COHOMOLOGICAL HALL ALGEBRAS OF ONE-DIMENSIONAL SHEAVES ON SURFACES: AMALGAMATION AND PBW-TYPE THEOREMS

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ABSTRACT. In this paper we discuss a PBW-type theorem for (equivariant) nilpotent cohomological Hall algebras (COHAs) associated with a smooth surface X and a reduced subscheme $Z \subset X$, stating a conjecturally relation between the nilpotent COHAs associated with X and the irreducible components of Z and the nilpotent COHAs associated with X and Z. We prove this relation when X is a minimal resolution of a Kleinian singularity and when X is an elliptic surface with singular fiber of type DE.

CONTENTS

1. INTRODUCTION

Let X be a smooth quasi-projective complex surface and let $j\colon Z \hookrightarrow X$ be the inclusion of a closed subscheme. Assume that there is a torus T acting on X such that Z is T-invariant[1]. In [DPS+25b], we introduce the T-equivariant nilpotent cohomological Hall algebra $\mathbf{HA}^T_{X,Z}$.

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[1] For a uniform treatment, we allow that $T = \{1\}$.

Let us assume that $Z = Z_1 \cup Z_2$. The functoriality with respect to the diagram of embeddings

$$\begin{array}{ccccc} & & Z_1 & & \\ & \nearrow & & \searrow & \\ Z_1 \cap Z_2 & & & & Z_1 \cup Z_2 \\ & \searrow & & \nearrow & \\ & & Z_2 & & \end{array}$$

yields a diagram of algebra morphisms

$$\begin{array}{ccccc} & & \mathbf{HA}^T_{X,Z_1} & & \\ & \nearrow & & \searrow & \\ \mathbf{HA}^T_{X,Z_1\cap Z_2} & & & & \mathbf{HA}^T_{X_Z} \\ & \searrow & & \nearrow & \\ & & \mathbf{HA}^T_{X,Z_2} & & \end{array}$$

For instance, in the particular case that Z_1 and Z_2 are (-2)-rational curves intersecting transversally at a single point p, the results of [DPS+25a] in types A_1 and A_2, together with the fact that $\mathbf{HA}^T_{X,p} \simeq \mathbb{Y}^+_\infty(\widehat{\mathfrak{gl}}_1)$ for any point $p \in X$, see [SV13, Dav24], yields a diagram

$$\begin{array}{ccccc} & & \mathbb{Y}^+_\infty(\widehat{\mathfrak{gl}}_2) & & \\ & \nearrow & & \searrow & \\ \mathbb{Y}^+_\infty(\widehat{\mathfrak{gl}}_1) & & & & \mathbb{Y}^+_\infty(\widehat{\mathfrak{gl}}_3) \\ & \searrow & & \nearrow & \\ & & \mathbb{Y}^+_\infty(\widehat{\mathfrak{gl}}_2) & & \end{array}$$

of morphisms of double loop positive halves of affine Yangians, which can be shown to be embeddings.

It is natural to wonder ho one can build the algebra $\mathbf{HA}^T_{X,Z}$ from the algebras $\mathbf{HA}^T_{X,Z_i}$ with $i = 1, 2$. Under some very mild assumptions, any coherent sheaf $\mathcal{F}$ set-theoretically supported on Z has a unique subsheaf $\mathcal{F}_1 \subset \mathcal{F}$ supported on Z_1 for which $\mathcal{F}/\mathcal{F}_1$ is set-theoretically supported on Z_2 and pure of dimension one at $Z_1 \cap Z_2$. This induces a stratification of the moduli stack $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z)$ according to the class of $\mathcal{F}_1$, which we construct and study in detail in this paper. Though all the strata are locally closed, there are in general no open or closed strata (typically as in an infinite chain of $\mathbb{P}^1$'s).

The Hall multiplication induces a continuous map

$$\mathfrak{a} \colon \mathbf{HA}^T_{X,Z_2} \widehat{\otimes}_{\mathsf{H}^\bullet_T} \mathbf{HA}^T_{X,Z_1} \longrightarrow \mathbf{HA}^T_{X,Z} .$$

Furthermore, this map factors naturally through a second map

$$\bar{\mathfrak{a}} \colon \mathbf{HA}^T_{X,Z_2} \widehat{\otimes}_{\mathbf{HA}^T_{X,Z_1\cap Z_2}} \mathbf{HA}^T_{X,Z_1} \longrightarrow \mathbf{HA}^T_{X,Z} . \tag{1.1}$$

In particular, one has a commutative diagram

$$\begin{array}{ccc} \mathbf{HA}^T_{X,Z_2} \widehat{\otimes}_{\mathsf{H}^\bullet_T} \mathbf{HA}^T_{X,Z_1} & & \\ \downarrow \tau & \searrow^{\mathfrak{a}} & \\ \mathbf{HA}^T_{X,Z_2} \widehat{\otimes}_{\mathbf{HA}^T_{X,Z_1\cap Z_2}} \mathbf{HA}^T_{X,Z_1} & \xrightarrow[\bar{\mathfrak{a}}]{} & \mathbf{HA}^T_{X,Z} \end{array}$$

which yields a third map

$$\ker(\tau) \longrightarrow \ker(\mathfrak{a}) . \tag{1.2}$$

There is no hope, in general, for the map (1.1) to be topologically surjective. Indeed, assume that Z is the ravioli –union of two $\mathbb{P}^1$s which intersect transversally in two points– and consider the moduli of sheaves of length one. In that case, $\mathsf{H}_{-1}(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \delta))$ is non-trivial while

$\mathsf{H}_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_i};\delta))$ is concentrated in even degrees for $i=1,2$. We make the following somewhat bold conjecture:

Conjecture. *Assume that both Z_1 and $Z_2 \smallsetminus (Z_1\cap Z_2)$ are cohomologically pure. Then, the maps* (1.1) *and* (1.2) *are topologically surjective.*

This essentially means that the morphism (1.1) is an isomorphism of topological vector spaces. One can view such a statement as a geometric form of a PBW theorem for the cohomological Hall algebra $\mathbf{HA}^T_{X,Z}$.

Now, we describe more precisely the results of this paper and state the conjecture in full generality.

1.1. **General framework.** Assume that Z is connected, purely[2] one-dimensional, proper, and reduced. Let

$$Z^\bullet\colon \quad \emptyset =: Z^{-1}\subset Z^0\subset \cdots \subset Z^s := Z\ , \tag{1.3}$$

be a T-invariant stratification by closed subschemes for $s\geq 0$, where for $1\leq i\leq s$, the subscheme Z^i is reduced and purely one-dimensional, while Z^0 is zero-dimensional. For each $0\leq i\leq s$, let Z_i° denote the locally closed stratum $Z_i^\circ := Z^i\smallsetminus Z^{i-1}$ and let Z_i denote its scheme-theoretic closure in Z, i.e., it is the smallest closed subscheme of Z containing Z_i° as an open subscheme. Let also Z_i^{i-1} be the scheme-theoretic intersection of Z^{i-1} and Z_i in X for $0\leq i\leq s$. In particular, $Z_0^\circ = Z^0 = Z_0$ and $Z_0^{-1}=\emptyset$.

Let $\langle Z_i\rangle \subset \mathsf{NS}(X)$ be the subgroup generated by the classes of the irreducible components of Z_i for $1\leq i\leq s$, and let $\langle Z\rangle\subset \mathsf{NS}(X)$ be the subgroup generated by the classes of irreducible components of Z.

In this Part, we shall investigate the relation between the T-equivariant COHA $\mathbf{HA}^T_{X,Z}$ and the T-equivariant COHAs $\mathbf{HA}^T_{X,Z_i}$ for $1\leq i\leq s$. More precisely, working under certain assumptions, it will be shown that the stratification (1.3) and the Hall multiplication yield a natural map

$$\mathfrak{a}_\gamma\colon \mathbf{HA}^T_{X,Z_s}(\gamma_s)\widehat{\otimes}_{\mathsf{H}^\bullet_T}\ \cdots\ \widehat{\otimes}_{\mathsf{H}^\bullet_T}\mathbf{HA}^T_{X,Z_1}(\gamma_1)\longrightarrow \mathbf{HA}^T_{X,Z}(\gamma)$$

for any $\gamma\in\langle Z\rangle$, where $\gamma_i\in\langle Z_i\rangle$ are uniquely determined by the relation $\gamma=\sum_{i=1}^s\gamma_i$. Here, $\mathbf{HA}^T_{X,Z}(\gamma)$ is the degree γ part of $\mathbf{HA}^T_{X,Z}$ with respect to the grading given by first Chern classes belonging to $\langle Z\rangle$, and similarly for $\mathbf{HA}^T_{X,Z_i}(\gamma_i)$ where $i=1,\ldots,s$.

Furthermore, this map factors naturally through a second map

$$\overline{\mathfrak{a}}_\gamma\colon \mathbf{HA}^T_{X,Z_s}(\gamma_s)\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_s^{s-1}}}\ \cdots\ \widehat{\otimes}_{\mathbf{HA}^T_{X,Z_2^1}}\mathbf{HA}^T_{X,Z_1}(\gamma_1)\longrightarrow \mathbf{HA}^T_{X,Z}(\gamma)\ ,$$

which will called in the following the *amalgamation map*. In particular, one has a commutative diagram

$$\begin{array}{ccc}
\mathbf{HA}^T_{X,Z_s}(\gamma_s)\widehat{\otimes}_{\mathsf{H}^\bullet_T}\ \cdots\ \widehat{\otimes}_{\mathsf{H}^\bullet_T}\mathbf{HA}^T_{X,Z_1}(\gamma_1) & \xrightarrow{\ \mathfrak{a}_\gamma\ } & \\
\big\downarrow{\scriptstyle \tau_\gamma} & & \\
\mathbf{HA}^T_{X,Z_s}(\gamma_s)\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_s^{s-1}}}\ \cdots\ \widehat{\otimes}_{\mathbf{HA}^T_{X,Z_2^1}}\mathbf{HA}^T_{X,Z_1}(\gamma_1) & \xrightarrow[\overline{\mathfrak{a}}_\gamma]{} & \mathbf{HA}^T_{X,Z}(\gamma)
\end{array}$$

which yields a third map

$$\ker(\tau_\gamma)\longrightarrow \ker(\mathfrak{a}_\gamma)\ . \tag{1.4}$$

[2]i.e., the structure sheaf $\mathcal{O}_Z$ is a pure one-dimensional sheaf as coherent sheaf of $\mathcal{O}_X$-modules.

Finally, by taking the direct sum with respect to $\gamma \in \langle Z \rangle$, we introduce also the following *amalgamation map*

$$\bar{\mathfrak{a}}\colon \mathbf{HA}^T_{X,Z_s} \widehat{\otimes}_{\mathbf{HA}^T_{X,Z_s^{s-1}}} \cdots \widehat{\otimes}_{\mathbf{HA}^T_{X,Z_2^1}} \mathbf{HA}^T_{X,Z_1} \longrightarrow \mathbf{HA}^T_{X,Z} . \tag{1.5}$$

The main conjecture we formulate is:

Conjecture 1.1.

(1) *For any $\gamma \in \langle Z \rangle$, the map $\bar{\mathfrak{a}}_\gamma$ is topologically surjective, i.e., the image is dense with respect to the quasi-compact topology*[3] *on the target.*

(2) *For any $\gamma \in \langle Z \rangle$, the map (1.4) is also topologically surjective.*

Claim (1) is proved in Theorem 4.21, subject to a list of assumptions.

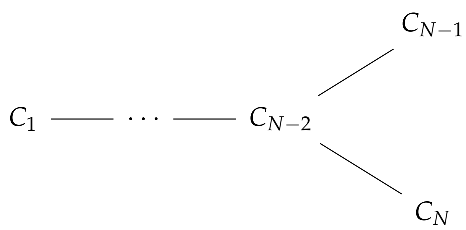
Remark 1.2. One can formulate a motivic variant of Conjecture 1.1 and the motivic variant of Claim (1) is proved in Theorem 4.22, also subject to a list of assumptions. △

We expect that the conjecture holds with the assumption now being that $Z_i \smallsetminus (Z_i \cap Z_{\leq i-1})$ is cohomologically pure for all i. These assumptions are in particular verified for Kleinian resolutions of singularities as well as for elliptic surfaces with a singular fiber of affine type DE. Note that this assumption also holds in the case of a tree of projective lines, or more generally in the case of a smooth projective curve Z_1 with finitely trees of $\mathbb{P}^1$ branching out of Z_1.

