{Coh}_{\mathsf{ps}}(X)$ is a closed immersion. Since X satisfies the perfect resolution property (cf. Example 4.36), this follows from Proposition 4.45. □

5. Nilpotent cohomological Hall algebra

In this section we introduce the cohomological Hall algebra associated to the category of coherent sheaves on a smooth quasi-projective surface over a field k of characteristic zero, set-theoretically supported on a fixed closed subscheme. We shall call it the *nilpotent* COHA.

5.1. **The Λ-graded 2-Segal structure on the Waldhausen construction.** Given an integer n we let $[n]$ denote the linearly ordered poset $\{0 < 1 < \cdots < n\}$ and we set

$$\mathsf{T}_n := \mathsf{Fun}([1], [n]) .$$

The collection of the various T_n determines a functor

$$\mathsf{T}_\bullet \colon \mathbf{\Delta} \longrightarrow \mathsf{Cat}_\infty .$$

Given a stable ∞-category $\mathcal{C}$, we set

$$\mathcal{S}_n \subseteq \mathsf{Fun}(\mathsf{T}_n, \mathcal{C})$$

be the full subcategory spanned by those functors $F \colon \mathsf{T}_n \to \mathcal{C}$ satisfying the following two conditions:

(1) $F(i,i) \simeq 0$;
(2) for every $0 \leqslant i < j \leqslant n-1$, the square

$$\begin{array}{ccc} F(i,j) & \longrightarrow & F(i+1,j) \\ \downarrow & & \downarrow \\ F(i,j+1) & \longrightarrow & F(i+1,j+1) \end{array}$$

is a pullback in $\mathcal{C}$.

The ∞-categories $\mathcal{S}_n\mathcal{C}$ depends simplicially on n, i.e., they assemble into a simplicial object

$$\mathcal{S}_\bullet\mathcal{C} \colon \mathbf{\Delta}^{\mathsf{op}} \longrightarrow \mathsf{Cat}_\infty ,$$

known as the *Waldhausen construction* of $\mathcal{C}$. One of the main observations of [DK19] is that this satisfies the 2-Segal condition. The functoriality of this construction allow to replace $\mathcal{C}$ by a functor

$$\mathfrak{C} \colon \mathsf{dAff}_k^{\mathsf{op}} \longrightarrow \mathsf{Cat}_\infty^{\mathsf{st}} ,$$

giving rise to a 2-Segal object in derived prestacks

$$\mathcal{S}_\bullet\mathfrak{C} \colon \mathbf{\Delta}^{\mathsf{op}} \longrightarrow \mathsf{PreSt}_k ,$$

which we refer to as the Waldhausen construction of $\mathscr{C}$, Besides, this construction is obviously functorial in $\mathscr{C}$.

Notation 5.1. Given a stable ∞-category $\mathcal{C}$, we refer to $F \in \mathcal{S}_n\mathcal{C}$ as an *n-flag of objects in* $\mathcal{C}$. Given $0 \leqslant i \leqslant j \leqslant n$, we set

$$\mathrm{ev}_{i,j}(F) \coloneqq F(i,j) \ ,$$

We also denote by

$$\partial_i(F) \in \mathcal{S}_{n-1}\mathcal{C}$$

the $(n-1)$-flag obtained restricting F along the morphism $[n-1] \to [n]$ in $\mathbf{\Delta}$ that misses i. In particular, when $n = 2$ we have the overlap of notation

$$\partial_0 = \mathrm{ev}_{1,2} \ , \qquad \partial_1 = \mathrm{ev}_{0,2} \ , \qquad \partial_2 = \mathrm{ev}_{0,1} \ .$$

We refer to the collection of objects $\{\mathrm{ev}_{i,i+1}(F)\}_{0 \leqslant i \leqslant n-1}$ as the *diagonal of* F. Sending an n-flag to its diagonal gives rise to natural transformation

$$\delta_{\mathcal{C},n} \colon \mathcal{S}_n\mathcal{C} \longrightarrow \prod_{i=0}^{n-1} \mathcal{C} \ .$$

When $\mathcal{C}$ is replaced by a functor $\mathscr{C} \colon \mathsf{dAff}_k^{\mathsf{op}} \to \mathsf{Cat}_\infty$, this induces a natural transformation

$$\delta_{\mathscr{C},n} \colon \mathcal{S}_n\mathscr{C} \longrightarrow \prod_{i=0}^{n-1} \mathscr{C}$$

of derived Cat_∞-valued prestacks. ⊘

We apply this to the setting of §4. Specifically we fix a quasi-compact and quasi-separated derived formal scheme $\mathcal{X}$ and we consider the functor

$$\mathsf{dAff}_k^{\mathsf{op}} \longrightarrow \mathsf{Cat}_\infty$$

sending S to $\mathsf{Coh}^{\mathsf{nil}}_{S,\mathsf{ps}}(\mathcal{X} \times S)$. Applying the Waldhausen construction and passing to the maximal ∞-groupoid at the end, we obtain a 2-Segal object

$$\mathcal{S}_\bullet\widetilde{\mathbf{Coh}}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X}) \colon \mathbf{\Delta} \longrightarrow \mathsf{PreSt}_{\mathbb{C}} \ .$$

We set

$$\mathcal{S}_\bullet\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X}) \coloneqq {}^{\mathsf{conv}}\big(\mathcal{S}_\bullet\widetilde{\mathbf{Coh}}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X})\big) \ .$$

When $n = 1$, there is a canonical identification

$$\mathcal{S}_1\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X}) \simeq \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X}) \ .$$

When $\mathcal{X} \simeq \widehat{X}_Z$ is a formal completion, we saw that under mild assumptions on X this derived stack is indgeometric or even admissible. We now generalize these results to arbitrary n. The key ingredient is the following:

Theorem 5.2. *Let X be a quasi-compact and quasi-separated derived scheme over a field k of characteristic zero and let $j \colon Z \hookrightarrow X$ be a closed immersion. Then for every $n \geqslant 1$ the square*

$$\begin{array}{ccc} \mathcal{S}_n\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z) & \longrightarrow & \mathcal{S}_n\mathbf{Coh}_{\mathsf{ps}}(X) \\ \big\downarrow & & \big\downarrow \\ \prod_{i=1}^n \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z) & \longrightarrow & \prod_{i=1}^n \mathbf{Coh}_{\mathsf{ps}}(X) \end{array}$$

whose vertical morphisms send an n-flag to its diagonal, is a pullback.

Proof. It follows from Theorem 4.53–(1) that both horizontal arrows are fully faithful on S points for every $S \in \mathsf{dAff}_k$. Thus, the same goes for the induced morphism from $\mathcal{S}_n\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$ to the fiber product. We are therefore left to argue for its essential surjectivity. Notice that since both source and target are convergent, we can restrict without loss of generality to the case where $S \in {}^{<\infty}\mathsf{dAff}$. Unraveling the definition of the Waldhausen construction, we reduce ourselves to check that if

$$\mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}''$$

is a fiber sequence in $\mathsf{Coh}_{S,\mathsf{ps}}(X \times S)$ and both $\mathcal{F}'$ and $\mathcal{F}''$ belong to the essential image of $\mathsf{Coh}^{\mathsf{nil}}_{S,\mathsf{ps}}(\widehat{X}_Z)$, then the same goes for $\mathcal{F}$, and this is immediate from the characterization of the essential image over an eventually coconnective base given in Theorem 4.53–(3). ☐

Corollary 5.3. *Let X be a quasi-compact and quasi-separated derived scheme over a field k of characteristic zero and let $j\colon Z \hookrightarrow X$ be a closed immersion. Then*

(1) if X universally satisfies the perfect resolution property, then for every integer $n \geqslant 0$ the derived stack $\mathcal{S}_n\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$ is indgeometric;

(2) if X is quasi-projective scheme admitting a projective compactification Y that contains Z as a closed subscheme, then for every integer $n \geqslant 0$ the indgeometric derived stack $\mathcal{S}_n\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$ is admissible.

Proof. Assumption (1) guarantees via Theorem 4.39 that the product $\prod_{i=0}^{n-1}\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$ is indgeometric. Then Theorem 5.2 guarantees that the pullback of a presentation for the above product via the diagonal δ_n provides a presentation for $\mathcal{S}_n\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$.

We now prove (2). First, recall from [PS23, Corollary 4.6] that $\mathcal{S}_n\mathbf{Coh}_{\mathsf{ps}}(X)$ is geometric. Then (2) follows combining Theorems 4.65 and 5.2 with Corollary 2.26. ☐

Fix now a free abelian group of finite rank $(\Lambda, +)$ and a map

$$v\colon \pi_0(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X})) \longrightarrow \Lambda\,,$$

which we assume to be additive in the sense of [AHLH23, §7.3]. We use the notion of Λ-graded 2-Segal object of [DPS22, §I.6] to study the compatibility between the group structure of Λ and the Hall multiplication. Given $\mathbf{v} \in \Lambda$, we let

$$\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X}; \mathbf{v})$$

be the corresponding union of connected components. More generally, given an element $\underline{\mathbf{v}} = (\mathbf{v}_{0,1}, \mathbf{v}_{1,2}, \ldots, \mathbf{v}_{n-1,n}) \in \Lambda^n$, we let

$$\mathcal{S}_n\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X}; \underline{\mathbf{v}})$$

be the open and closed substack of $\mathcal{S}_n\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X})$ parametrizing n-flags whose diagonal is of type $\underline{\mathbf{v}}$. A routine verification similar to [PS23, Lemma 4.1] shows that this satisfies the Λ-graded 2-Segal condition. Since the convergent completion functor ${}^{\mathsf{conv}}(-)$ commutes with limits (see Recollection 2.3), it automatically follows that $\mathcal{S}_\bullet\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(X)$ is again a 2-Segal object in $\mathsf{PreSt}_{\mathbb{C}}$. In particular, [DPS22, Proposition I.6.4] shows that $\mathcal{S}_\bullet\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X})$ determines an $\mathbb{E}_1$-monoid in $\mathsf{Corr}^{\times_\Lambda}(\Lambda\text{-}\mathsf{dSt}_{\mathbb{C}})$. In what follows, we tacitly consider $\mathcal{S}_\bullet\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X})$ as a Λ-graded 2-Segal derived stack.

5.2. **Functoriality.** We now discuss the functorial dependency of $\mathcal{S}_\bullet\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X})$ on the derived formal scheme $\mathcal{X}$. Ultimately, this relies on the functoriality of $\mathsf{QCoh}^{\mathsf{nil}}$, so recall from §4.2 that for any morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of derived formal schemes we have by construction a well defined pushforward functor

$$f_*\colon \mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X}) \longrightarrow \mathsf{QCoh}^{\mathsf{nil}}(\mathcal{Y})\,.$$

When f is schematic, Proposition 4.18 provides a well defined left adjoint

$$f^*\colon \mathsf{QCoh}^{\mathsf{nil}}(\mathcal{Y}) \longrightarrow \mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X}) .$$

Notice that if $S \in \mathsf{dAff}_k$ is a derived affine scheme, then $\mathcal{X} \times S$ and $\mathcal{Y} \times S$ are again derived formal schemes. Besides:

(1) Assume that f is ind-finite, that is that it admits a presentation as $\mathrm{colim}_\alpha f_\alpha$ where $f_\alpha\colon X_\alpha \to Y_\alpha$ is a finite morphism between derived quasi-compact and quasi-separated schemes (and $\{X_\alpha\}$ and $\{Y_\alpha\}$ are presentations for the formal schemes $\mathcal{X}$ and $\mathcal{Y}$). In this case, Lemma 4.30 implies that $f_{S,*}$ descends to a morphism

$$f_{S,*}\colon \mathsf{Coh}^{\mathsf{nil}}_{S,\mathsf{ps}}(\mathcal{X} \times S) \longrightarrow \mathsf{Coh}^{\mathsf{nil}}_{S,\mathsf{ps}}(\mathcal{Y} \times S)$$

for every $S \in \mathsf{dAff}$. In turn, this provides a well defined morphism

$$^{(\bullet)}\mathbf{f}_*\colon \mathcal{S}_\bullet\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X}) \longrightarrow \mathcal{S}_\bullet\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{Y}) .$$

When $\bullet = 1$, we simply write $\mathbf{f}_*$ instead of $^{(1)}\mathbf{f}_*$.

(2) Assume that f is schematic. Then for every test scheme S, $f_{S,*}\colon \mathsf{QCoh}^{\mathsf{nil}}(\mathcal{X} \times S) \to \mathsf{QCoh}^{\mathsf{nil}}(\mathcal{Y} \times S)$ admits a left adjoint f_S^*. If in addition f is proper and flat then this left adjoint descends to a morphism

$$f_S^*\colon \mathsf{Coh}^{\mathsf{nil}}_{S,\mathsf{ps}}(\mathcal{Y} \times S) \longrightarrow \mathsf{Coh}^{\mathsf{nil}}_{S,\mathsf{ps}}(\mathcal{X} \times S) .$$

In turn, this provides a well defined morphism

$$^{(\bullet)}\mathbf{f}^*\colon \mathcal{S}_\bullet\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{Y}) \longrightarrow \mathcal{S}_\bullet\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\mathcal{X}) .$$

When $\bullet = 1$, we simply write $\mathbf{f}^*$ instead of $^{(1)}\mathbf{f}^*$.

We will be particularly interested in the following example.

Example 5.4. Let X be a quasi-compact and quasi-separated derived scheme over a field k of characteristic zero and let

$$\begin{array}{ccc} Z_1 & \xrightarrow{j_{12}} & Z_2 \\ & {\scriptstyle j_1}\searrow \quad \swarrow{\scriptstyle j_2} & \\ & X & \end{array}$$

be closed immersions. This induces an ind-finite morphism

$$\hat{\jmath}_{12}\colon \widehat{X}_{Z_1} \longrightarrow \widehat{X}_{Z_2} ,$$

and therefore a morphism

$$^{(\bullet)}\hat{\jmath}_{12,*}\colon \mathcal{S}_\bullet\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}) \longrightarrow \mathcal{S}_\bullet\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}) .$$

△

In the setting of the above example, the morphism of 2-Segal derived stacks is strict in the following sense:

Lemma 5.5. *In the setting of Example 5.4 the square*

$$\begin{array}{ccc} \mathcal{S}_2\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}) & \xrightarrow{^{(2)}\hat{\jmath}_{12,*}} & \mathcal{S}_2\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}) \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\ \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}) & \xrightarrow{\hat{\jmath}_{12,*} \times \hat{\jmath}_{12,*}} & \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}) \end{array}$$

is a pullback.

Proof. This is an immediate consequence of Theorem 5.2 and the transitivity property for pullback squares. □

5.3. **The main existence theorem.** We are now ready to establish the existence of the nilpotent cohomological Hall algebra.

We start with some general considerations. Let X be a n-dimensional smooth scheme over a field k of characteristic zero and let $j\colon Z \hookrightarrow X$ be the inclusion of a closed subscheme. Assume that X admits a projective compactification $\overline{X}$ that contains Z as a closed subscheme. We let $N(X)$ be the numerical Grothendieck group of the category of properly supported coherent sheaves on X, and we set

$$N_{\leqslant 1}(X) \coloneqq \mathsf{Im}(K_0(\mathsf{Coh}_{\leqslant 1}(X)) \longrightarrow N(X)) .$$

We take $\Lambda \coloneqq N_{\leqslant 1}(X)$. We have the following.

Lemma 5.6.

(1) the map

$$\partial_0 \times \partial_2 \colon \mathcal{S}_2\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z) \longrightarrow \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$$

is representable by linear stacks and its tor-amplitude is contained in $[1-n, 1]$. In particular, if X is a surface then $\partial_0 \times \partial_2$ is derived lci.

(2) Asume that X is a surface. Then the map

$$\partial_1 \colon \mathcal{S}_2\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z) \longrightarrow \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$$

of Λ-graded derived stacks is locally rpas (in the sense of Definitions 3.2 and 3.3).

Proof. Point (1) is an immediate consequence of Theorem 5.2 and [PS23, Proposition 3.11]. As for point (2), we already know that the analogous statement holds for $\mathbf{Coh}_{\mathsf{ps}}(X)$, since in the latter case it is representable by Quot schemes, which are known to be proper as soon as the Hilbert polynomial is fixed. Therefore, it is enough to argue that the square

$$\begin{array}{ccc}
\mathcal{S}_2\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z) & \xrightarrow{{}^{(2)}\hat{\jmath}_{12,*}} & \mathcal{S}_2\mathbf{Coh}_{\mathsf{ps}}(X) \\
\downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial_1} \\
\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z) & \xrightarrow{\hat{\jmath}_{12,*}} & \mathbf{Coh}_{\mathsf{ps}}(X)
\end{array}$$

is a pullback square. This can be done after evaluating on a test scheme $S \in \mathsf{dAff}$, and since all the stacks that appear are convergent, we can assume without loss of generality that $S \in {}^{<\infty}\mathsf{dAff}$. It follows from Theorem 4.53–(1) that both the top and the bottom horizontal maps are fully faithful. So, unraveling the definitions we are reduced to show that if

$$\mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}''$$

is a fiber sequence in $\mathsf{Coh}_{S,\mathsf{ps}}(X \times S)$ and $\mathcal{F}$ lies in the essential image of $\hat{\jmath}_{12,*}$, the same goes for $\mathcal{F}'$ and $\mathcal{F}''$. Since S is eventually coconnective, the criterion given in Theorem 4.53–(1) reduces us to show that

$$\mathcal{F}'|_{X\times S \smallsetminus Z \times S} \simeq \mathcal{F}''|_{X\times S\smallsetminus Z\times S} \simeq 0 .$$

Since both $\mathcal{F}'$ and $\mathcal{F}''$ are S-flat, this is automatic. □

Assume now that X admits a projective compactification $\overline{X}$ that contains Z as a closed subscheme (notice that this is automatic if X is quasi-projective and Z is itself proper). Then Corollary 5.2 implies that each $\mathcal{S}_n\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$ is admissible. Fixing a motivic formalism $\mathbf{D}^*$ as in

[DPS22, §II.1.1], $\mathcal{A} \in \mathsf{CAlg}(\mathbf{D}(\mathsf{Spec}(k)))$ and an abelian subgroup $\Gamma \subset \mathsf{Pic}(\mathbf{D}(S))$, we can consider

$$\mathsf{H}_0^{\mathbf{D},\Gamma}(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z);\mathcal{A}) \in \mathsf{Pro}(\mathsf{Ab}^{\mathsf{gr}}) .$$

This is a pro-Γ-graded abelian group. In practice, we will consider it as a topological abelian group with the topology induced from the pro-structure. We refer to this topology as *the quasi-compact topology* (see Remark 3.1).