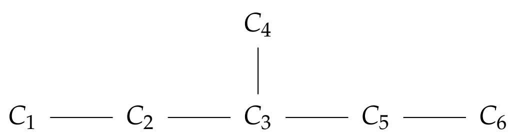
1.2. **First example: ADE quivers.** Let $\pi\colon X \to X_{\mathsf{con}}$ be a minimal resolution of an ADE singularity and let T be an algebraic torus acting on X for which $\pi^{-1}(0)$ is T-invariant. Let $Z := \pi^{-1}(0)_{\mathsf{red}}$. To define a stratification of Z of the form (1.3), we need to recall the dual intersection graphs associated to such fibers on a case by case basis. We shall denote by C_i the irreducible components of Z, with $1 \leq i \leq N$.

- Type A_N, with $N \geq 1$:

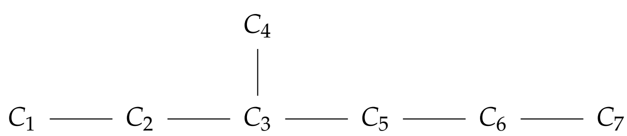
- Type D_N, with $N \geq 4$:



Type E_6:



Type E_7:



[3] in the sense of [DPS+25b, Remark 3.1].

Type E_8:

$$\begin{array}{ccccccccc} & & & & C_4 & & & & \\ & & & & | & & & & \\ C_1 & \text{———} & C_2 & \text{———} & C_3 & \text{———} & \cdots & \text{———} & C_8 \end{array}$$

For the A series we will assume $N \geq 2$ below, since in the case $N = 1$ amalgamation is trivial.

Set $Z^N := Z$, and let Z^{i-1} be the scheme-theoretic closure of the complement $Z^i \smallsetminus C_i$ for any $1 \leq i \leq N$. Note that $Z^0 = \emptyset$. We get the following stratification of Z:

$$\emptyset = Z^0 \subset Z^1 \subset \cdots \subset Z^N := Z\,. \tag{1.6}$$

Theorem (Theorem 7.16)**.** *Conjecture 1.1–(1) holds for X and Z, with the stratification (1.6).*

Now consider the non-equivariant case, i.e., $T = \{1\}$. Let $\mathcal{Q}$ be an affine quiver and let Kr be the Kronecker quiver. Following [DPS+25a, §III.7.8], introduce the Lie algebras

$${}_{\mathcal{Q}}\mathfrak{n}^{+}_{\mathsf{ell}} := \mathfrak{n}_{\mathsf{f}}[s^{\pm 1}, t] \oplus s^{-1}\mathfrak{h}_{\mathsf{f}}[s^{-1}, t] \oplus \bigoplus_{n>0,\, k<0} \mathbb{Q}c_{k,n}\,,$$

where $\mathfrak{n}_{\mathsf{f}}$ and $\mathfrak{h}_{\mathsf{f}}$ are, respectively, the positive nilpotent part and the Cartan part of the simple Lie algebra associated to X_{con}, and

$${}_{\mathsf{Kr}}\mathfrak{n}^{+}_{\mathsf{ell}} := \mathsf{Span}_{\mathbb{Q}}\{e \otimes s^{\pm \ell}, e \otimes t^{\ell}, h \otimes s^{-\ell-1}, h \otimes s^{-1}t^{\ell}, c_{k,n} \,|\, n > 0,\, k < 0,\, \ell \in \mathbb{N}\}\,,$$

where, e, f, h are the standard generators of $\mathfrak{sl}(2)$. Moreover, we denote by $\widehat{\mathsf{U}}(-)$ the completion of the enveloping algebra introduced in [DPS+25a, Lemma III.7.25].

[DPS+25a, Theorem III.7.26] yield

$$\mathbf{HA}_{X,Z} \simeq \widehat{\mathsf{U}}({}_{\mathcal{Q}}\mathfrak{n}^{+}_{\mathsf{ell}}) \quad \text{and} \quad \mathbf{HA}_{X,Z_i} \simeq \mathbf{HA}_{T^*\mathbb{P}^1,\mathbb{P}^1} \simeq \widehat{\mathsf{U}}({}_{\mathsf{Kr}}\mathfrak{n}^{+}_{\mathsf{ell}})\,,$$

for any $i = 1, \ldots, N+1$. Moreover, for $i = 2, \ldots, N+1$, [DPS+25a, Corollary III.8.3] yields

$$\mathbf{HA}_{X,Z_i^{i-1}} \simeq (\mathbf{HA}_{T^*\mathbb{P}^1,\mathbb{P}^1})_{\mathbb{N}\delta} \simeq \mathsf{U}(h\,s^{-1}[s^{-1}, t] \oplus K_-)\,,$$

where

$$K_- := \bigoplus_{n>0,\, k<0} \mathbb{Q}c_{k,n}\,.$$

The non-equivariant version of Theorem 7.16 yields the following.

Corollary. *The amalgamation map (1.5) in the non-equivariant case reduces to a topologically surjective map*

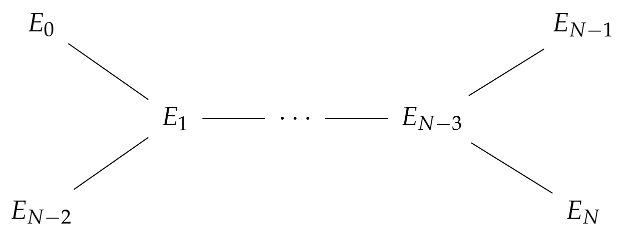
$$\overline{\mathfrak{a}}\colon \widehat{\mathsf{U}}({}_{\mathsf{Kr}}\mathfrak{n}^{+}_{\mathsf{ell}}) \widehat{\otimes}_{\mathsf{U}(h\,s^{-1}[s^{-1},t]\oplus K_-)} \cdots \widehat{\otimes}_{\mathsf{U}(h\,s^{-1}[s^{-1},t]\oplus K_-)} \widehat{\mathsf{U}}({}_{\mathsf{Kr}}\mathfrak{n}^{+}_{\mathsf{ell}}) \longrightarrow \widehat{\mathsf{U}}({}_{\mathcal{Q}}\mathfrak{n}^{+}_{\mathsf{ell}})\,,$$

where on the left-hand-side one has the completed tensor product of N copies of $\widehat{\mathsf{U}}({}_{\mathsf{Kr}}\mathfrak{n}^{+}_{\mathsf{ell}})$ over $\mathsf{U}(h\,s^{-1}[s^{-1}, t] \oplus K_-)$.

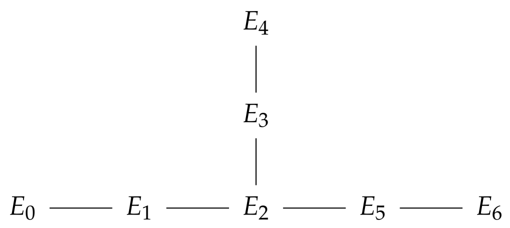
1.3. **Second example: affine DE quivers.** Let $\pi\colon X \to C$ be a smooth projective elliptic surface with a section $\sigma\colon C \to X$ over a smooth projective curve C. We assume that X is relatively minimal over C in the sense of Definition 8.1. Moreover, let us assume that one of the singular fibers of π, denoted by $E = \pi^{-1}(o)$, is of affine type DE. In addition, π will have other singular fibers according to the Kodaira classification. If the torus T is non-trivial, we will also assume that X admits a torus action $T \times X \to X$ which preserves the singular fiber E. An example of elliptic surface which admit nontrivial torus actions are provided in Remark 8.4.

Set $Z := E_{\mathsf{red}}$. To introduce a stratification of Z of the form of (1.3), we need to recall the dual intersection graphs associated to such fibers on a case by case basis. We shall denote by E_i the irreducible components of Z. Recall that $E_i \simeq \mathbb{P}^1$. Then, one has:

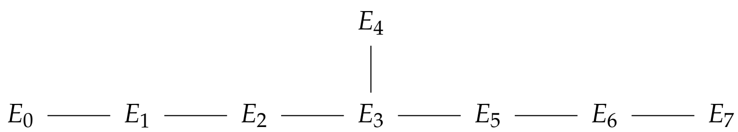
- Type affine D:



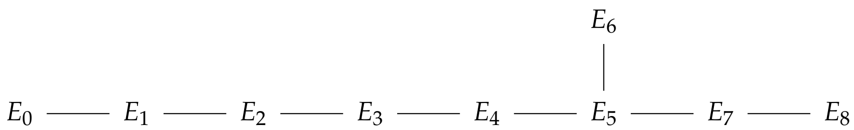
- Type affine E_6:



- Type affine E_7:



- Type affine E_8:



Set $Z^{N+1} := Z$. For any $1 \leq i \leq N+1$, let Z^{i-1} be the scheme-theoretic closure of the complement $Z^i \smallsetminus E_{i-1}$. We get the following stratification of Z:

$$\emptyset = Z^0 \subset Z^1 \subset \cdots \subset Z^{N+1} := Z\ . \tag{1.7}$$

We obtain the following.

Theorem (Theorem 8.52)**.** *Conjecture 1.1–(1) holds for X and Z, with the stratification (1.7).*

In particular, the amalgamation map (1.5) in the non-equivariant case reduces to a topologically surjective map

$$\overline{\mathfrak{a}}\colon \widehat{\mathsf{U}}({}_{\mathsf{Kr}}\mathfrak{n}^+_{\mathsf{ell}})\widehat{\otimes}_{\mathsf{U}(h\, s^{-1}[s^{-1},t]\oplus K_-)} \cdots \widehat{\otimes}_{\mathsf{U}(h\, s^{-1}[s^{-1},t]\oplus K_-)}\widehat{\mathsf{U}}({}_{\mathsf{Kr}}\mathfrak{n}^+_{\mathsf{ell}}) \longrightarrow \mathbf{HA}_{X,Z}\ ,$$

where on the left-hand-side one has the completed tensor product of $N+1$ copies of $\widehat{\mathsf{U}}({}_{\mathsf{Kr}}\mathfrak{n}^+_{\mathsf{ell}})$ over $\mathsf{U}(h\, s^{-1}[s^{-1},t] \oplus K_-)$.

Notation. We shall denote the classical k-thickening of Z inside X by $Z^{(k)}_{\mathsf{cl}}$. This is the classical truncation $\mathsf{t}_0(Z^{(k)})$ of the derived k-thickening of Z inside X.

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2. SUPPORT FILTRATION

In this section, we shall introduce the support filtration for a coherent sheaf and its relative version for a family.

2.1. **Definition of the support filtration.** Let X be a smooth quasi-projective complex surface. Let Z be a closed subscheme of X, which is connected, purely[4] one-dimensional, proper, and reduced. Let

$$Z^\bullet \colon \quad \emptyset =: Z^{-1} \subset Z^0 \subset Z^1 \subset \cdots \subset Z^s := Z\,, \tag{2.1}$$

be a stratification by closed subschemes for $s \geq 0$, where for $1 \leq i \leq s$, the subscheme Z^i is reduced and purely one-dimensional, while Z^0 is zero-dimensional. For each $0 \leq i \leq s$, let Z_i° denote the locally closed stratum $Z_i^\circ := Z^i \smallsetminus Z^{i-1}$ and let Z_i denote its scheme-theoretic closure in Z, i.e., it is the smallest closed subscheme of Z containing Z_i° as an open subscheme. Let also Z_i^{i-1} be the scheme theoretic intersection of Z^{i-1} and Z_i in X for $0 \leq i \leq s$. In particular, $Z_0^\circ = Z^0 = Z_0$ and $Z_0^{-1} = \emptyset$.

We make the following assumptions:

Assumption 0.1 (Assumptions on Z)**.**

(1) The connected components $Z_{i,\alpha}$ of Z_i are smooth subschemes of X, for $1 \leq i \leq s$ and $1 \leq \alpha \leq c_i$.

(2) Z^0 is contained in the smooth locus of Z.

(3) The equivalence classes in $\mathsf{NS}(X)$ associated to the connected components $Z_{i,\alpha}$ are linearly independent over $\mathbb{Q}$ for $1 \leq i \leq s$ and $1 \leq \alpha \leq c_i$.

⊘

Remark 2.1. Under the stated assumptions, the subschemes Z_i are smooth purely one-dimensional for $1 \leq i \leq s$. Hence they are effective Cartier divisors on X for all $1 \leq i \leq s$. Moreover, since Z^0 is contained in the smooth locus, it is an effective Cartier divisor on Z. .