Theorem 5.7. *Let X be a smooth surface over a field k of characteristic zero and let $j\colon Z \hookrightarrow X$ be the inclusion of a closed subscheme. Assume that X admits a projective compactification $\overline{X}$ that contains Z as a closed subscheme. Then for every choice of an R-linear motivic formalism $\mathbf{D}^*$ and every oriented $\mathcal{A} \in \mathsf{CAlg}(\mathbf{D}(\mathsf{Spec}(k)))$ and every abelian subgroup $\Gamma \subset \mathsf{Pic}(\mathbf{D}(\mathsf{Spec}(k)))$ closed under Thom twists, the Λ-graded pro-R-module $\mathsf{H}_0^{\mathbf{D},\Gamma}(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z);\mathcal{A})$ is an associative algebra, which we shall denote by $\mathbf{HA}^{\mathbf{D},\Gamma}_{\widehat{X}_Z,\mathcal{A}}$, with multiplication given by pull-push along the correspondence*

$$\begin{array}{ccc} & \mathcal{S}_2\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z) & \\ {\scriptstyle \partial_0\times\partial_2}\swarrow & & \searrow{\scriptstyle \partial_1} \\ \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)\times\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z) & & \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z) \end{array}$$

In particular, seeing $\mathbf{HA}^{\mathbf{D},\Gamma}_{\widehat{X}_Z,\mathcal{A}}$ as a topological abelian group, the Hall multiplication defined above becomes continuous with respect to the quasi-compact topology. Moreover:

(1) let $j'\colon Z' \hookrightarrow X'$ be another closed embedding into a smooth surface over k satisfying the above assumption. Then, an isomorphism of formal completions $\widehat{X'}_{Z'} \simeq \widehat{X}_Z$ induces a functorial and continuous isomorphism of algebras

$$\mathbf{HA}^{\mathbf{D},\Gamma}_{\widehat{X'}_{Z'},\mathcal{A}} \simeq \mathbf{HA}^{\mathbf{D},\Gamma}_{\widehat{X}_Z,\mathcal{A}} .$$

(2) Let $j'\colon Z' \hookrightarrow Z$ be a nested closed subscheme inside X. Then, the morphism

$$\hat{\jmath}'_*\colon \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z'}) \longrightarrow \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)$$

induces a continuous morphism of algebras

$$j'_*\colon \mathbf{HA}^{\mathbf{D},\Gamma}_{\widehat{X}_{Z'},\mathcal{A}} \longrightarrow \mathbf{HA}^{\mathbf{D},\Gamma}_{\widehat{X}_Z,\mathcal{A}} .$$

(3) Let $(\mathbf{D}',\mathcal{A}',\Gamma')$ be a second motivic formalism equipped with an oriented ring of coefficients $\mathcal{A}'$ and an abelian subgroup Γ' closed under Thom twists. A transformation

$$(s,\phi)\colon (\mathbf{D},\mathcal{A},\Gamma) \longrightarrow (\mathbf{D}',\mathcal{A}',\Gamma')$$

as in [DPS22, Theorem II.1.34] *induces a morphism of algebras*

$$\mathbf{HA}^{\mathbf{D},\Gamma}_{\widehat{X}_Z,\mathcal{A}} \longrightarrow \mathbf{HA}^{\mathbf{D}',\Gamma'}_{\widehat{X}_Z,\mathcal{A}'} .$$

Proof. We start by establishing the existence of the Hall multiplication. To begin with, using [DPS22, Proposition I.6.4] we can convert the Λ-graded 2-Segal object $\mathcal{S}_\bullet\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)$ constructed in §5.1 into an $\mathbb{E}_1$-monoid object in $\mathsf{Corr}^{\times\Lambda}(\Lambda\text{-}\mathsf{indGeom}_k^{\mathsf{adm}})$. Besides, Lemma 5.6 and the 2-Segal property allow to promote this into an $\mathbb{E}_1$-monoid object in $\mathsf{Corr}^{\times\Lambda}(\Lambda\text{-}\mathsf{indGeom}_k^{\mathsf{adm}})_{\mathsf{qc.lci}\cap\mathsf{fconn,lrpas}}$. Since $\mathsf{H}_0^{\mathbf{D},\Gamma}(-;\mathcal{A})$ is a lax-monoidal functor in virtue of Theorem 3.9, we deduce that $\mathbf{HA}^{\mathbf{D},\Gamma}_{\widehat{X}_Z,\mathcal{A}}$ acquires the structure of an associative algebra, whose product is given by the prescribed rule.

Concerning the extra statements, (1) is simply a consequence of the fact that $\mathcal{S}_\bullet\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)$, as a derived stack, only depends on the formal completion $\widehat{X}_Z$. The ambient scheme X was only used to guarantee, via Theorem 5.2 and Lemma 5.6 that $\partial_2 \times \partial_1$ is representable by finitely connected (in the sense of Definitions 3.2 and 3.3), quasi-compact, lci derived geometric stacks, and that ∂_1 is locally rpas (in the sense of Definitions 3.2 and 3.3).

We now prove (2). To begin with, observe that Corollary 4.66 implies that both

$$ {}^{\mathrm{red}}\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z'}) \qquad \text{and} \qquad {}^{\mathrm{red}}\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z}) $$

are closed inside $\mathbf{Coh}_{\mathsf{ps}}(X)$. In particular, $\hat{\jmath}_*$ is a nil-closed immersion, and therefore it induces a well defined continuous pushforward map

$$ j'_*\colon \mathbf{HA}^{\mathbf{D},\Gamma}_{\widehat{X}_{Z'},\mathcal{A}} \longrightarrow \mathbf{HA}^{\mathbf{D},\Gamma}_{\widehat{X}_{Z},\mathcal{A}} . $$

The compatibility with the Hall multiplication is a direct consequence of the functorial behavior of $\mathsf{H}_0^{\mathbf{D},\Gamma}(-;\mathcal{A})$ and of Lemma 5.5. Finally, statement (3) follows directly from the last part of Theorem 3.5. □

Remark 5.8. The Hall multiplication is compatible with the addition of Λ. In fact, the above proof exhibits $\mathbf{HA}^{\mathbf{D},\Gamma}_{\widehat{X}_Z,\mathcal{A}}$ as an algebra object in $\mathsf{Fun}(\Lambda, \mathsf{Mod}_R)$, where the latter is equipped with Day's convolution. This is the ultimate reason we used the language of Λ-graded 2-Segal objects. △

We are mostly interested in the following three examples.

Example 5.9.

(1) when $\mathbf{D}^* := \mathbf{DM}^*_{\mathbb{Q}}$ is the rational Voevodsky's formalism of [DPS22, Example II.1.8] and $\mathcal{A} := \mathsf{H}\mathbb{Q}$ is the motivic Eilenberg-MacLane $\mathbb{E}_\infty$-ring spectrum, and $\Gamma := \mathbb{Z}\langle 1\rangle$, we write

$$ \mathbf{HA}^{\mathsf{mot}}_{X,Z} := \mathbf{HA}^{\mathbf{D},\Gamma}_{\widehat{X}_Z,\mathsf{H}\mathbb{Q}} , $$

whose underlying topological abelian group is

$$ \mathsf{H}^{\mathsf{mot}}_\bullet(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z);0) := \mathsf{H}_0^{\mathbf{D},\Gamma}(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z);\mathsf{H}\mathbb{Q}) . $$

This recovers (motivically defined) Chow groups of $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)$. Thus, we refer to $\mathbf{HA}^{\mathsf{mot}}_{X,Z}$ as the *motivic cohomological Hall algebra of* $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)$.

By replacing $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)$ with $\mathbf{Coh}_{\mathsf{ps}}(X)$ and using the results of [PS23, §4] with Theorem 3.5, we also obtain the *motivic cohomological Hall algebra of* $\mathbf{Coh}_{\mathsf{ps}}(X)$.

(2) When $\mathbf{D}^* := \mathbf{DM}^*_{\mathbb{Q}}$ is the rational Voevodsky's formalism of [DPS22, Example II.1.8] and $\mathcal{A} := \mathsf{KGL}^{\mathsf{et}}$ is the étale hypersheafification of the algebraic K-theory spectrum, and $\Gamma := \mathbb{Z}\langle 1\rangle$, we write

$$ \mathbf{G}_{X,Z} := \mathbf{HA}^{\mathbf{D},\Gamma}_{\widehat{X}_Z,\mathsf{KGL}^{\mathsf{et}}} , $$

whose underlying topological abelian group is

$$ \mathbf{G}(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)) := \mathsf{H}_0^{\mathbf{D},\Gamma}(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z);\mathsf{KGL}^{\mathsf{et}}) . $$

This recovers the algebraic G-theory of $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)$. Taking its π_0, we obtain the *K-theoretical Hall algebra of* $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)$. By replacing $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)$ with $\mathbf{Coh}_{\mathsf{ps}}(X)$, we recover the (full) K-theoretical Hall algebra of $\mathbf{Coh}_{\mathsf{ps}}(X)$, as defined in [PS23, §5] (see also [KV23] for an equivalent realization of the K-theoretical Hall algebra of $\mathbf{Coh}_{\mathsf{ps}}(X)$).

(3) When $\mathbf{D}^* := {}^{\mathsf{top}}\mathbf{D}$ is the topological formalism of [DPS22, §II.1.11], $\mathcal{A} := \mathbb{Q}$, and $\Gamma := \mathbb{Z}\langle 1/2\rangle$ (see [DPS22, Remark II.1.24]), we simply write

$$ \mathbf{HA}_{X,Z} := \mathbf{HA}^{\mathbf{D},\Gamma}_{\widehat{X}_Z,\mathbb{Q}} , $$

whose a graded pro-$\mathbb{Q}$-vector space is

$$ \mathsf{H}^{\mathsf{BM}}_\bullet(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)) := \mathsf{H}_0^{\mathsf{BM},\Gamma}(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z);\mathbb{Q}) . $$

We shall call $\mathbf{HA}_{X,Z}$ the *cohomological Hall algebra of* $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)$.

We will also be interested in taking $\Gamma := \mathbb{Z}\langle 1\rangle$, in which case we write

$$\mathbf{HA}^{\mathsf{even}}_{X,Z} := \mathbf{HA}^{\mathbf{D},\Gamma}_{\widehat{X}_Z,\mathbb{Q}} \quad \text{and} \quad \mathsf{H}^{\mathsf{BM}}_{\mathsf{even}}(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)) := \mathsf{H}^{\mathbf{D},\Gamma}_0(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z);\mathbb{Q}) .$$

The natural inclusion $\mathbb{Z}\langle 1\rangle \subset \mathbb{Z}\langle 1/2\rangle$ induces a continuous morphism of algebras

$$\mathbf{HA}^{\mathsf{even}}_{X,Z} \longrightarrow \mathbf{HA}_{X,Z} \quad \text{and} \quad \mathsf{H}^{\mathsf{BM}}_{\mathsf{even}}(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)) \longrightarrow \mathsf{H}^{\mathsf{BM}}_{\bullet}(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)) .$$

By replacing $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$ with $\mathbf{Coh}_{\mathsf{ps}}(X)$, we recover the (even) cohomological Hall algebra of $\mathbf{Coh}_{\mathsf{ps}}(X)$, which has been defined in [KV23].

Also, the natural transformation $\mathbf{DM}_{\mathbb{Q}} \to {}^{\mathsf{top}}\mathbf{D}_{\mathbb{Q}}$ of [DPS22, Remark II.1.13] induces via Theorem 5.7–(3) a continuous morphism of algebras

$$\mathbf{HA}^{\mathsf{mot}}_{X,Z} \longrightarrow \mathbf{HA}^{\mathsf{even}}_{X,Z} ,$$

given by the cycle class map.

△

Remark 5.10. Assume that there is a torus T acting on X such that Z is T-invariant. Following the rough lines of [PS23, §4.3] we can adapt the above results in the T-equivariant setting as follows. As observed in Variant 4.25, the T-action descends to a T-action on $\widehat{X}_Z$, and the structural morphism $[\widehat{X}_Z/T] \to \mathrm{B}T$ is representable by derived formal schemes. Thus, $\mathsf{QCoh}^{\mathsf{nil}}([\widehat{X}_Z/T])$ is well defined, which in turn allows to define a 2-Segal object

$$\mathcal{S}_\bullet\mathsf{Coh}^{\mathsf{nil}}_{\mathsf{ps},\mathrm{B}T}([\widehat{X}_Z/T]) \in \mathsf{Fun}_{2\text{-}\mathsf{Seg}}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{dSt}_{/\mathrm{B}T}) .$$

By construction

$$\mathsf{Spec}(k) \times_{\mathrm{B}T} \mathcal{S}_\bullet\mathsf{Coh}^{\mathsf{nil}}_{\mathsf{ps},\mathrm{B}T}([\widehat{X}_Z/T]) \simeq \mathcal{S}_\bullet\mathsf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z) .$$

Therefore, Corollary 2.28 implies that each $\mathcal{S}_\bullet\mathsf{Coh}^{\mathsf{nil}}_{\mathsf{ps},\mathrm{B}T}([\widehat{X}_Z/T])$ is admissible. At this point, thus applying Theorem 3.5 we obtain a variant of Theorem 5.7 in the equivariant setting. In particular, there exists an associative algebra structure on $\mathsf{H}^{\mathbf{D},\Gamma}_0(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)/T;\mathcal{A})$, which we shall denote by $\mathbf{HA}^{\mathbf{D},\Gamma,T}_{\widehat{X}_Z,\mathcal{A}}$, and equivariant versions of statements (1), (2), and (3) hold. In particular, we denote by $\mathbf{HA}^{\mathsf{mot},T}_{X,Z}$, $\mathbf{G}^T_{X,Z}$, and $\mathbf{HA}^T_{X,Z}$ the equivariant versions of the examples discussed in Example 5.9.

△

Remark 5.11. Let C be a smooth projective curve over a field k and let $X := T^*C$ be its cotangent bundle. When Z is the zero section of X, Theorem 5.7 recovers the construction of the ($\mathbb{G}_m$-equivariant) (motivic) cohomological and K-theoretical Hall algebras of nilpotent Higgs sheaves on C (see [SS20, PS23], and [Min20] for the rank zero case). Furthermore, the generation theorem given in [SS20, Theorem 5.1] admits a natural interpretation in this setting. Namely, with the notations introduced in [SS20, §5], it says that the subalgebra generated by the subspaces $\mathsf{H}_*(\Lambda_{(\alpha)})$ is dense in $\mathbf{HA}_{T^*C,C}$. △

Appendix A. Set-theoreticity and pure sheaves

Let X be an projective scheme over a field k and let Z be a nonempty closed subscheme of X. The goal of this appendix is the proof of the following result.

Proposition A.1. *Let $\mathcal{E}$ be a pure coherent sheaf on X of dimension $m \geq 1$, set-theoretically supported on Z. Then, there exists a filtration*

$$0 =: \mathcal{E}_{\ell+1} \subset \mathcal{E}_\ell \subset \mathcal{E}_{\ell-1} \subset \cdots \subset \mathcal{E}_0 := \mathcal{E} ,$$

for $\ell \geq 1$, so that each subquotient is a pure m-dimensional sheaf with scheme-theoretic support contained in Z.

Before proving the proposition, we need some preliminary results.

Let $\mathcal{I}_Z \subset \mathcal{O}_X$ be the defining ideal sheaf of Z. For any $k \geq 1$, let $Z^{(k)}_{\mathsf{cl}} \subset X$ be the closed subscheme defined by the ideal sheaf $\mathcal{I}_Z^k \subset \mathcal{O}_X$. By convention, we set $\mathcal{I}_Z^0 := \mathcal{O}_X$. Then, $Z^{(k)}_{\mathsf{cl}} = \mathrm{t}_0(Z^{(k)})$ for any $k \in \mathbb{N}$, with $k \geq 1$.

Let $\mathcal{E}$ be a nontrivial coherent sheaf on X of pure dimension $m \geq 1$, with set-theoretic support contained in Z. Let $W \subset X$ be the scheme-theoretic support of $\mathcal{E}$, i.e., the closed subscheme determined by the annihilator ideal $\mathsf{Ann}(\mathcal{E}) \subset \mathcal{O}_X$. Since $\mathcal{E}$ is set-theoretically supported on Z, it follows that ${}^{\mathsf{red}}W$ is contained in Z as a closed subscheme.

Note that there exists a unique positive integer $k_{\mathcal{E}} \geq 1$ so that $\mathcal{I}_Z^k \subset \mathsf{Ann}(\mathcal{E})$ for all $k \geq k_{\mathcal{E}}$, while $\mathcal{I}_Z^k \not\subseteq \mathsf{Ann}(\mathcal{E})$ for all $k < k_{\mathcal{E}}$.

For any $k \geq 1$, let $\mathcal{I}_Z^k\mathcal{E}$ be the image of the canonical multiplication map $\mathcal{I}_Z^k \otimes \mathcal{E} \to \mathcal{E}$. Then one has a canonical exact sequence

$$0 \longrightarrow \mathcal{I}_Z^k\mathcal{E} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E} \otimes \mathcal{O}_{Z^{(k)}_{\mathsf{cl}}} \longrightarrow 0\,.$$

Furthermore, the subsheaf $\mathcal{I}_Z^k\mathcal{E} \subset \mathcal{E}$ is a nonzero purely m-dimensional subsheaf of $\mathcal{E}$ for any $1 \leq k < k_{\mathcal{E}}$. For any $k \geq 1$, let $\mathcal{Q}_k \subset \mathcal{E} \otimes \mathcal{O}_{Z^{(k)}}$ be the maximal subsheaf so that $\dim \mathsf{Supp}(\mathcal{Q}_k) < m$. Set $\mathcal{F}_k := \mathcal{E} \otimes \mathcal{O}_{Z^{(k)}_{\mathsf{cl}}} / \mathcal{Q}_k$.