Moreover, the scheme-theoretic intersection $Z_i^{i-1} := Z_i \cap Z^{i-1}$ in X is a zero-dimensional subscheme of X for $1 \leq i \leq s$. Furthermore, under the above assumptions, the set of irreducible

[4]i.e., the structure sheaf $\mathcal{O}_Z$ is a pure one-dimensional sheaf as coherent sheaf of $\mathcal{O}_X$-modules.

components of Z is $\{Z_{i,\alpha}\}$, for $1 \leq i \leq s$ and $1 \leq \alpha \leq c_i$. In particular, for fixed $1 \leq i \leq s$, the irreducible components $Z_{i,\alpha}$ are pairwise disjoint, for $1 \leq \alpha \leq c_i$, and $Z^0 = Z^0_1$ is disjoint from Z^1_2. △

Let $\langle Z_i \rangle \subset \mathsf{NS}(X)$ be the subgroup generated by the classes $[Z_{i,1}], \ldots, [Z_{i,c_i}]$ for $1 \leq i \leq s$, and, similarly, let $\langle Z \rangle \subset \mathsf{NS}(X)$ be the subgroup generated by the classes of irreducible components $[Z_{i,\alpha}]$, with $1 \leq i \leq s$ and $1 \leq \alpha \leq c_i$.

Remark 2.2. Let $\mathcal{E}$ be a pure one-dimensional coherent sheaf on X. Then, a representative of his first Chern class is given by the *Fitting support* of $\mathcal{E}$ (see e.g. [BBHR09, Proposition C.11]). Recall that the Fitting support contains the scheme-theoretic support of $\mathcal{E}$ and their underlying reduced structures coincide (see e.g. [Sta25, Lemma 69.5.3, Tag 0CZ3]). Thanks to Assumption 0.1–(3), $\mathsf{ch}_1(\mathcal{E}) \in \langle Z_i \rangle$ if and only if $\mathcal{E}$ is set-theoretically supported on Z_i for $1 \leq i \leq s$. △

Definition 2.3 (Support filtration)**.** Let $\mathcal{E}$ be a coherent sheaf on X set-theoretically supported on Z. The (support) $Z^\bullet$*-filtration* of $\mathcal{E}$ induced by the stratification (2.1) is the $(s+1)$-step filtration

$$\mathcal{E}^\bullet : \quad 0 =: \mathcal{E}^{-1} \subset \mathcal{E}^0 \subset \mathcal{E}^1 \subset \cdots \subset \mathcal{E}^s := \mathcal{E} \,,$$

where $\mathcal{E}^{i-1}$ is the maximal subsheaf of $\mathcal{E}^i$ with set-theoretic support contained in Z^{i-1}, for $1 \leq i \leq s$. ⊘

Definition 2.4. Let $Y \subset Z \subset X$ be closed subschemes of X, with Z pure one-dimensional and Y zero-dimensional. Let $\mathcal{F}$ be a coherent sheaf on X with set-theoretic support contained in Z. We say that $\mathcal{F}$ is *pure at* Y if it does not contain any nonzero subsheaf with set-theoretic support contained in Y_{red}. ⊘

Remark 2.5. Let $\mathcal{F}$ be a coherent sheaf on X, set-theoretically supported on Z, which is pure at Y. Then, $\mathcal{F}$ is either a one-dimensional sheaf, which does not contain any nonzero subsheaf with set-theoretic support contained in Y_{red} or a zero-dimensional sheaf supported on $Z \smallsetminus Y$.

Furthermore, the condition formulated in Definition 2.4 is equivalent to state that the set-theoretic intersection $\mathsf{Ass}(\mathcal{F}) \cap Y_{\mathsf{red}}$ is empty. Here, $\mathsf{Ass}(\mathcal{F})$ denotes the set of *associated points* of $\mathcal{F}$, whose definition can be found e.g. in [Sta25, Definition 31.2.1, Tag 02OI]. △

Proposition 2.6. *Let $\mathcal{E}$ be a coherent sheaf on X set-theoretically supported on Z and let $\mathcal{E}^\bullet$ be the $Z^\bullet$-filtration of $\mathcal{E}$. Then, the set-theoretic support of each subquotient $\mathcal{E}_i := \mathcal{E}^i/\mathcal{E}^{i-1}$ is contained in Z_i for $0 \leq i \leq s$. Moreover, $\mathcal{E}_i$ is pure at $Z_i^{i-1} = Z_i \cap Z^{i-1}$ for $1 \leq i \leq s$.*

Proof. Let $\mathcal{T}_i \subset \mathcal{E}_i$ be the maximal subsheaf of $\mathcal{E}_i$ with set-theoretic support contained in Z^{i-1}. Assume that $\mathcal{T}_i \neq 0$ for some $1 \leq i \leq s$. Set $\mathcal{E}_i' := \mathcal{E}_i/\mathcal{T}_i$. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}^{i-1} & \longrightarrow & \mathcal{E}^i & \longrightarrow & \mathcal{E}_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & \mathcal{E}_i' & \xrightarrow{\mathsf{id}} & \mathcal{E}_i' & \longrightarrow & 0 \end{array} ,$$

where all vertical arrows are surjective. Then the snake Lemma yields an exact sequence

$$0 \longrightarrow \mathcal{E}^{i-1} \longrightarrow \mathcal{K}^i \longrightarrow \mathcal{T}_i \longrightarrow 0 \,,$$

where $\mathcal{K}_i$ is the kernel of the middle vertical arrow. By assumption $\mathcal{E}^{i-1}$ and $\mathcal{T}_i$ are set-theoretically supported on Z^{i-1}, hence so is $\mathcal{K}^i$. However $\mathcal{E}^{i-1} \subset \mathcal{E}^i$ is maximal with this property, thus $\mathcal{E}^{i-1} = \mathcal{K}^i$ and $\mathcal{T}_i$ has to be the zero sheaf: this leads us to a contradiction. Thus, all the assertions follow. □

We have some immediate corollaries:

Corollary 2.7. *Under the same assumptions as in Proposition 2.6, one has*

$$\mathsf{Hom}_X(\mathcal{E}^{i-1}, \mathcal{E}_i) = 0$$

for all $1 \leq i \leq s$*. If* X *is locally K-trivial*[5]*, one also has*

$$\mathsf{Ext}^2_X(\mathcal{E}_i, \mathcal{E}^{i-1}) = 0 \,,$$

and if in addition $\mathcal{E}_i$ *is zero-dimensional, then*

$$\mathsf{Ext}^k_X(\mathcal{E}^{i-1}, \mathcal{E}_i) = 0$$

for $0 \leq k \leq 2$.

Let $\gamma \in \langle Z \rangle$. By Assumption 0.1–(3), there is a unique decomposition

$$\gamma = \sum_{i=1}^{s} \gamma_i$$

with $\gamma_i \in \langle Z_i \rangle$ for $1 \leq i \leq s$.

Corollary 2.8. *Under the same assumptions as in Proposition 2.6, if* $\mathsf{ch}_1(\mathcal{E}) = \gamma \in \langle Z \rangle$*, one has* $\mathsf{ch}_1(\mathcal{E}_i) = \gamma_i$*, with* $1 \leq i \leq s$.

By analogy with Harder-Narasimhan filtrations [HL10, Theorem 1.3.7], one has the following.

Lemma 2.9. *Let* $k \subset k'$ *be a field extension. Let* $(\mathcal{E}')^\bullet$ *be the* $(Z^\bullet)_{k'}$*-filtration of* $\mathcal{E} \otimes_k k'$*. Then* $(\mathcal{E}')^i = \mathcal{E}^i \otimes_k k'$.

Let us prove now a partial converse to Proposition 2.6.

Proposition 2.10. *Let* $\mathcal{E}$ *be a coherent sheaf on* X *set-theoretically supported on* Z*, let*

$$\mathcal{E}^\bullet\colon \quad 0 =: \mathcal{E}^{-1} \subset \mathcal{E}^0 \subset \cdots \subset \mathcal{E}^s := \mathcal{E}$$

be its $Z^\bullet$*-filtration, and let* $\mathcal{E}_i := \mathcal{E}^i / \mathcal{E}^{i-1}$ *be the associated subquotients. Let*

$$0 =: \mathcal{F}^{-1} \subset \mathcal{F}^0 \subset \cdots \subset \mathcal{F}^s := \mathcal{E}$$

be a filtration so that $\mathcal{F}^i$ *is set-theoretically supported on* Z^i *for all* $0 \leq i \leq s$ *and each subquotient* $\mathcal{F}_i := \mathcal{F}^i / \mathcal{F}^{i-1}$ *is set-theoretically supported on* Z_i *for* $0 \leq i \leq s$*. Then* $\mathcal{F}^i \subset \mathcal{E}^i$ *and the quotient* $\mathcal{Q}^i := \mathcal{E}^i / \mathcal{F}^i$ *is zero-dimensional for* $0 \leq i \leq s$*. Moreover, there is a canonical exact sequence*

$$0 \longrightarrow \mathcal{Q}^{s-1} \longrightarrow \mathcal{F}_s \longrightarrow \mathcal{E}_s \longrightarrow 0 \,.$$

In addition, suppose one of the following extra conditions holds:

(1) $\chi(\mathcal{F}^i) = \chi(\mathcal{E}^i)$ *for* $0 \leq i \leq s$*, or*

(2) each subquotient $\mathcal{F}_i := \mathcal{F}^i / \mathcal{F}^{i-1}$ *is pure at* Z_i^{i-1} *for* $0 \leq i \leq s$.

Then, $\mathcal{F}^i = \mathcal{E}^i$ *for all* $0 \leq i \leq s$.

Proof. First, the definition of the support filtration implies that $\mathcal{F}^i \subset \mathcal{E}^i$ for $0 \leq i \leq s$. Moreover, one has a decomposition

$$\mathsf{ch}_1(\mathcal{E}) = \sum_{i=0}^{s} \mathsf{ch}_1(\mathcal{F}_i) \,. \tag{2.2}$$

Since each $\mathcal{F}_i$ is set-theoretically supported on Z_i, one also has $\mathsf{ch}_1(\mathcal{F}_i) \in \langle Z_i \rangle$ for $1 \leq i \leq s$ and $\mathsf{ch}_1(\mathcal{F}_0) = 0$. By Assumption 0.1–(3) on Z, Corollary 2.8, and the uniqueness of the decomposition (2.2), this implies that

$$\mathsf{ch}_1(\mathcal{F}_i) = \mathsf{ch}_1(\mathcal{E}_i)$$

[5] i.e., there exists an open subscheme $U \subset X$ containing Z so that $\omega_X|_U \simeq \mathcal{O}_U$.

for $0 \leq i \leq s$. Hence, also

$$\mathsf{ch}_1(\mathcal{F}^i) = \mathsf{ch}_1(\mathcal{E}^i)$$

for $0 \leq i \leq s$. Thus, the quotient $\mathcal{Q}^i := \mathcal{E}^i/\mathcal{F}^i$ is zero-dimensional. Moreover, one has a canonical commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}^{i-1} & \longrightarrow & \mathcal{F}^i & \longrightarrow & \mathcal{F}_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E}^{i-1} & \longrightarrow & \mathcal{E}^i & \longrightarrow & \mathcal{E}_i & \longrightarrow & 0 \end{array} \tag{2.3}$$

for all $1 \leq i \leq s$, where the first two vertical arrows from the left-hand-side are the canonical inclusions. Then, the snake lemma yields an injective morphism

$$\mathcal{T}_i \longrightarrow \mathcal{Q}^{i-1} \tag{2.4}$$

for all $1 \leq i \leq s$, where $\mathcal{T}_i$ is the kernel of the right vertical arrow. For $i = s$ the central vertical arrow is the identity. Hence in this case, the snake lemma yields an exact sequence

$$0 \longrightarrow \mathcal{Q}^{s-1} \longrightarrow \mathcal{F}_s \longrightarrow \mathcal{E}_s \longrightarrow 0\,.$$

This proves the first part of the proposition.

Next suppose the additional Condition (1) holds. Clearly, this implies $\mathcal{Q}^i = 0$ for all $0 \leq i \leq s$, which proves the claim.

Finally suppose that the additional Condition (2) holds. Then, the claim will be proven by descending induction on $1 \leq i \leq s$. Suppose $\mathcal{F}^i = \mathcal{E}^i$. Then, the injective morphism (2.4) is an isomorphism and the snake lemma yields an exact sequence

$$0 \longrightarrow \mathcal{Q}^{i-1} \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{E}_i \longrightarrow 0\,.$$

Since $\mathcal{F}_i$ is pure at Z_i^{i-1} by assumption, we get that $\mathcal{Q}^{i-1} = 0$. Hence by applying the snake lemma to the commutative diagram (2.3), we get $\mathcal{E}^{i-1}/\mathcal{F}^{i-1} = 0$. □

2.2. **Relative support filtrations for families.** By using the notion of support filtration for families, in this section we shall construct a stratification $\{\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)^{\mathsf{m}}\}$ of the moduli stack $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)$ indexed by $\mathsf{m} = (m_0,\ldots,m_s) \in \mathbb{Z}^{s+1}$ which has similar properties to the Harder-Narasimhan filtration.

We shall use the notion of families of properly supported nilpotent coherent sheaves introduced in [DPS+25b, Definitions 4.27[6] and 4.28]. First recall the following:

Definition 2.11. Let T be a scheme locally of finite type over $\mathbb{C}$. Let $\mathcal{E}$ be a flat family of sheaves on X parametrized by T. We say that $\mathcal{E}$ *has invariants* (γ, n), with $\gamma \in \langle Z\rangle$ and $n \in \mathbb{Z}$, if $\mathsf{ch}_1(\mathcal{E}_t) = \gamma$ and $\chi(\mathcal{E}_t) = n$ for any $t \in T$. ⊘

Let us introduce the notion of relative support filtration.

Definition 2.12. Let T be a scheme locally of finite type over $\mathbb{C}$. Let $\mathcal{E}$ be a flat family of sheaves on X, set-theoretically supported on Z, parametrized by T. A *relative $Z^\bullet$-filtration for $\mathcal{E}$* is a filtration

$$\mathcal{E}^\bullet\colon \quad 0 = \mathcal{E}^{-1} \subset \mathcal{E}^0 \subset \cdots \subset \mathcal{E}^s = \mathcal{E}$$

in the abelian category of coherent $\mathcal{O}_{X\times T}$-modules so that

(1) each subquotient $\mathcal{E}^i/\mathcal{E}^{i-1}$ is T-flat for $0 \leq i \leq s$, and

(2) the induced filtration $0 = \mathcal{E}_t^{-1} \subset \mathcal{E}_t^0 \subset \cdots \subset \mathcal{E}_t^s$ is the support $Z_t^\bullet$-filtration of $\mathcal{E}_t$ for any $t \in T$.

⊘

[6] Recall that by [DPS+25b, Theorem 4.53], being nilpotent is equivalent to being set-theoretically supported on Z.

Remark 2.13. Note that Condition (1) in Definition 2.12 implies that $\mathcal{E}^i$ is T-flat for $0 \leq i \leq s$. Thus, Condition (2) is well-posed. △

Remark 2.14. Recall that any $\gamma \in \langle Z \rangle$ has a unique decomposition $\gamma = \sum_{i=1}^{s} \gamma_i$ with $\gamma_i \in \langle Z_i \rangle$. Let T be a scheme locally of finite type over $\mathbb{C}$. Let $\mathcal{E}$ be a flat family of sheaves on X, set-theoretically supported on Z, parametrized by T with invariants (γ, n). Suppose that $\mathcal{E}$ admits a relative $Z^\bullet$-filtration $\mathcal{E}^\bullet$ and let $\mathcal{E}_i$ denote the associated subquotients for $0 \leq i \leq s$. Then, Corollary 2.8 implies that $\mathsf{ch}_1(\mathcal{E}_{i,t}) = \gamma_i$ for $1 \leq i \leq s$ and for all $t \in T$. △

For any $1 \leq i \leq s$, set

$$\gamma^i \coloneqq \sum_{j=1}^{i} \gamma_j \ .$$

Set also $\gamma^0 = \gamma_0 \coloneqq 0$.

Definition 2.15. Let T be a scheme locally of finite type over $\mathbb{C}$. Let $\mathcal{E}$ be a flat family of sheaves on X, set-theoretically supported on Z, parametrized by T, and suppose that $\mathcal{E}$ admits a relative $Z_\bullet$-filtration

$$\mathcal{E}^\bullet\colon \quad 0 =\colon \mathcal{E}^{-1} \subset \mathcal{E}^0 \subset \cdots \subset \mathcal{E}^s \coloneqq \mathcal{E} \ .$$

We say that $\mathcal{E}^\bullet$ is *of type* $(\mathsf{n}, \mathsf{k}) \in \mathbb{Z}^{s+1} \times (\mathbb{Z}_{>0})^{s+1}$, with

$$\mathsf{n} \coloneqq (n^0, \dots, n^s) \quad \text{and} \quad \mathsf{k} \coloneqq (k_0, \dots, k_s) \ ,$$

if $\mathcal{E}^i$ corresponds to a map $T \to \mathfrak{Coh}((Z^i)^{(k_i)}_{\mathsf{cl}}; \gamma^i, n^i)$ for $0 \leq i \leq s$. ⊘

Fix invariants (γ, n). The next goal is to construct a stratification of the moduli stack $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n)$ so that the universal family admits a relative $Z^\bullet$-filtration on each stratum. It will be first shown that the function

$$t \longmapsto (\chi(\mathcal{E}^0_t), \dots, \chi(\mathcal{E}^s_t))$$

is constructible, by analogy to the similar property of Harder-Narasimhan filtrations proven in [Sha77].

For future reference, note the following consequence of the proof of [DPS⁺25b, Proposition 4.45].

Corollary 2.16. *Let $Y \subset X$ be a closed proper subscheme. Let T be a scheme locally of finite type over $\mathbb{C}$ and let $\mathcal{F}$ be a T-flat family of coherent sheaves on X. Let t, t_0 be points in T so that t_0 is a specialization of t, i.e., $t_0 \in \overline{\{t\}}$. Suppose $\mathcal{F}_t$ is scheme-theoretically supported on $Y \times \{t\}$. Then $\mathcal{F}_{t_0}$ is scheme-theoretically supported on $Y \times \{t_0\}$.*

Lemma 2.17. *Let T be a scheme locally of finite type over $\mathbb{C}$. Let $\mathcal{E}$ be a flat family of coherent sheaves on X, set-theoretically supported on Z, parametrized by T, with invariants (γ, n). Let t, t_0 be points in T so that t_0 is a specialization of t. Let $\mathcal{E}^\bullet_t$ and $\mathcal{E}^\bullet_{t_0}$ be the $Z^\bullet$-filtrations of $\mathcal{E}_t$ and $\mathcal{E}_{t_0}$ respectively. Assume that $\mathcal{E}^i_t$ is scheme-theoretically supported on $(Z^i_t)^{(k_i)}_{\mathsf{cl}}$ for $1 \leq i \leq s$ and for some $k_i \geq 1$. Then, the following hold for $0 \leq i \leq s$:*

(1) $\chi(\mathcal{E}^i_t) \leq \chi(\mathcal{E}^i_{t_0})$.

(2) $\mathcal{E}^i_{t_0}$ *is scheme-theoretically supported on* $(Z^i_{t_0})^{(\ell_i)}_{\mathsf{cl}}$*, for* $0 \leq i \leq s$ *and for some* $\ell_i \geq k_i$*. Moreover, if* $\chi(\mathcal{E}^i_{t_0}) = \chi(\mathcal{E}^i_t)$*, then* $\ell_i = k_i$.

Proof. Let us start by proving (1). We shall use arguments similar to those in the proofs of [Sha77, Proposition 9 and Theorem 3]. First, it suffices to prove the claim for $T = \mathsf{Spec}(R)$, where R is a DVR over $\mathbb{C}$. Let t and t_0 denote the generic and the closed point, respectively.

Set $\mathcal{Q}_t^i := \mathcal{E}_t/\mathcal{E}_t^i$. By [Gro95, Lemma 3.7] there exists a unique quotient $\mathcal{E} \twoheadrightarrow \mathcal{G}^i$, flat over T, so that $\mathcal{G}_t^i = \mathcal{Q}_t^i$. Let $\mathcal{F}^i := \ker(\mathcal{E} \twoheadrightarrow \mathcal{G}^i)$, which is also flat over T. By construction, $\mathcal{F}_t^i = \mathcal{E}_t^i$ is scheme-theoretically supported on $(Z_t^i)_{\mathsf{cl}}^{(k_i)}$ and

$$\mathsf{ch}_1(\mathcal{F}_t^i) = \gamma^i \ .$$

By Corollary 2.16, $\mathcal{F}_{t_0}^i$ is scheme-theoretically on $(Z_{t_0}^i)_{\mathsf{cl}}^{(k_i)}$. In particular, $\mathcal{F}_{t_0}^i \subset \mathcal{E}_{t_0}^i$. Since $\mathcal{F}^i$ is T-flat, one also has

$$\mathsf{ch}_1(\mathcal{F}_{t_0}^i) = \gamma^i \ .$$

Furthermore, by Corollary 2.8, we have

$$\mathsf{ch}_1(\mathcal{E}_{t_0}^i) = \gamma^i$$

as well. Hence, $\mathcal{Q} := \mathcal{E}_{t_0}^i/\mathcal{F}_{t_0}^i$ is zero-dimensional, which implies

$$\chi(\mathcal{F}_{t_0}^i) \leq \chi(\mathcal{E}_{t_0}^i) \ .$$

Since $\mathcal{F}^i$ is flat over T, one has

$$\chi(\mathcal{F}_{t_0}^i) = \chi(\mathcal{F}_t^i) = \chi(\mathcal{E}_t^i) \ .$$

In conclusion, $\chi(\mathcal{E}_t^i) \leq \chi(\mathcal{E}_{t_0}^i)$. Moreover, $\mathcal{E}_{t_0}^i$ is scheme-theoretically supported on $(Z_{t_0}^i)_{\mathsf{cl}}^{(\ell_i)}$, for $0 \leq i \leq s$ and for some $\ell_i \geq k_i$.

Statement (2) follows from the previous arguments by using the short exact sequence

$$0 \longrightarrow \mathcal{F}_{t_0}^i \longrightarrow \mathcal{E}_{t_0}^i \longrightarrow \mathcal{Q} \longrightarrow 0 \ .$$

If $\chi(\mathcal{E}_{t_0}^i) = \chi(\mathcal{E}_t^i)$, one gets $\mathcal{Q} = 0$. Therefore, $\ell_i = k_i$. □

For the next step, note the following consequence of [DPSV23, Corollary C.5].

Lemma 2.18. *Let T be a scheme of locally finite type and let $\mathcal{Q}$ be a fixed zero-dimensional sheaf on X. Let $\mathcal{F}$ be a flat family of sheaves on X, set-theoretically supported on Z, parametrized by T. Then, the set of points $t \in T$ so that $\mathsf{Hom}_{X_t}((p_T^\ast\mathcal{Q})_t, \mathcal{F}_t) = 0$ is open in T, where $p_T\colon X \times T \to X$ denotes the projection.*

By analogy to [Sha77, Lemma 5], one next proves the following.

Lemma 2.19. *Let T be a reduced irreducible scheme of finite type and let $\xi \in T$ denote the generic point. Let $\mathcal{E}$ be a flat family of coherent sheaves on X, set-theoretically supported on Z, parametrized by T. Let $\mathcal{E}_\xi^\bullet$ be the $Z^\bullet$-filtration of $\mathcal{E}_\xi$. Then, there exists an open subscheme $U \subset T$ so that $\mathcal{E}|_U$ admits a relative $Z^\bullet$-filtration $\mathcal{E}_U^\bullet$ whose restriction to X_ξ coincides with $\mathcal{E}_\xi^\bullet$.*

Proof. Let $f\colon \mathsf{Fl} \to T$ be the relative flag scheme over T whose functor of points associates to any morphism $\phi\colon W \to T$ a flag of $\mathcal{O}_{X\times W}$-modules

$$0 =: \mathcal{F}^{-1} \subset \mathcal{F}^0 \subset \cdots \subset \mathcal{F}^s := \phi^\ast\mathcal{E}$$

so that all subquotients are W-flat and have the same numerical invariants as the subquotients of $\mathcal{E}_\xi^\bullet$, respectively[7]. For ease of exposition, for any open subscheme $V \subset T$, set $\mathsf{Fl}_V := \mathsf{Fl} \times_T V$ and $f_V\colon \mathsf{Fl}_V \to V$ be the canonical morphism induced by f.