Lemma A.2. *The following hold:*

1. *(1) $\mathcal{F}_k$ is a nonzero pure coherent sheaf of dimension m for all $k \geq 1$.*
2. *(2) For any $1 \leq k < k_{\mathcal{E}} - 1$ there is an epimorphism $f_{k+1} \colon \mathcal{F}_{k+1} \to \mathcal{F}_k$, which fits into a commutative diagram*

$$\begin{array}{ccc} \mathcal{E} \otimes \mathcal{O}_{Z^{(k+1)}_{\mathsf{cl}}} & \longrightarrow & \mathcal{F}_{k+1} \\ \big\downarrow & & \big\downarrow f_k \\ \mathcal{E} \otimes \mathcal{O}_{Z^{(k)}_{\mathsf{cl}}} & \longrightarrow & \mathcal{F}_k \end{array} \quad . \tag{A.1}$$

Moreover, $\ker(f_k)$ *is a nonzero subsheaf of* $\mathcal{F}_{k+1}$ *annihilated by* $\mathcal{I}_Z$.

Proof. We start by proving (1). Clearly, it suffices to prove the claim for $1 \leq k \leq k_{\mathcal{E}}$. Moreover, given the construction of $\mathcal{F}_k$, it suffices to prove that the set-theoretic support of $\mathcal{E} \otimes \mathcal{O}_{Z^{(k)}_{\mathsf{cl}}}$ coincides with the set-theoretic support of $\mathcal{E}$ for any $1 \leq k \leq k_{\mathcal{E}}$. Assume that this is not the case and let p be a point in $\mathsf{Supp}(\mathcal{E}) \smallsetminus \mathsf{Supp}(\mathcal{E} \otimes \mathcal{O}_{Z^{(k)}_{\mathsf{cl}}})$ for some $1 \leq k \leq k_{\mathcal{E}}$. Let $\mathcal{J} \subset \mathcal{O}_{Z^{(k_{\mathcal{E}})}_{\mathsf{cl}}}$ be the defining ideal of the closed subscheme $Z^{(k)}_{\mathsf{cl}} \subset Z^{(k_{\mathcal{E}})}_{\mathsf{cl}}$. Note that $\mathcal{J}$ is nilpotent. By assumption, the stalk of the tensor product $\mathcal{E} \otimes \mathcal{O}_{Z^{(k)}}$ at p vanishes. This implies that $\mathcal{J}\mathcal{E}_p = \mathcal{E}_p$. Since $\mathcal{J}$ is nilpotent, Nakayama's lemma [Sta25, Tag 07RC, Lemma 10.20.1] implies that stalk $\mathcal{E}_p$ vanishes, leading to a contradiction. In conclusion, each quotient $\mathcal{F}_k$ is nonzero, pure, and of dimension m.

Now, we prove (2). First, note that claim (1) implies that the composition

$$\mathcal{E} \otimes \mathcal{O}_{Z^{(k+1)}_{\mathsf{cl}}} \longrightarrow \mathcal{E} \otimes \mathcal{O}_{Z^{(k)}} \longrightarrow \mathcal{F}_k$$

factors through the epimorphism $\mathcal{E} \otimes \mathcal{O}_{Z^{(k+1)}_{\mathsf{cl}}} \to \mathcal{F}_{k+1}$. Therefore, one obtains indeed a commutative diagram as in (A.1) for any $k \geq 1$, where the induced morphism $f_k \colon \mathcal{F}_{k+1} \to \mathcal{F}_k$ is surjective.

We next show that $\ker(f_k) \neq 0$ for all $1 \leq k < k_{\mathcal{E}}$. Assume that this is not the case for some $1 \leq k < k_{\mathcal{E}}$. Then, f_k is an isomorphism. Moreover, recall that $\mathcal{I}^k\mathcal{E}$ is a nonzero pure m-dimensional subsheaf of $\mathcal{E}$. Hence, its set-theoretic support is a non-empty m-dimensional closed subspace of $|{}^{\mathsf{red}}W|$. Let

$$U := \mathsf{Supp}(\mathcal{I}_Z^k\mathcal{E}) \smallsetminus \mathsf{Supp}(\mathcal{Q}_k)\,,$$

and note that U is non-empty since $\dim \mathsf{Supp}(\mathcal{Q}_i) < m$ for $i = k, k+1$. Let $p \in U$ be an arbitrary point. By construction, the stalk of $\mathcal{F}_i$ at p coincides with the stalk of $\mathcal{E} \otimes \mathcal{O}_{Z_{\mathsf{cl}}^{(i)}}$ for $i = k, k+1$. Therefore, the canonical epimorphism

$$\mathcal{E}_p \otimes \mathcal{O}_{Z_{\mathsf{cl}}^{(i+1)}, p} \longrightarrow \mathcal{E}_p \otimes \mathcal{O}_{Z_{\mathsf{cl}}^{(i)}, p}$$

is an isomorphism. Given the canonical commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_Z^{k+1}\mathcal{E} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E} \otimes \mathcal{O}_{Z_{\mathsf{cl}}^{(k+1)}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \mathsf{id}} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_Z^{k}\mathcal{E} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E} \otimes \mathcal{O}_{Z_{\mathsf{cl}}^{(k)}} & \longrightarrow & 0 \end{array} , \tag{A.2}$$

this implies that the natural injection

$$\mathcal{I}_p^{k+1}\mathcal{E}_p \longrightarrow \mathcal{I}_p^{k}\mathcal{E}_p$$

is an isomorphism. Then, by Nakayama's lemma [Sta25, Tag 07RC, Lemma 10.20.1], there exists $f \in \mathcal{I}_p$ so that $(1-f)\mathcal{I}_p^k\mathcal{E}_p = 0$. Since $p \in \mathsf{Supp}(\mathcal{I}_Z^k\mathcal{E})$, one has $\mathcal{I}_p^k\mathcal{E}_p \neq 0$. Hence $(1-f)$ is not a unit in $\mathcal{O}_{X,p}$. This implies that it must belong to the maximal ideal $\mathfrak{m}_p \subset \mathcal{O}_{X,p}$ by [Sta25, Tag 00E9,Lemma 10.18.2]. By *loc.cit.*, one also has $\mathcal{I}_p \subset \mathfrak{m}_p$, which leads to $1 \in \mathfrak{m}_p$; clearly a contradiction. In conclusion, $\ker(f_k) \neq 0$ for all $1 \leq k < k_{\mathcal{E}}$.

Furthermore, applying the snake lemma to diagram (A.2), one obtains an isomorphism

$$\ker(f_k) \longrightarrow \mathcal{I}_Z^k\mathcal{E} / \mathcal{I}_Z^{k+1}\mathcal{E} .$$

This implies that $\ker(f_k)$ is annihilated by $\mathcal{I}_Z$. □

Proof of Proposition A.1. For any $k \geq 1$, let $\mathcal{E}_k \subset \mathcal{E}$ be the kernel of the canonical epimorphism $\mathcal{E} \to \mathcal{F}_k$. Then, the filtration

$$0 =: \mathcal{E}_{k_{\mathcal{E}}} \subset \cdots \subset \mathcal{E}_0 := \mathcal{E}$$

is so that each subquotient $\mathcal{E}_k/\mathcal{E}_{k-1}$ is a nonzero pure sheaf of dimension m, annihilated by $\mathcal{I}_Z$. Indeed, for any $1 \leq k < k_{\mathcal{E}}$, diagram (A.1) in Lemma A.2 yields the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}_{k+1} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F}_{k+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \mathsf{id}} & & \downarrow{\scriptstyle f_k} & & \\ 0 & \longrightarrow & \mathcal{E}_{k} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F}_{k} & \longrightarrow & 0 \end{array} ,$$

where the rows are exact and $\mathcal{E}_{k+1} \subset \mathcal{E}_k$ as subsheaves of $\mathcal{E}$. Moreover, applying the snake lemma, one obtains an isomorphism $\ker(f_k) \simeq \mathcal{E}_k/\mathcal{E}_{k+1}$. Then, the claim follows from Lemma A.2–(2). □

APPENDIX B. IND-OBJECTS

B.1. **Restricted presentations for ind-morphisms.** Fix an ∞-category $\mathcal{C}$ and let P be a property of morphisms in $\mathcal{C}$. Recall from [Lur09, Proposition 5.3.5.15] that for every ∞-category $\mathcal{C}$, there is a canonical equivalence

$$\mathsf{Ind}(\mathcal{C}^{\Delta^1}) \simeq \mathsf{Ind}(\mathcal{C})^{\Delta^1} .$$

Reviewing morphisms in $\mathcal{C}$ as objects in $\mathcal{C}^{\Delta^1}$, [DPS22, Definition II.7.2] allows to talk about *ind-P* morphisms in $\mathsf{Ind}(\mathcal{C})$; concretely, a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ in $\mathsf{Ind}(\mathcal{C})$ is ind-P if there exists a presentation $\{f_\alpha\colon X_\alpha \to Y_\alpha\}_{\alpha \in I}$ for the morphism f such that for every index $\alpha \in I$ the map $f_\alpha\colon X_\alpha \to Y_\alpha$ satisfies the property P.

Example B.1.