Note that the flag $\mathcal{E}_\xi^\bullet$ determines a section of f over the generic point. We will first prove that this section extends to a section of f over an open subscheme $V \subset T$. Under the current assumptions, T is integral, hence the local ring $\mathcal{O}_{T,\xi}$ is canonically isomorphic to the fraction field $K(T)$, which also coincides with the residual field $\kappa(\xi)$. Then, [Gro60, Proposition 6.5.1-(ii)] proves that the given section $\mathsf{Spec}(\kappa(\xi)) \to \mathsf{Fl}_\xi$ extends to a morphism $s\colon W \to \mathsf{Fl}_W$ over some open subscheme $W \subset T$. Let $g\colon W \to W$ be the composition $g := f_W \circ s$. By construction, the localization of g at the generic point $\xi \in W$ coincides with the identity morphism $\mathsf{Spec}(\kappa(\xi)) \to$

[7]The construction of the relative flag scheme can be found e.g. in [HL10, §2.A.1].

$\mathsf{Spec}(\kappa(\xi))$. Then, [Gro60, Proposition 6.5.1-(i)] proves that there exists an open neighborhood $V \subset W$ so that $g|_V = f|_V \circ s|_V$ coincides with the identity morphism id_V. In conclusion, $s_V := s|_V$ is a section of f_V.

Let $\mathcal{F}^\bullet$ denote the flag of $\mathcal{O}_{X\times V}$-modules associated to the section s_V. By construction, the subsheaves $\mathcal{F}^i$ are T-flat. Since each $\mathcal{F}^i_\xi = \mathcal{E}^i_\xi$ is set-theoretically supported on Z^i_ξ, Corollary 2.16 implies that $\mathcal{F}^i_v$ is set theoretically supported on Z^i_v for $v \in V$. Moreover, by Proposition 2.6, the i-th subquotient of $\mathcal{E}^\bullet_\xi$ is set-theoretically supported on $Z_i \times \{\xi\}$, and it is furthermore pure on $Z_i^{i-1} \times \xi$ for $1 \leq i \leq s$. Then, Corollary 2.16 implies that the i-th subquotient $\mathcal{F}_{i,v}$ is set-theoretically supported on $Z_i \times \{v\}$ for $1 \leq i \leq s$ and $v \in V$. For any $p \in Z_i^{i-1}$, the set of points $t \in V$ such that $\mathsf{Hom}_X((p_V^*\mathcal{O}_p)_t, \mathcal{F}_{i,t}) = 0$ is open by Lemma 2.18, where $p_V\colon X \times V \to X$ is the canonical projection. Since Z_i^{i-1} is zero-dimensional by Assumption (2), it follows that there exists an open subscheme $U \subset V$ so that for any point $u \in U$ the sheaf $\mathcal{F}_{i,u}$ is pure at $Z_i^{i-1} \times \{u\}$, for $1 \leq i \leq s$.

Let $\mathcal{E}^\bullet_u$ be the canonical $Z^\bullet$-filtration of $\mathcal{E}_u$ for any point $u \in U$, and let $\mathcal{E}_{u,i}$ denote the associated subquotients for $1 \leq i \leq s$. As shown above, the $\mathcal{O}_{X_u}$-modules $\mathcal{F}^i_u$ and $\mathcal{F}_{i,u}$ satisfy the same support conditions as $\mathcal{E}^i_u$ and $\mathcal{E}_{u,i}$, respectively, for all $0 \leq i \leq s$. Since the assumptions and Condition (2) of Proposition 2.10 hold, the filtrations $\mathcal{E}^\bullet_u$ and $\mathcal{F}^\bullet|_{X_u}$ coincide for any point $u \in U$. □

Now, let $\mathcal{E}$ be a flat family of coherent sheaves on X parametrized by a locally noetherian scheme T, set-theoretically supported on Z. For any $t \in T$, let $\mathcal{E}^\bullet_t$ denote the $Z^\bullet$-filtration of $\mathcal{E}_t$. For any $\mathsf{m} = (m_0, \ldots, m_s) \in \mathbb{Z}^{s+1}$, let $|T|^{\leq m_0, \ldots, \leq m_s} \subset |T|$ be the subset determined by the conditions

$$\chi(\mathcal{E}^i_t) \leq m_i \quad \text{for } 0 \leq i \leq s\ .$$

Let $|T|^{m_0,\ldots,m_s} \subset |T|^{\leq m_0, \ldots, \leq m_s}$ be the subset determined by the conditions

$$\chi(\mathcal{E}^i_t) = m_i \quad \text{for } 0 \leq i \leq s\ .$$

Let $\prec$ be the partial order relation on $\mathbb{Z}^{s+1}$ defined by

$$\mathsf{m} \prec \mathsf{m}' \Leftrightarrow m_i \leq m_i' \quad \text{for all } 1 \leq i \leq s \text{ and } \mathsf{m}' - \mathsf{m} \neq 0\ .$$

For ease or exposition, we will write

$$|T|^{\preceq \mathsf{m}} := |T|^{\leq m_0, \ldots, \leq m_s} \quad \text{and} \quad |T|^{\mathsf{m}} := |T|^{m_0,\ldots,m_s}\ .$$

Then Lemmas 2.17 and 2.19 yield:

Proposition 2.20. *Let T be a scheme locally of finite type over $\mathbb{C}$. Let $\mathcal{E}$ be a flat family of sheaves on X, set-theoretically supported on Z, parametrized by T, with invariants (γ, n). Let $\mathcal{E}^\bullet_t$ be the $Z^\bullet$-filtration of $\mathcal{E}_t$ for any point $t \in T$. Then, the function*

$$t \longmapsto (\chi(\mathcal{E}^0_t), \ldots, \chi(\mathcal{E}^s_t))$$

is constructible and upper semicontinuous on $|T|$.

Moreover, for any $\mathsf{m} \in \mathbb{Z}^{s+1}$, the subset $|T|^{\preceq \mathsf{m}} \subset |T|$ is open in $|T|$, while the subset $|T|^{\mathsf{m}} \subset |T|^{\preceq \mathsf{m}}$ is closed in $|T|^{\preceq \mathsf{m}}$.

Proof. First, upper continuity follows from Lemma 2.17. As explained in [AB13, Example A.4], in order to prove constructibility, it suffices to assume that T is reduced, irreducible, and of finite type. Then constructibility follows from Lemma 2.19. □

The next step is to construct a natural scheme structure on each stratum $|T|^{\mathsf{m}}$ by analogy to [Nit11, Theorem 5]. As opposed to *loc. cit.* in the present context, these strata will in fact carry natural ind-scheme structures. The proof will use the result below, which follows from the second part of the proof of [DPS+25b, Proposition 4.45].

Lemma 2.21. *Let $Y \subset X$ be a closed proper subscheme. Let T be a locally noetherian scheme and let $\mathcal{E}$ be a T-flat family of coherent sheaves on X. Then, there exists a unique closed subscheme $T_Y \subset T$ so that given an arbitrary morphism $\phi\colon W \to T$, the pullback $\phi^*\mathcal{E}$ is scheme-theoretically supported on $Y \times W$ if and only if ϕ factors through the closed immersion $T_Y \to T$.*

Remark 2.22. For ease of exposition, under the assumptions of Lemma 2.21, we will say that $T_Y \subset T$ is the unique closed subscheme of T so that the restriction $\mathcal{E}|_{X\times T_Y}$ is universally scheme-theoretically supported on Y. △

Now, let T be a connected locally noetherian scheme. For any $\mathsf{m} = (m_0, \dots, m_s) \in \mathbb{Z}^{s+1}$ and $\mathsf{k} = (k_0, \dots, k_s) \in (\mathbb{Z}_{>0})^{s+1}$ let

$$|T|^{\mathsf{m}}_{\mathsf{k}} \subset |T|^{\mathsf{m}}$$

be the set of points $t \in T$ where $\chi(\mathcal{E}^i_t) = m_i$ and $\mathcal{E}^i_t$ is scheme-theoretically supported on $(Z^i_t)^{(k_i)}_{\mathsf{cl}}$ for all $0 \le i \le s$. Note that $|T|^{\mathsf{m}}_{\mathsf{k}}$ is a closed subset of $|T|^{\preceq \mathsf{m}}$ by Proposition 2.20 and Corollary 2.16. Moreover, one clearly has

$$|T|^{\mathsf{m}} = \bigcup_{\mathsf{k}\in(\mathbb{Z}_{>0})^{s+1}} |T|^{\mathsf{m}}_{\mathsf{k}} .$$

Proposition 2.23. *Let T be a connected and locally noetherian scheme. Let $\mathcal{E}$ be a flat family of coherent sheaves on X, set-theoretically supported on Z, parametrized by T, with invariants (γ, n), where $n = m_s$. Then, there exists a unique closed subscheme $T^{\mathsf{m}}_{\mathsf{k}} \subset T^{\preceq \mathsf{m}}$ so that:*

(1) the underlying topological space of $T^{\mathsf{m}}_{\mathsf{k}}$ is $|T|^{\mathsf{m}}_{\mathsf{k}}$.

(2) A base change morphism $\phi\colon W \to T$ factors through the locally closed immersion $T^{\mathsf{m}}_{\mathsf{k}} \subset T$ if and only if $(\mathsf{id} \times \phi)^*\mathcal{E}$ admits a relative $Z^\bullet$-filtration of type (m, k).

(3) For any base change morphism $\phi\colon W \to T$, one has $W^{\mathsf{m}}_{\mathsf{k}} = \phi^{-1}(T^{\mathsf{m}}_{\mathsf{k}})$.

(4) If a relative $Z^\bullet$-filtration for $\mathcal{E}$, of type (m, k) exists, then it is unique.

(5) Given $\mathsf{k}' \in (\mathbb{Z}_{>0})^{s+1}$ with $k_i \le k'_i$ for $1 \le i \le s$, there is a canonical closed embedding $T^{\mathsf{m}}_{\mathsf{k}} \hookrightarrow T^{\mathsf{m}}_{\mathsf{k}'}$, which is naturally compatible with the base change properties (3) and (4).

Proof. First note that the subset $|T|^{\preceq \mathsf{m}} \subset |T|$ is open, hence it has a canonical open subscheme structure $T^{\preceq \mathsf{m}} \subset T$. Let $\mathcal{E}^{\preceq \mathsf{m}}$ denote the restriction of $\mathcal{E}$ to $T^{\preceq \mathsf{m}} \times X$.

The proof will proceed by descending induction on $0 \le i \le s$. By Lemma 2.21, there exists a unique closed subscheme $T^{m_s}_{k_s} \subset T^{\preceq \mathsf{m}}$ so that the restriction $\mathcal{E}^{\preceq m}|_{X\times T^{m_s}_{k_s}}$ is universally scheme-theoretically supported on $(Z_s)^{(k_s)}_{\mathsf{cl}}$. By construction, this subscheme satisfies properties (1)–(5).

Now, assume that $T^{m_i,\dots,m_s}_{k_i,\dots,k_s} \subset T^{\prec m}$ is a closed subscheme so that the restriction

$$\mathcal{E}^{m_i,\dots,m_s}_{k_i,\dots,k_s} := \mathcal{E}^{\prec \mathsf{m}}|_{X\times T^{m_i,\dots,m_s}_{k_i,\dots,k_s}}$$

satisfies the inductive hypothesis for the partial stratification

$$\varnothing \subset Z^i \subset \cdots \subset Z^s := Z . \tag{2.5}$$

In particular, there exists a unique relative filtration

$$0 \subset \mathcal{F}^i \subset \cdots \subset \mathcal{F}^s := \mathcal{E}^{m_i,\dots,m_s}_{k_i,\dots,k_s}$$

over $T^{m_i,\dots,m_s}_{k_i,\dots,k_s}$ with respect to the stratification (2.5). Moreover, each $\mathcal{F}^j$ is scheme-theoretically supported on $(Z^j)^{(k_j)}_{\mathsf{cl}} \times T^{m_i,\dots,m_s}_{k_i,\dots,k_s}$ for $i \le j \le s$.

Recall that γ has a unique decomposition

$$\gamma = \sum_{i=0}^{s} \gamma_i$$

with $\gamma_i \in \langle Z_i \rangle$ for $1 \le i \le s$ and $\gamma_0 = 0$. Let $n_i = m_i - m_{i-1}$ and let

$$\pi \colon \mathsf{Quot}(\mathcal{F}^i; \gamma_i, n_i) \longrightarrow T^{m_i,\ldots,m_s}_{k_i,\ldots,k_s}$$

denote the relative Quot scheme assigning to any morphism $\phi \colon W \to T^{m_i,\ldots,m_s}_{k_i,\ldots,k_s}$ the set of isomorphism classes of W-flat quotients $(\mathrm{id} \times \phi)^* \mathcal{F}^i \twoheadrightarrow \mathcal{G}$ on $X \times W$ with invariants (γ_i, n_i). Let

$$0 \longrightarrow \mathcal{K} \longrightarrow (\mathrm{id} \times \pi)^* \mathcal{F}^i \longrightarrow \mathcal{G} \longrightarrow 0$$

denote the universal quotient. Note that all terms in the above exact sequence are flat over the Quot scheme.