(1) If the morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is representable by P-morphisms, in the sense that for every $Y \in \mathcal{C}$ and every morphism $Y \to \mathcal{Y}$ in $\mathsf{Ind}(\mathcal{C})$ the fiber product $Y \times_{\mathcal{Y}} \mathcal{X}$ belongs to $\mathcal{C}$ and the canonical map $Y \times_{\mathcal{Y}} \mathcal{X} \to Y$ satisfies the property P, then f is ind-P.

(2) Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$ which, seen as a morphism in $\mathsf{Ind}(\mathcal{C})$, is ind-$P$. Then f is a retract of a P-morphism. In particular, if P-morphisms are stable under retracts, we see that a morphism in $\mathcal{C}$ satisfies P if and only if it is ind-P as a morphism in $\mathsf{Ind}(\mathcal{C})$.

△

Definition B.2. Let $\mathcal{A}$ be an ∞-category and let $F\colon \mathcal{A} \to \mathcal{C}$ be a diagram. We say that F is a *P-diagram* if it takes every arrow in $\mathcal{A}$ to P-morphisms in $\mathcal{C}$.
⊘

The following result establishes the existence of *simultaneous presentations* for specific diagrams.

Proposition B.3. *Assume that $\mathcal{C}$ has finite limits and let P be a property of morphisms in $\mathcal{C}$ which is stable under fiber products. For $k \in \{1,2\}$, let $\mathcal{A} := \Lambda^2_k$ be the corresponding 2-horn. For a diagram $F\colon \mathcal{A} \to \mathsf{Ind}(\mathcal{C})$, the following statements are equivalent:*

(1) F is an ind-P-diagram;

(2) there exists a filtered diagram $I \to \mathsf{Fun}(\mathcal{A}, \mathcal{C})_{/F}$, sending $\alpha \in \mathcal{A}$ to a P-diagram F_α such that the canonical map

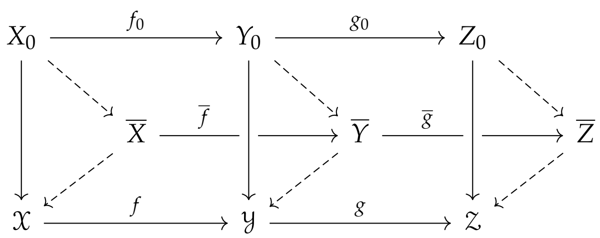
$$\operatorname*{colim}_{\alpha} F_\alpha \longrightarrow F$$

is an equivalence in $\mathsf{Ind}(\mathcal{C})$.

Proof. We deal with the case $k = 1$. The case $k = 2$ is similarly dealt with. First of all, the implication (2) ⇒ (1) is trivial. Let us prove the converse. Let therefore F be an ind-P-diagram, which we represent as

$$\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z} \ ,$$

where both f and g are ind-P morphisms. Using [Lur09, Proposition 5.3.5.15], we find a canonical equivalence $\mathsf{Ind}(\mathsf{Fun}(\Lambda^2_1, \mathcal{C})) \simeq \mathsf{Fun}(\Lambda^2_1, \mathsf{Ind}(\mathcal{C}))$. Applying [DPS22, Lemma II.7.4], we are reduced to check that the following factorization problem has always a solution: for every solid commutative diagram

 , (B.1)

where the top line is a compact object in $\mathsf{Fun}(\Lambda^2_1, \mathsf{Ind}(\mathcal{C}))$ (that is, X_0, Y_0 and Z_0 are compact in $\mathsf{Ind}(\mathcal{C})$), there exists a P-diagram $\overline{F}\colon \Lambda^2_1 \to \mathcal{C}$ (represented as the middle line in the above diagram) and a factorization as indicated. Since g is ind-P, we can fix a P-presentation as filtered colimit of a diagram $\{g_\alpha\colon Y_\alpha \to Z_\alpha\}_{\alpha \in J}$ of P-morphisms. Since $g_0\colon Y_0 \to Z_0$ is compact in $\mathsf{Ind}(\mathcal{C})^{\Delta^1}$, we deduce the existence of an index α and a factorization of the front right square in (B.1) as

$$\begin{array}{ccccc} Y_0 & \longrightarrow & Y_\alpha & \longrightarrow & \mathcal{Y} \\ \big\downarrow{\scriptstyle g_0} & & \big\downarrow{\scriptstyle g_\alpha} & & \big\downarrow{\scriptstyle g} \\ Z_0 & \longrightarrow & Z_\alpha & \xrightarrow{f_\alpha} & \mathcal{Z} \end{array} .$$

On the other hand, $f\colon \mathcal{X} \to \mathcal{Y}$ is also an ind-P-morphism. Thus, we can fix a P-presentation $\{f_\alpha\colon X_\beta \to \widetilde{Y}_\beta\}_{\beta\in J'}$. Since source and target of the composite morphism $f_0'\colon X_0 \to Y_\alpha$ are compact in $\mathsf{Ind}(\mathcal{C})$, we deduce the existence of an index $\beta \in J'$ and of a factorization

$$\begin{array}{ccccc} X_0 & \longrightarrow & X_\beta & \longrightarrow & \mathcal{X} \\ \downarrow{\scriptstyle f_0'} & & \downarrow{\scriptstyle f_\beta} & & \downarrow{\scriptstyle f} \\ Y_\alpha & \longrightarrow & \widetilde{Y}_\beta & \longrightarrow & \mathcal{Y} \end{array} .$$

Set $X_{\alpha,\beta} := Y_\alpha \times_{\widetilde{Y}_\beta} X_\beta$. Since $\mathcal{C}$ has finite limits, this is an object in $\mathcal{C}$; moreover since P-morphisms are stable under pullbacks, the morphism $f_{\alpha,\beta}\colon X_{\alpha,\beta} \to Y_\alpha$ satisfies P. Combining what we have found so far, we see that the diagram

$$X_{\alpha,\beta} \xrightarrow{f_{\alpha,\beta}} Y_\alpha \xrightarrow{g_\alpha} Z_\alpha$$

solves the factorization problem (B.1). □

Variant B.4. In the setting of the previous proposition, let P and Q be two different properties of morphisms of $\mathcal{C}$, and assume that at least one of the two is stable under fiber products. Say that a diagram $F\colon \Lambda^2_1 \to \mathcal{C}$ depicted as

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a (P,Q)-diagram if f satisfies P and g satisfies Q. Then for a diagram

$$\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$$

in $\mathsf{Ind}(\mathcal{C})$ the following statements are equivalent:

(1) f is ind-P and g is ind-Q;

(2) the above diagram admits a presentation as filtered colimit of $\mathcal{C}$-valued (P,Q)-diagrams.

A similar statement holds replacing Λ^2_1 by Λ^2_2. ⊘

Remark B.5. If $\mathcal{C}$ admits finite colimits and P-morphisms are closed under pushout, then a dual argument shows that in the statements of Proposition B.3 and Variant B.4 one can replace Λ^2_2 by Λ^2_0. However, these assumptions will not be satisfied in our framework. △

Corollary B.6. *Let $\mathcal{C}$ be an ∞-category and let P be a property of morphisms in $\mathcal{C}$. Then:*

(1) If for every $X \in \mathcal{C}$ the identity id_X satisfies P, then for every $\mathcal{X} \in \mathsf{Ind}(\mathcal{C})$ the identity $\mathrm{id}_{\mathcal{X}}$ is ind-P.

Assume furthermore that $\mathcal{C}$ has finite limits and that P-morphisms are stable under pullback. Then:

(2) Ind-P-morphisms are stable under pullbacks.

(3) If P-morphisms are closed under composition, then the same goes for ind-P-morphisms.

(4) Assume that P-morphism have the following closure property: given a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & Z & \end{array} ,$$

then if g satisfies P, then f satisfies P if and only if h satisfies P. Then ind-P morphism have the same closure property.

Proof. If id_X satisfies P for every $X \in \mathcal{C}$, then it is obvious that $\mathrm{id}_{\mathcal{X}}$ is ind-P for every $\mathcal{X} \in \mathrm{Ind}(\mathcal{C})$. This proves (1).

For (2), let

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow f \\ \mathcal{Z} & \xrightarrow{g} & \mathcal{Y} \end{array}$$

be a pullback square in $\mathrm{Ind}(\mathcal{C})$ where f is in ind-P. Applying Variant B.4 (with $Q =$ all the class of all morphisms of $\mathcal{C}$), we can find a filtered diagram

$$\left\{ Z_\alpha \xrightarrow{g_\alpha} Y_\alpha \xleftarrow{f_\alpha} X_\alpha \right\}_{\alpha \in I}$$

whose colimit coincides with

$$\mathcal{Z} \xrightarrow{g} \mathcal{Y} \xrightarrow{f} \mathcal{X}$$

and for which f_α satisfies P for every index $\alpha \in I$. Thus, the map $\mathcal{W} \to \mathcal{Z}$ can be presented as the colimit of the P-morphisms $Z_\alpha \times_{Y_\alpha} X_\alpha \to Z_\alpha$, which satisfy P by assumptions.

We now prove (3). Let $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$ be two ind-P morphisms. Applying Proposition B.3 with $k = 1$, we can find a simultaneous presentation

$$\left\{ X_\alpha \xrightarrow{f_\alpha} Y_\alpha \xrightarrow{g_\alpha} Z_\alpha \right\}_{\alpha \in I}$$

for the two morphisms f and g such that f_α and g_α both satisfy P for every index $\alpha \in I$. The conclusion is therefore obvious.

For (4), we proceeds similarly, applying Proposition B.3 with $k = 2$. □

B.2. **Restricted presentations for ind-objects.** In this section, we discuss presentations of ind-objects, following [DPS22, §II.7].

We fix as usual an ∞-category $\mathcal{C}$ and a property P of morphisms in $\mathcal{C}$. We make the following global assumption:

Assumption 2. Let $\mathcal{C}$ be an ∞-category with finite limits and let P be a property of morphisms in $\mathcal{C}$ such that:

(1) every identity of $\mathcal{C}$ satisfies P;

(2) P-morphisms are stable under composition and retracts;

(3) given morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ such that g satisfies P, then f satisfies P if and only if $g \circ f$ satisfies P.

⊘

Notice that (1) guarantees that every object $X \in \mathcal{C}$ is ind-P.

Remark B.7. Under the above assumption, let $\mathcal{C}_P$ be the associated (non full) subcategory of $\mathcal{C}$ and consider the induced functor $\mathrm{Ind}(\mathcal{C}_P) \to \mathrm{Ind}(\mathcal{C})$. Unraveling the definition, we see that an object belongs to the essential image of this functor if and only if it is ind-P. Moreover, this functor take every morphism to an ind-P morphism. In other words, it factors through the non-full subcategory $\mathrm{Ind}_P(\mathcal{C})_{\text{ind-}P}$. △

Lemma B.8. *Let $\mathcal{X} \in \mathrm{Ind}(\mathcal{C})$ be an ind-object and let $\{X_\alpha\}_{\alpha \in I}$ be a presentation for $\mathcal{X}$. The following statements are equivalent:*

(1) $\{X_\alpha\}_{\alpha \in I}$ is a P-presentation;

(2) for every $\alpha \in I$, the morphism $X_\alpha \to \mathcal{X}$ is an ind-P morphism;

Proof. First assume that $\{X_\alpha\}_{\alpha \in I}$ is a P-presentation. Then for every $\alpha \in I$, the map $X_\alpha \to \mathcal{X}$ can be written as colimit of the maps $\{X_\alpha \to X_\beta\}_{\beta \in I_{\alpha/}}$. Since every transition map satisfies P, we conclude that $X_\alpha \to \mathcal{X}$ is an ind-P morphism.

Conversely, suppose that for every $\alpha \in I$, the map $X_\alpha \to \mathcal{X}$ is ind-P. Let $\alpha \to \beta$ be a morphism in $\mathcal{C}$, and consider the commutative triangle

$$\begin{array}{ccc} X_\alpha & \longrightarrow & X_\beta \\ & \searrow \quad \swarrow & \\ & \mathcal{X} & \end{array} .$$

Since P is stable under pullbacks and satisfies the 2-out-of-3 property, Corollary B.6–(4) guarantees that the map $X_\alpha \to X_\beta$ is ind-P. Since P-morphisms are closed under retracts, Example B.1–(2) guarantees that $X_\alpha \to X_\beta$ satisfies P as well. Thus, $\{X_\alpha\}_{\alpha \in P}$ is a P-presentation. □

Corollary B.9. *For an ind-object $\mathcal{X} \in \mathsf{Ind}(\mathcal{C})$, the following statements are equivalent:*

(1) $\mathcal{C}$ is an ind-P object;

(2) the full subcategory $\mathcal{C}^{\text{ind-}P}_{/\mathcal{X}}$ of $\mathcal{C} \times_{\mathsf{Ind}(\mathcal{C})} \mathsf{Ind}(\mathcal{C})_{/\mathcal{X}}$ spanned by ind-P morphisms $X \to \mathcal{X}$ is filtered and the inclusion $\mathcal{C}^{\text{ind-}P}_{/\mathcal{X}} \hookrightarrow \mathcal{C}_{/\mathcal{X}}$ is colimit-cofinal.

Proof. The implication (2) ⇒ (1) follows directly from Lemma B.8.

Conversely, fix a P-presentation $\{X_\alpha\}_{\alpha \in I}$. Reasoning as in [DPS22, Lemma II.7.4], we reduce to the following factorization statement: for every compact object $\overline{X} \in \mathsf{Ind}(\mathcal{C})$ and every morphism $f \colon \overline{X} \to \mathcal{X}$, there exists a factorization of f as

$$\overline{X} \xrightarrow{f'} X' \xrightarrow{f''} \mathcal{X} \quad ,$$

where $X' \in \mathcal{C}$ and f'' is ind-P. Since $\overline{X}$ is compact, there exists $\alpha \in I$ such that f factors through the structural map $X_\alpha \to \mathcal{X}$. On the other hand, Lemma B.8 guarantees that this map is ind-P. Thus, the conclusion follows. □

Proposition B.10. *Under Assumption 2, the induced functor*

$$\mathsf{Ind}(\mathcal{C}_P) \to \mathsf{Ind}_P(\mathcal{C})_{\text{ind-}P} \tag{B.2}$$

is an equivalence. In particular, if $f \colon \mathcal{X} \to \mathcal{Y}$ is an ind-P morphism between ind-P objects, then there exists a presentation $\{f_\alpha \colon X_\alpha \to Y_\alpha\}_{\alpha \in I}$ such that for every $\alpha \in I$ the morphism f_α satisfies P and both $\{X_\alpha\}_{\alpha \in I}$ and $\{Y_\alpha\}_{\alpha \in I}$ are P-presentations for $\mathcal{X}$ and $\mathcal{Y}$, respectively.

Proof. In virtue of Remark B.7, we see that the functor under consideration is essentially surjective. It is therefore enough to check that it is fully faithful. Observe that the inclusion $\mathcal{C}_P \hookrightarrow \mathcal{C}$ is faithful, that is for every $X, Y \in \mathcal{C}_P$, the morphism

$$\mathsf{Map}_{\mathcal{C}_P}(X, Y) \longrightarrow \mathsf{Map}_{\mathcal{C}}(X, Y)$$

is (-1)-truncated. Let now $\mathcal{X}$ and $\mathcal{Y}$ be two objects in $\mathsf{Ind}(\mathcal{C}_P)$ and fix two P-presentations $\{X_\alpha\}_{\alpha \in I}$ and $\{Y_\beta\}_{\beta \in J}$. Then we have

$$\mathsf{Map}_{\mathsf{Ind}(\mathcal{C}_P)}(\mathcal{X}, \mathcal{Y}) \simeq \lim_\alpha \operatorname*{colim}_\beta \mathsf{Map}_{\mathcal{C}_P}(X_\alpha, Y_\beta)$$

and

$$\mathsf{Map}_{\mathsf{Ind}(\mathcal{C})}(\mathcal{X}, \mathcal{Y}) \simeq \lim_\alpha \operatorname*{colim}_\beta \mathsf{Map}_{\mathcal{C}}(X_\alpha, Y_\beta) .$$

Since (-1)-truncated morphisms are stable under arbitrary limits and under filtered colimits, it follows that the map

$$\mathsf{Map}_{\mathsf{Ind}(\mathcal{C}_P)}(\mathcal{X}, \mathcal{Y}) \longrightarrow \mathsf{Map}_{\mathsf{Ind}(\mathcal{C})}(\mathcal{X}, \mathcal{Y})$$

is (-1)-truncated as well, i.e. $\mathrm{Ind}(\mathcal{C}_P) \to \mathrm{Ind}(\mathcal{C})$ is a faithful functor. Since the same is true for $\mathrm{Ind}(\mathcal{C})_{\text{ind-}P} \hookrightarrow \mathrm{Ind}(\mathcal{C})$, we deduce that the functor (B.2) is faithful as well. To complete the proof, it is therefore enough to check that the functor in consideration is full.

Let therefore $f \colon \mathcal{X} \to \mathcal{Y}$ be an ind-P morphism between ind-P objects. Let $(\mathcal{C}^{\Delta^1})^{\text{ind-}P}_{/f}$ be the full subcategory of $(\mathcal{C}^{\Delta^1})_{/f} := \mathcal{C}^{\Delta^1} \times_{\mathrm{Ind}(\mathcal{C}^{\Delta^1})} \mathrm{Ind}(\mathcal{C}^{\Delta^1})_{/f}$ spanned by those squares