Applying Lemma 2.21, let $\mathsf{Q}_{k_{i-1}}$ be the unique closed subscheme of $\mathsf{Quot}(\mathcal{F}^i; \gamma_i, n_i)$ so that the restriction $\mathcal{K}|_{X \times \mathsf{Q}_{k_{i-1}}}$ is universally scheme-theoretically supported on $(Z^{i-1})^{(k_{i-1})}_{\mathsf{cl}}$. For any $t \in T^{m_i,\ldots,m_s}_{k_i,\ldots,k_s}$, let $\mathcal{F}^{i-1}_t \subset \mathcal{F}^i_t$ be the maximal subsheaf of $\mathcal{F}^i_t$ with set-theoretic support contained in Z^{i-1}_t. Then, let

$$|T|^{m_{i-1},\ldots,m_s}_{k_{i-1},\ldots,k_s} \subset |T^{m_i,\ldots,m_s}_{k_i,\ldots,k_s}|$$

be the closed subset determined by the conditions

- $\chi(\mathcal{F}^{i-1}_t) = m_{i-1}$, and
- $\mathcal{F}^{i-1}_t$ is scheme-theoretically supported on $(Z^{i-1}_t)^{(k_{i-1})}_{\mathsf{cl}}$.

We shall show that π induces an isomorphism $|\mathsf{Q}_{k_{i-1}}| \to |T|^{m_{i-1},\ldots,m_s}_{k_{i-1},\ldots,k_s}$. The first step is to show that $\pi(q) \in |T|^{m_{i-1},\ldots,m_s}_{k_{i-1},\ldots,k_s}$ for any $q \in \mathsf{Q}_{k_{i-1}}$. Let $t = \pi(q)$ and let

$$0 \longrightarrow \mathcal{K}_q \longrightarrow \mathcal{F}^i_q \longrightarrow \mathcal{G}_q \longrightarrow 0$$

denote the quotient associated to q. Note that $\mathcal{K}_q$ is scheme-theoretically supported on $(Z^{i-1}_q)^{(k_{i-1})}_{\mathsf{cl}}$, and it has invariants (γ^{i-1}, m_{i-1}). Let $\mathcal{F}^{i-1}_q$ be the maximal subsheaf of $\mathcal{F}^i_q$ with set-theoretic support contained in Z^{i-1}_q. Then $\mathcal{F}^{i-1}_q$ has invariants (γ^{i-1}, m'_{i-1}), where $m'_{i-1} \le m_{i-1}$ since $t \in |T|^{\preceq \mathsf{m}}$. Since $\mathcal{K}_q$ is set-theoretically supported on Z^{i-1}_q, it is a subsheaf of $\mathcal{F}^{i-1}_q$. Since they have identical first Chern class, the quotient is zero-dimensional, which implies

$$m_{i-1} = \chi(\mathcal{K}_q) \le m'_{i-1}$$

Therefore, $m'_{i-1} = m_{i-1}$, hence $\mathcal{K}_q = \mathcal{F}^{i-1}_q$. This proves that indeed $t = \pi(q) \in |T|^{m_{i-1},\ldots,m_s}_{k_{i-1},\ldots,k_s}$.

Conversely, let $t \in |T|^{m_{i-1},\ldots,m_s}_{k_{i-1},\ldots,k_s}$. Then the quotient $\mathcal{F}^i_t / \mathcal{F}^{i-1}_t$ determines a unique point $q \in \mathsf{Quot}(\mathcal{F}^i; \gamma_i, n_i)$ so that $\pi(q) = t$. Moreover, By Lemma 2.21, $q \in |\mathsf{Q}_{k_{i-1}}|$, since $\mathcal{F}^{i-1}_t$ is scheme-theoretically supported on $(Z^{i-1}_t)^{(k_{i-1})}_{\mathsf{cl}} \times \{t\}$. In conclusion, the projection π determines a set bijection

$$|\mathsf{Q}_{k_{i-1}}| \longrightarrow |T|^{m_{i-1},\ldots,m_s}_{k_{i-1},\ldots,k_s} \,. \tag{2.6}$$

The next goal is to show that the natural morphism $\mathsf{Q}_{k_{i-1}} \to T^{m_i,\ldots,m_s}_{k_i,\ldots,k_s}$ is a closed immersion. First note that the residue field extension $k(t) \subset k(q)$ determined by the morphism $\mathsf{Q}_{k_{i-1}} \to T^{m_i,\ldots,m_s}_{k_i,\ldots,k_s}$ is trivial by Lemma 2.9. Now, consider the commutative diagram

$$\begin{array}{ccc} \mathsf{Q}_{k_{i-1}} & \xrightarrow{\ \theta\ } & \mathsf{Quot}(\mathcal{F}^i; \gamma_i, n_i) \\ & \searrow^{\pi_{i-1}} & \downarrow \pi \\ & & T^{m_i,\ldots,m_s}_{k_i,\ldots,k_s} \end{array} ,$$

where θ is the canonical closed immersion. Let $T^{m_{i-1},\ldots,m_s}_{k_{i-1},\ldots,k_s} \subset T^{m_i,\ldots,m_s}_{k_i,\ldots,k_s}$ denote the scheme-theoretic image of π_{i-1}. Recall that this is a closed subscheme of the target so that

- π_{i-1} factors through the closed immersion $T^{m_{i-1},\ldots,m_s}_{k_{i-1},\ldots,k_s} \to T^{m_i,\ldots,m_s}_{k_i,\ldots,k_s}$, and
- if $W \subset T^{m_i,\ldots,m_s}_{k_i,\ldots,k_s}$ is another closed subscheme so that π_{i-1} factors through the associated closed immersion, then W contains $T^{m_{i-1},\ldots,m_s}_{k_{i-1},\ldots,k_s}$ as a closed subscheme (cf. [Sta25, Tag 01R5]).

The topological space $|T^{m_{i-1},\ldots,m_s}_{k_{i-1},\ldots,k_s}|$ is the closure of $\pi(|\mathsf{Q}_{k_{i-1}}|) \subset |T^{m_i,\ldots,m_s}_{k_i,\ldots,k_s}|$ by [Sta25, Tag 01R5, Lemma 29.6.3]. Since $|T^{m_{i-1},\ldots,m_s}_{k_{i-1},\ldots,k_s}| \subset |T^{m_i,\ldots,m_s}_{k_i,\ldots,k_s}|$ is closed, given the bijection (2.6), it follows that

$$|T^{m_{i-1},\ldots,m_s}_{k_{i-1},\ldots,k_s}| = |T|^{m_{i-1},\ldots,m_s}_{k_{i-1},\ldots,k_s} .$$

Finally, note that the map π is unramified at any $q \in |\mathsf{Q}_{k_1}|$ by the same argument as in the proof of [Nit11, Theorem 5]. The relative tangent space of the morphism $\pi \colon \mathrm{Quot}(\mathcal{F}^i;\gamma_i,n_i) \to T^{m_i,\ldots,m_s}_{k_i,\ldots,k_s}$ at a point $q \in \pi^{-1}(t)$ is

$$T_q(\pi^{-1}(t)) \simeq \mathrm{Hom}_{X_q}(\mathcal{K}_q, \mathcal{G}_q) .$$

As shown above, $\mathcal{K}_q \subset \mathcal{F}_q$ is the maximal subsheaf with set-theoretic support contained in Z_{i-1}. Then Corollary 2.7 shows that

$$\mathrm{Hom}_{X_q}(\mathcal{K}_q, \mathcal{G}_q) = 0 .$$

Since the closed embedding θ is unramified, it follows that the composition $\pi_{i-1} \colon \mathsf{Q}_{k_1} \to T^{m_i,\ldots,m_s}_{k_i,\ldots,k_s}$ is also unramified. In conclusion π_{i-1} is proper, injective, unramified, and induces trivial residual field extensions. Hence it is a closed embedding by [Nit11, Lemma 4]. The closed subscheme $T^{m_{i-1},\ldots,m_s}_{k_{i-1},\ldots,k_s} \subset T^{\preceq \mathsf{m}}$ satisfies property (1) by construction.

The remaining properties follow from Lemma 2.21 and the functoriality properties of Quot schemes. □

Let us fix $\gamma \in \langle Z \rangle$, $n \in \mathbb{Z}$, and $k \in \mathbb{Z}$, with $k \geq 1$. For any $\mathsf{m} \in \mathbb{Z}^{s+1}$ with $m_s = n$, let

$$\left|\mathfrak{Coh}(Z^{(k)}_{\mathsf{cl}};\gamma,n)\right|^{\preceq \mathsf{m}} \subset \left|\mathfrak{Coh}(Z^{(k)}_{\mathsf{cl}};\gamma,n)\right|$$

denote the subset of points ξ so that $\chi(\mathcal{E}^i_\xi) \leq m_i$ for $0 \leq i \leq s$, which is open by Proposition 2.20. Denote by $\mathfrak{Coh}(Z^{(k)}_{\mathsf{cl}};\gamma,n)^{\preceq \mathsf{m}}$ the associated open substack of $\mathfrak{Coh}(Z^{(k)}_{\mathsf{cl}};\gamma,n)$.

For any $\mathsf{k} \in (\mathbb{Z}_{>0})^{s+1}$ with $k_s = k$, let

$$\left|\mathfrak{Coh}(Z^{(k)}_{\mathsf{cl}};\gamma,n)\right|^{\mathsf{k}}_{\mathsf{m}} \subset \left|\mathfrak{Coh}(Z^{(k)}_{\mathsf{cl}};\gamma,n)\right|$$

denote the subset of points ξ so that $\chi(\mathcal{E}^i_\xi) = m_i$ for $1 \leq i \leq s$, and $\mathcal{E}^i_\xi$ is scheme-theoretically supported on $(Z^i_\xi)^{(k_i)}_{\mathsf{cl}}$, for $1 \leq i \leq s$.

Corollary 2.24.

(1) *For any $\mathsf{m} \in \mathbb{Z}^{s+1}$, with $m_s = n$, and any $\mathsf{k} \in (\mathbb{Z}_{>0})^{s+1}$, with $k_s = k$, there exists a locally closed substack $\mathfrak{Coh}(Z^{(k)}_{\mathsf{cl}};\gamma,n)^{\mathsf{m}}_{\mathsf{k}}$ of $\mathfrak{Coh}(Z^{(k)}_{\mathsf{cl}};\gamma,n)$ such that*

$$\left|\mathfrak{Coh}(Z^{(k)}_{\mathsf{cl}};\gamma,n)^{\mathsf{m}}_{\mathsf{k}}\right| = \left|\mathfrak{Coh}(Z^{(k)}_{\mathsf{cl}};\gamma,n)\right|^{\mathsf{m}}_{\mathsf{k}} .$$

(2) For any $\mathsf{k} \preceq \mathsf{k}'$*, one has a commutative diagram*

$$\begin{array}{ccc} \mathfrak{Coh}(Z_{\mathsf{cl}}^{(k)};\gamma,n)_{\mathsf{k}}^{\mathsf{m}} & \xrightarrow{\jmath_{\mathsf{k},\mathsf{k}'}^{\mathsf{m}}} & \mathfrak{Coh}(Z_{\mathsf{cl}}^{(k)};\gamma,n)_{\mathsf{k}'}^{\mathsf{m}} \\ & \searrow \quad \swarrow & \\ & \mathfrak{Coh}(Z_{\mathsf{cl}}^{(k)};\gamma,n) & \end{array} , \tag{2.7}$$

with $\jmath_{\mathsf{k},\mathsf{k}'}^{\mathsf{m}}$ *a closed immersion.*

(3) For each pair (m,k)*, the restriction* $\mathcal{E}_{\mathsf{k}}^{\mathsf{m}}$ *of the universal family to* $\mathfrak{Coh}(Z_{\mathsf{cl}}^{(k)};\gamma,n)_{\mathsf{k}}^{\mathsf{m}} \times X$ *has a relative canonical* $Z^\bullet$*-filtration of type* (m,k)*. Moreover, these relative filtrations are naturally compatible via pull-back to the closed embeddings* $\jmath_{\mathsf{k},\mathsf{k}'}^{\mathsf{m}}$*.*

For each $\mathsf{m} \in \mathbb{Z}^{s+1}$, the data

$$\left(\mathfrak{Coh}(Z_{\mathsf{cl}}^{(k)};\gamma,n)_{\mathsf{k}}^{\mathsf{m}},\ \jmath_{\mathsf{k},\mathsf{k}'}^{\mathsf{m}}\right)$$

with $\mathsf{k},\mathsf{k}' \in (\mathbb{Z}_{>0})^{s+1}$, with $k_s = k$ and $k'_s = k$, form a direct system of geometric (classical) stacks locally of finite presentation over $\mathbb{C}$. We define

$$\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z;\gamma,n)^{\mathsf{m}} := \operatorname*{colim}_{\mathsf{k}} \mathfrak{Coh}(Z_{\mathsf{cl}}^{(k)};\gamma,n)_{\mathsf{k}}^{\mathsf{m}} .$$

We also define

$$\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z;\gamma,n)^{\preceq\mathsf{m}} := \operatorname*{colim}_{k} \mathfrak{Coh}(Z_{\mathsf{cl}}^{(k)};\gamma,n)_{k}^{\preceq\mathsf{m}} ,$$

with respect to the closed immersions $\mathfrak{Coh}(Z_{\mathsf{cl}}^{(k)};\gamma,n)^{\preceq\mathsf{m}} \to \mathfrak{Coh}(Z_{\mathsf{cl}}^{(k')};\gamma,n)^{\preceq\mathsf{m}}$ for $k \leq k'$. Thanks to the commutative diagram (2.7), we also have a closed immersion

$$\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z;\gamma,n)^{\mathsf{m}} \longrightarrow \mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z;\gamma,n)^{\preceq\mathsf{m}} ,$$

as well as an open immersion

$$\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z;\gamma,n)^{\preceq\mathsf{m}} \longrightarrow \mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z;\gamma,n) ,$$

thanks to the description of $\widehat{X}_Z$ in terms of $Z_{\mathsf{cl}}^{(k)}$ in [DPS+25b, Remark 4.6]. We denote by $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z;\gamma,n)^{\preceq\mathsf{m}}$ the corresponding derived enhancement.

3. A closed stratification induced by the support filtration

In this section we consider a one-step stratification

$$Z^\bullet : \quad \emptyset =: Z^{-1} = Z^0 \subset Z^1 \subset Z^2 := Z$$

as in §2 satisfying the Assumptions 0.1. In particular, Z and Z^1 are purely one-dimensional. Recall that Z_i is the scheme-theoretic closure of $Z_i^\circ := Z^i \smallsetminus Z^{i-1}$, and the scheme-theoretic intersection $Z_i^{i-1} := Z^{i-1} \cap Z_i$ is zero-dimensional for $i = 1,2$. In particular $Z_1 = Z^1$, hence we will set $Z_{1,2} := Z_2^1$. Then the support filtration of any coherent sheaf $\mathcal{E}$ with set-theoretic support contained in Z reduces to $\mathcal{E}^1 \subset \mathcal{E}$. Note that $\mathcal{E}_1 = \mathcal{E}^1$ and set $\mathcal{E}_2 := \mathcal{E}/\mathcal{E}_1$. Furthermore, since Z and Z_i, for $i = 1,2$, are reduced, all thickenings $Z_{\mathsf{cl}}^{(k)}$ and $(Z_i)_{\mathsf{cl}}^{(k)}$, for $i = 1,2$, are effective divisors on X for all $k \geq 1$.

For ease of exposition, we will use only lower indices in the following and the $Z^\bullet$-filtration will be referred to as the Z_1-filtration.

3.1. **Construction.** Let us fix $\gamma \in \langle Z \rangle$ and $n \in \mathbb{Z}$. Let $a \in \mathbb{Z}$.

Fix $k \in \mathbb{Z}$, with $k \geq 1$. Let $\mathfrak{Coh}(Z_{\mathsf{cl}}^{(k)}; \gamma, n)_k^{\geqslant a}$ be the scheme-theoretic image of the locally closed immersion

$$\bigsqcup_{\substack{n_1 \in \mathbb{Z} \\ n_1 \geq a}} \mathfrak{Coh}(Z_{\mathsf{cl}}^{(k)}; \gamma, n)_k^{n_1} \longrightarrow \mathfrak{Coh}(Z_{\mathsf{cl}}^{(k)}; \gamma, n) ,$$

defined as in [Sta25, Tag 04XM, Definition 38.1]. Its existence is proven in [Sta25, Tag 04XM, Lemma 38.3].

For $\ell > k$, we have a closed immersion

$$\mathfrak{Coh}(Z_{\mathsf{cl}}^{(k)}; \gamma, n)_k^{\geqslant a} \longrightarrow \mathfrak{Coh}(Z_{\mathsf{cl}}^{(\ell)}; \gamma, n)_\ell^{\geqslant a} .$$

We set

$$\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma, n)^{\geqslant a} \coloneqq \operatorname*{colim}_k \mathfrak{Coh}(Z_{\mathsf{cl}}^{(k)}; \gamma, n)_k^{\geqslant a} .$$

Then we have a canonical closed immersion

$$f^{\geqslant a} \colon \mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma, n)^{\geqslant a} \longrightarrow \mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma, n) .$$

By construction, the complement of $\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma, n)^{\geqslant a}$ in $\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma, n)$ is the open substack $\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma, n)^{<a} \subset \mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma, n)$.

Moreover, for any pair $a, b \in \mathbb{Z}$, $a < b$, let $\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma, n)^{a,b}$ be the locally closed substack defined by the pullback diagram

$$\begin{array}{ccc} \mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma, n)^{a,b} & \longrightarrow & \mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma, n)^{\geqslant a} \\ \big\downarrow & & \big\downarrow f^{\geqslant a} \\ \mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma, n)^{<b} & \longrightarrow & \mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma, n) \end{array} . \tag{3.1}$$

3.2. **Closed stratification and convolution maps.** Assume that there is a torus T acting on X such that Z is T-invariant[8]. Set $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{ext}}(\widehat{X}_Z) \coloneqq \mathcal{S}_2\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z)$, where the latter is introduced in [DPS+25b, §5.1]. Its connected components labeled by topological invariants will be denoted in the usual way.

Given a decomposition $(\gamma, n) = (\gamma_1, n_1) + (\gamma_2, n_2)$, consider the convolution diagram

$$\begin{array}{ccc} & \overset{p_{2,1}}{\nearrow} & \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma, n) \\ & & \big\uparrow p \\ \mathbf{Coh}_{2,1}^{\mathsf{ext}}(\widehat{X}_Z; (\gamma_1, n_1), (\gamma_2, n_2)) & \longrightarrow & \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{ext}}(\widehat{X}_Z; (\gamma_1, n_1), (\gamma_2, n_2)) \\ \big\downarrow q_{2,1} & & \big\downarrow q \\ \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_2}; \gamma_2, n_2) \times \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_1}; \gamma_1, n_1) & \longrightarrow & \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma_2, n_2) \times \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma_1, n_1) \end{array} \tag{3.2}$$

where $q_{2,1} \coloneqq \mathsf{ev}_2 \times \mathsf{ev}_0$ is derived lci, while $p_{2,1} \coloneqq \mathsf{ev}_1$ is locally rpas[9] by [DPS+25b, Lemma 5.6].

For any $n_1 \in \mathbb{Z}$, we define the substacks

$$\mathfrak{Coh}_{2,1}^{\mathsf{ext}}(\widehat{X}_Z; (\gamma_1, n_1), (\gamma_2, n_2))^{n_1} \subset \mathfrak{Coh}_{2,1}^{\mathsf{ext}}(\widehat{X}_Z; (\gamma_1, n_1), (\gamma_2, n_2))^{\geqslant n_1} \subset \mathfrak{Coh}_{2,1}^{\mathsf{ext}}(\widehat{X}_Z; (\gamma_1, n_1), (\gamma_2, n_2))$$

[8]Recall that $T = \{1\}$ is allowed, in which case one recovers the non-equivariant theory.

[9]In the sense of [DPS+25b, Definitions 3.2 and 3.3].

through the pull-back squares

$$\begin{array}{ccc}
\mathfrak{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^{n_1} & \xrightarrow{p^{n_1}_{2,1}} & \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)^{n_1} \\
\Big\downarrow \varepsilon^{n_1} & & \Big\downarrow f^{n_1} \\
\mathfrak{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^{\geqslant n_1} & \xrightarrow{p^{\geqslant n_1}_{2,1}} & \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)^{\geqslant n_1} \\
\Big\downarrow \varepsilon^{\geqslant n_1} & & \Big\downarrow f^{\geqslant n_1} \\
\mathfrak{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2)) & \xrightarrow{p_{2,1}} & \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)
\end{array} ,$$

where f^{n_1} is the canonical open immersion.

Furthermore, as a consequence of Lemma 2.18, note that one has an open substack

$$j^\circ\colon \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2,n_2)^\circ \longrightarrow \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2,n_2)$$

consisting of sheaves $\mathcal{E}_2$ which are pure at $Z_{1,2}$. We then define the open substack

$$\mathbf{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^\circ \subset \mathbf{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))$$

through the pull-back square

$$\begin{array}{ccc}
\mathbf{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^\circ & \longrightarrow & \mathbf{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2)) \\
\Big\downarrow q^\circ_{2,1} & & \Big\downarrow q_{2,1} \\
\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2,n_2)^\circ \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1,n_1) & \xrightarrow{j^\circ\times \mathsf{id}} & \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2,n_2) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1,n_1)
\end{array} .$$

Now, consider the following diagram:

$$\begin{array}{ccc}
 & & \mathfrak{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^{n_1} \\
 & \overset{\phi}{\swarrow} & \Big\downarrow \varepsilon^{n_1} \\
 & & \mathfrak{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^{\geqslant n_1} \\
 & & \Big\downarrow \varepsilon^{\geqslant n_1} \\
\mathfrak{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^\circ & \longrightarrow & \mathfrak{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))
\end{array} ,$$

excluding the dotted arrow.

Lemma 3.1. *The following hold:*

(i) $\varepsilon^{\geqslant n_1}$ *is a canonical equivalence and* $p^{\geqslant n_1}_{2,1}$ *is representable and proper.*

(ii) $p^{n_1}_{2,1}$ *is an equivalence.*

(iii) There is a natural equivalence

$$\phi\colon \mathfrak{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^{n_1} \to \mathfrak{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^\circ$$

so that the enhanced diagram including the dotted arrow is commutative.

(iv) $q^\circ_{2,1}$ *is a vector bundle stack of rank* $\gamma_1\cdot\gamma_2$.

(v) The following relations hold:

$$(p_{2,1})_* = f^{\geqslant n_1}_* \circ (p^{\geqslant n_1}_{2,1})_* \circ (\varepsilon^{\geqslant n_1})^* , \tag{3.3}$$

$$(f^{n_1})^* \circ (p^{\geqslant n_1}_{2,1})_* \circ (\varepsilon^{\geqslant n_1})^* \circ q^!_{2,1} = (p^{n_1}_{2,1})_* \circ \phi^* \circ (q^\circ_{2,1})^! \circ (\mathsf{id}\times j^\circ)^* , \tag{3.4}$$

in T-equivariant (motivic) Borel-Moore homology.