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ Y & \longrightarrow & \mathcal{Y} \end{array}$$

whose horizontal morphisms are ind-P. Since ind-P morphism satisfy the 2-out-of-3 property and since P-morphisms are closed under retracts, it automatically follows in this situation that $g \colon X \to Y$ satisfies P. It follows that the forgetful functor

$$(\mathcal{C}^{\Delta^1})^{\text{ind-}P}_{/f} \longrightarrow \mathcal{C}^{\Delta^1}$$

can be canonically factored through $(\mathcal{C}_P)^{\Delta^1}$. Thus, it is enough to check that $(\mathcal{C}^{\Delta^1})^{\text{ind-}P}_{/f}$ is filtered and that the inclusion

$$(\mathcal{C}^{\Delta^1})^{\text{ind-}P}_{/f} \hookrightarrow (\mathcal{C}^{\Delta^1})_{/f}$$

is colimit-cofinal. As usual, it is enough to show that the following factorization problem can always be solved: for every solid diagram

$$\begin{array}{ccccc} \overline{X} & & \longrightarrow & & \mathcal{X} \\ & \searrow & & \nearrow{\scriptstyle u} & \\ \big\downarrow{\scriptstyle g} & & X' & & \big\downarrow{\scriptstyle f} \\ & & \big\downarrow{\scriptstyle g'} & & \\ \overline{Y} & & \longrightarrow & & \mathcal{Y} \\ & \searrow & & \nearrow{\scriptstyle v} & \\ & & Y' & & \end{array},$$

where $\overline{X}$ and $\overline{Y}$ are compact objects in $\mathrm{Ind}(\mathcal{C})$, there exists a factorization as indicated in the diagram, where furthermore u and v are ind-P. To see this, start by choosing a P-presentation $\{X_\alpha\}_{\alpha \in I}$ for $\mathcal{X}$. Since $\overline{X}$ is compact, we can find $\alpha \in I$ such that the map $\overline{X} \to \mathcal{X}$ factors through $X_\alpha \to \mathcal{X}$. Fix at the same time a P-presentation $\{Y_\beta\}_{\beta \in J}$ for $\mathcal{Y}$. Since both $\overline{Y}$ and X_α are compact in $\mathrm{Ind}(\mathcal{C})$, we can find a factorization of the original square as

$$\begin{array}{ccccc} \overline{X} & \longrightarrow & X_\alpha & \xrightarrow{u} & \mathcal{X} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \overline{Y} & \longrightarrow & Y_\beta & \xrightarrow{v} & \mathcal{Y} \end{array}.$$

Since we started with P-presentations for $\mathcal{X}$ and $\mathcal{Y}$, Lemma B.8 guarantees that u and v are ind-P morphisms, whence the conclusion. □

Remarkably, this result allows to improve Corollary B.6–(4) without assuming the existence of pushouts in $\mathcal{C}$ (cf. Remark B.5).

Corollary B.11. *Assume in addition to Assumption 2 that P-morphisms satisfy the* 2*-out-of-*3 *property. Then the same goes for ind-P morphisms.*

Proof. Let

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "h"'] & & \mathcal{Y} \arrow[dl, "g"] \\ & \mathcal{Z} & \end{tikzcd} \tag{B.3}$$

be a commutative triangle in $\mathsf{Ind}(\mathcal{C})$. Assume that $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are ind-P. If g is ind-P, then Corollary B.6–(4) shows that f is ind-P if and only if h is ind-P. We are therefore left to check that if both h and f are ind-P, then the same goes for g. To do this, we apply [DPS22, Lemma II.7.4] with $\mathcal{E} = \mathcal{C}^{\Delta^2}$, saying a 2-simplex of the form (B.3) in $\mathcal{C}$ satisfies Q if f, g and h satisfy the property P. We are therefore led to consider the following factorization problem: given a morphism ϕ in $\mathsf{Ind}(\mathcal{C}^{\Delta^2})$

$$\begin{tikzcd} & & & \overline{X} \arrow[r] \arrow[dlll, "\phi_0"'] \arrow[dr] & \overline{Y} \arrow[dlll, "\phi_1"'] \arrow[d] \\ \mathcal{X} \arrow[r, "f"] \arrow[dr, "h"'] & \mathcal{Y} \arrow[d, "g"] & & & \overline{Z} \arrow[dlll, "\phi_2"] \\ & \mathcal{Z} & & & \end{tikzcd} \tag{B.4}$$

where $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ are compact in $\mathsf{Ind}(\mathcal{C})$, we can factor α through a 2-simplex in $\mathcal{C}$ satisfying Q. Since f is ind-P, Proposition B.10 provides a presentation $\{f_\alpha \colon X_\alpha \to Y_\alpha\}_{\alpha \in I}$ such that every f_α satisfies P and for which the structural morphisms $X_\alpha \to \mathcal{X}$ and $Y_\alpha \to \mathcal{Y}$ are ind-P. Compactness of $\overline{X}$ and $\overline{Y}$ implies that there exists an index $\alpha \in I$ such that the morphism ϕ factors through the 2-simplex

$$\begin{tikzcd} X_\alpha \arrow[rr, "f_\alpha"] \arrow[dr, "h_\alpha"'] & & Y_\alpha \arrow[dl, "g_\alpha"] \\ & \mathcal{Z} & \end{tikzcd} .$$

Fix now a P-presentation $\{Z_\beta\}_{\beta \in J}$ for $\mathcal{Z}$. Compactness of X_α, Y_α, $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ implies therefore that there exists an index $\beta \in J$ and a commutative diagram

$$\begin{tikzcd} & & & \overline{X} \arrow[r] \arrow[dlll] \arrow[dr] & \overline{Y} \arrow[dlll] \arrow[d] \\ X_\alpha \arrow[r, "f_\alpha"] \arrow[dr, "h_{\alpha,\beta}"'] & Y_\alpha \arrow[d, "g_{\alpha,\beta}"] & & & \overline{Z} \arrow[dlll] \\ & Z_\beta & & & \end{tikzcd}$$

factorizing the original diagram (B.4). By construction, f_α satisfies P. To conclude, it is enough to argue that both $h_{\alpha,\beta}$ and $g_{\alpha,\beta}$ satisfy P. Since P-morphism satisfy the 2-out-of-3 property, it is enough to argue that $h_{\alpha,\beta}$ satisfies P. However, the diagram

$$\begin{tikzcd} X_\alpha \arrow[rr, "h_{\alpha,\beta}"] \arrow[dr, "h_\alpha"'] & & Z_\beta \arrow[dl, "j_\beta"] \\ & \mathcal{Z} & \end{tikzcd}$$

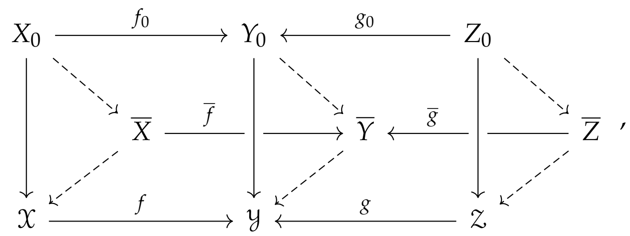
is commutative, and both h_α and j_β are ind-P. Thus, Corollary B.6–(4) guarantees that $h_{\alpha,\beta}$ satisfies P. The conclusion follows. □

Corollary B.12. *Assume in addition to Assumption 2 that $\mathcal{C}$ has finite limits and that P is stable under fiber products. Then $\mathrm{Ind}_P(\mathcal{C})$ is closed under fiber products in $\mathcal{C}$.*

Proof. Consider a span

$$\mathcal{X} \xrightarrow{f} \mathcal{Y} \xleftarrow{g} \mathcal{Z}$$

in $\mathrm{Ind}_P(\mathcal{C})$. We claim that whenever given a solid commutative diagram



(B.5)

where X_0, Y_0 and Z_0 are compact in $\mathrm{Ind}(\mathcal{C})$, there exists the dashed factorization through a span in $\mathcal{C}$ with the property that the morphisms $\overline{X} \to \mathcal{X}$, $\overline{Y} \to \mathcal{Y}$ and $\overline{Z} \to \mathcal{Z}$ are ind-P. Combining [DPS22, Lemma II.7.4] and Lemma B.8, this implies that the original span admits a simultaneous presentation

$$\{X_\alpha \xrightarrow{f_\alpha} Y_\alpha \xleftarrow{g_\alpha} Z_\alpha\}_{\alpha \in I}$$

which restricts to P-presentations $\{X_\alpha\}_{\alpha\in I}$, $\{Y_\alpha\}_{\alpha\in I}$ and $\{Z_\alpha\}_{\alpha\in I}$ for $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ respectively. At this point, since $\mathcal{C}$ has finite limits and P is closed under fiber products, we deduce that

$$\{X_\alpha \times_{Y_\alpha} Z_\alpha\}_{\alpha \in I}$$

is a P-presentation for $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.

We are therefore left to prove the above claim. Choose P-presentations $\{\overline{X}_\alpha\}_{\alpha\in I_1}$, $\{\overline{Y}_\beta\}_{\beta\in I_2}$ and $\{\overline{Z}_\gamma\}_{\gamma\in I_3}$ for $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, respectively. Since X_0 and Z_0 are compact, we can find indexes $\alpha \in I_1$ and $\gamma \in I_3$ together with factorizations of $X_0 \to \mathcal{X}$ and of $Z_0 \to \mathcal{Z}$ through $\overline{X}_\alpha$ and $\overline{Z}_\gamma$, respectively. Furthermore, since Y_0, $\overline{X}_\alpha$ and $\overline{Z}_\gamma$ are compact, we can find $\beta \in I_2$ such that the natural morphisms $\overline{X}_\alpha \longrightarrow \mathcal{Y}$, $Y_0 \longrightarrow \mathcal{Y}$ and $\overline{Z}_\gamma \longrightarrow \mathcal{Y}$ factor through $\overline{Y}_\beta$. Thus, the induced span $\overline{X}_\alpha \to \overline{Y}_\beta \leftarrow \overline{Z}_\gamma$ solves the factorization problem (B.5). □

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