Proof. We start by proving (i). Since $p_{2,1}$ is a proper representable map, so is $p_{2,1}^{\geqslant n_1}$. We shall show that $\varepsilon^{\geqslant n_1}$ is an equivalence. Given a scheme T locally of finite type over $\mathbb{C}$, the groupoid $\mathfrak{Coh}_{2,1}^{\mathsf{ext}}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))(T)$ consists of T-flat families of short exact sequences

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}_2 \longrightarrow 0$$

with

$$\mathcal{F}_i \in \mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_i};\gamma_i,n_i)(T)$$

for $1 \leq i \leq 2$. For any point $t \in T$, let $\mathcal{E}_{t,1} \subset \mathcal{E}_t$ be the $(Z_1)_t$-filtration of $\mathcal{E}_t$ and let $\mathcal{E}_{t,2} := \mathcal{E}_t/\mathcal{E}_{t,1}$. Since $\mathcal{F}_{1,t}$ is set-theoretically supported on $(Z_1)_t$, the injective morphism $\mathcal{F}_{1,t} \to \mathcal{E}_t$ factors through the inclusion $\mathcal{E}_{t,1} \subset \mathcal{E}_t$ and we get the exact sequences

$$0 \longrightarrow \mathcal{F}_{1,t} \longrightarrow \mathcal{E}_{t,1} \longrightarrow \mathcal{T} \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{F}_{2,t} \longrightarrow \mathcal{E}_{t,2} \longrightarrow 0\,.$$

Furthermore, by Corollary 2.8, one has $\mathsf{ch}_1(\mathcal{E}_{t,1}) = \mathsf{ch}_1(\mathcal{F}_{1,t})$. Therefore, $\mathcal{T}$ is zero-dimensional. This implies that

$$\chi(\mathcal{E}_{t,1}) \geq \chi(\mathcal{F}_{1,t}) = n_1\,,$$

hence the T-flat family

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}_2 \longrightarrow 0$$

belongs to $\mathfrak{Coh}_{2,1}^{\mathsf{ext}}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^{\geqslant n_1}(T)$.

This proves that the groupoid $\mathfrak{Coh}_{2,1}^{\mathsf{ext}}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))(T)$ is canonically equivalent to $\mathfrak{Coh}_{2,1}^{\mathsf{ext}}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^{\geqslant n_1}(T)$. Hence $\varepsilon^{\geqslant n_1}$ is an equivalence.

We prove (ii). Let T be a scheme locally of finite type over $\mathbb{C}$. By construction, an object of the groupoid $\mathfrak{Coh}_{2,1}^{\mathsf{ext}}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^{n_1}(T)$ consists of:

(1) a T-flat family of exact sequences
$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}_2 \longrightarrow 0\,, \tag{3.5}$$
with $\mathcal{F}_i \in \mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_1};\gamma_i,n_i)(T)$, for $1 \leq i \leq 2$,

(2) an additional T-flat family of exact sequences
$$0 \longrightarrow \mathcal{E}'_1 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E}'_2 \longrightarrow 0 \tag{3.6}$$
where $\mathcal{E}' \in \mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z;\gamma,n)(T)$ and $\mathcal{E}'_1 \subset \mathcal{E}'$ is a relative Z_1-filtration for $\mathcal{E}'$ over T, with $\chi(\mathcal{E}'_{1,t}) = n_1$ for any $t \in T$, and

(3) an isomorphism $f\colon \mathcal{E} \to \mathcal{E}'$.

By analogy to the proof of statement (i) above, note that the composition

$$\mathcal{F}_{1,t} \longrightarrow \mathcal{E}_t \xrightarrow{f_t} \mathcal{E}'_t$$

maps $\mathcal{F}_{1,t}$ isomorphically onto a subsheaf of $\mathcal{E}'_{t,1}$ for any $t \in T$. Moreover, again, $\mathsf{ch}_1(\mathcal{F}_{1,t}) = \mathsf{ch}_1(\mathcal{E}'_{t,1})$, which implies that $\mathcal{E}'_{t,1}/\mathcal{F}_{1,t}$ is zero-dimensional. Under the current assumptions, we also have

$$\chi(\mathcal{E}'_{t,1}) = \chi(\mathcal{F}_{1,t}) = n_1\,.$$

Hence $\mathcal{E}'_{t,1}/\mathcal{F}_{1,t} = 0$, i.e., the injection $\mathcal{F}_{1,t} \to \mathcal{E}'_{t,1}$ is an isomorphism. However, Proposition 2.23–(4) shows that the relative support filtration is unique, hence the isomorphism f extends to an isomorphism of the two flat families of exact sequences (3.5) and (3.6). This proves that the functor $p_{2,1}^{n_1}(T)$ is an equivalence of groupoids.

Now, we prove (iii). Again, let T be a scheme locally of finite type over $\mathbb{C}$. The proof of statement (ii) above shows that the groupoid $\mathfrak{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^{n_1}(T)$ consists of flat families of exact sequences

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0$$

so that $\mathcal{E}_1 \subset \mathcal{E}$ is a relative Z_1-filtration for $\mathcal{E}$. In particular, $\mathcal{E}_{2,t}$ is pure at $(Z_{2,1})_t$ for any $t \in T$. By construction, this groupoid coincides canonically with $\mathfrak{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^{\circ}(T)$.

Claim (iv) follows from [PS23, Proposition 3.6] and Corollary 2.7. Finally, we prove (v). Relation (3.3) follows from the identity

$$(p_{2,1})_* \circ \varepsilon_*^{\geqslant n_1} = f_*^{\geqslant n_1} \circ (p_{2,1}^{\geqslant n_1})_* \,,$$

keeping in mind that $\varepsilon_*^{\geqslant n_1}$ is an isomorphism, and its inverse is $(\varepsilon^{\geqslant n_1})^*$. Relation (3.4) follows by similar diagram manipulations. □

3.3. **An open exhaustion compatible with the closed stratification.** In this section, we shall introduce certain open exhaustions of the stack $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)$ and study their behavior relative to support stratifications.

Fix invariants (γ,n). Recall that by [DPS+25b, Theorem 4.65] the derived stack $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)$ is admissible in the sense of [DPS+25b, Definition 2.16]. In particular, there exists a (possibly transfinite) sequence

$$\emptyset = \mathfrak{U}_0(\gamma,n) \hookrightarrow \mathfrak{U}_1(\gamma,n) \hookrightarrow \cdots \mathfrak{U}_k(\gamma,n) \hookrightarrow \mathfrak{U}_{k+1}(\gamma,n) \hookrightarrow \cdots$$

of open Zariski immersions between quasi-compact quasi-separated indgeometric derived stacks, whose colimit in PreSt_k is $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)$. By definition, for any k the underlying reduced stack ${}^{\mathsf{red}}\mathfrak{U}_k(\gamma,n)$ of $\mathfrak{U}_k(\gamma,n)$ is a quasi-compact quasi-separated geometric classical stack.

We shall denote also by $\{\mathfrak{U}_k(\gamma,n)\}_{k\in I}$ the corresponding open exhaustion of the classical truncation $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)$ of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)$.

Let $j_i\colon Z_i \to X$ be the canonical closed embedding for $i=1,2$. We denote by $\{\mathfrak{U}_{i,k}(\gamma_i,n_i)\}_{k\in I}$ the open exhaustion of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_i};\gamma_i,n_i)$ given by the pullback square

$$\begin{array}{ccc} \mathfrak{U}_{i,k}(\gamma_i,n_i) & \longrightarrow & \mathfrak{U}_k(\gamma_i,n_i) \\ \downarrow & & \downarrow \\ \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_i};\gamma_i,n_i) & \xrightarrow{(\hat{\jmath}_i)_*} & \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma_i,n_i) \end{array} .$$

We denote also by $\{\mathfrak{U}_{i,k}(\gamma_i,n_i)\}_{k\in I}$ the corresponding open exhaustion of the classical truncation $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_i};\gamma_i,n_i)$. We shall say that $\{\mathfrak{U}_{i,k}(\gamma_i,n_i)\}_{k\in I}$ is the *induced* exhaustion from the fixed exhaustion $\{\mathfrak{U}_k(\gamma_i,n_i)\}_{k\in I}$ for $i=1,2$.

Furthermore, for each $k \in I$ let $\mathfrak{U}_{2,k}(\gamma_2,n_2)^{\circ}$ be the classical stack defined by

$$\begin{array}{ccc} \mathfrak{U}_{2,k}(\gamma_2,n_2)^{\circ} & \xrightarrow{u_k^{\circ}} & \mathfrak{U}_{2,k}(\gamma_2,n_2) \\ \downarrow & & \downarrow u_k \\ \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2,n_2)^{\circ} & \longrightarrow & \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2,n_2) \end{array} .$$

Define the classical stacks $\mathfrak{Y}_k((\gamma_1, n_1), (\gamma_2, n_2))$ and $\mathfrak{Y}_k((\gamma_1, n_1), (\gamma_2, n_2))^\circ$ by the pullback squares

$$\begin{array}{ccc} \mathfrak{Y}_k((\gamma_1,n_1),(\gamma_2,n_2))^\circ & \longrightarrow & \mathfrak{U}_{2,k}(\gamma_2,n_2)^\circ \times \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1,n_1) \\ \downarrow{\scriptstyle y_k^\circ} & & \downarrow{\scriptstyle u_k^\circ \times \mathsf{id}} \\ \mathfrak{Y}_k((\gamma_1,n_1),(\gamma_2,n_2)) & \longrightarrow & \mathfrak{U}_{2,k}(\gamma_2,n_2) \times \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1,n_1) \\ \downarrow{\scriptstyle y_k} & & \downarrow{\scriptstyle u_k \times \mathsf{id}} \\ \mathfrak{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^{\geqslant n_1} & \xrightarrow{q_{2,1}\circ \varepsilon^{\geqslant n_1}} & \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2,n_2) \times \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1,n_1) \end{array} .$$

In order to study the properties of the Hall multiplication map associated to the diagram (3.2), we need to assume the existence of another open exhaustion of $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)$ satisfying certain conditions. We need first to fix some notation.

Notation 3.2. Let $\{\mathfrak{S}_k(\gamma,n)\}_{k\in I}$ be an arbitrary open exhaustion of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)$ consisting of indgeometric derived stacks. For any $n_1 \in \mathbb{Z}$, let $\mathfrak{S}_k(\gamma,n)^{\geqslant n_1}$ and $\mathfrak{S}_k(\gamma,n)^{\,n_1}$ be the classical stacks defined by the pullback squares

$$\begin{array}{ccc} \mathfrak{S}_k(\gamma,n)^{\,n_1} & \longrightarrow & \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)^{\,n_1} \\ \downarrow{\scriptstyle s_k^{n_1}} & & \downarrow{\scriptstyle f^{\,n_1}} \\ \mathfrak{S}_k(\gamma,n)^{\geqslant n_1} & \xrightarrow{s_k^{\geqslant n_1}} & \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)^{\geqslant n_1} \\ \downarrow & & \downarrow{\scriptstyle f^{\geqslant n_1}} \\ \mathfrak{S}_k(\gamma,n) & \xrightarrow{s_k} & \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n) \end{array} ,$$

where, by abuse of notation, we denote the truncation of $\mathfrak{S}_k(\gamma,n)$ by the same symbol.

Fix an invariant (γ,n), together with a decomposition $(\gamma_1,n_1)+(\gamma_2,n_2)=(\gamma,n)$. Let

$$\mathfrak{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^{\geqslant n_1}_{\sigma_k} \quad \text{and} \quad \mathfrak{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^{n_1}_{\sigma_k}$$

be the classical stacks defined by the pullback squares

$$\begin{array}{ccc} \mathfrak{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^{n_1}_{\sigma_k} & \longrightarrow & \mathfrak{S}_k(\gamma,n)^{\,n_1} \\ \downarrow{\scriptstyle \eta^{\,n_1}} & & \downarrow{\scriptstyle s_k^{n_1}} \\ \mathfrak{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^{\geqslant n_1}_{\sigma_k} & \longrightarrow & \mathfrak{S}_k(\gamma,n)^{\geqslant n_1} \\ \downarrow{\scriptstyle \eta^{\geqslant n_1}} & & \downarrow{\scriptstyle s_k^{\geqslant n_1}} \\ \mathfrak{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z;(\gamma_1,n_1),(\gamma_2,n_2))^{\geqslant n_1} & \xrightarrow{p_{2,1}^{\geqslant n_1}} & \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)^{\geqslant n_1} \end{array} .$$

⊘

Assumption 0.2 (Existence of an open exhaustion compatible with $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)^{\geqslant n_1}$)**.** There exists an open exhaustion $\{\mathfrak{S}_k(\gamma,n)\}_{k\in I}$ of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)$ consisting of indgeometric derived stacks satisfying the following properties:

(1) For any $a \in \mathbb{Z}$ and any $k \in I$, there exists a unique constant $b \in \mathbb{Z}$ so that:
  (i) the open immersion $s_k^{\geqslant a}$ factors through the open immersion $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)^{a,c} \to \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)^{\geqslant a}$ for any $c \geq b$, and
  (ii) b is minimal with this property.

(2) For any invariant (γ, n), together with a decomposition $(\gamma_1, n_1) + (\gamma_2, n_2) = (\gamma, n)$, and any $k \in I$, there exist a canonical open immersion

$$\psi_k^{\geqslant n_1} \colon \mathfrak{Coh}_{2,1}^{\mathsf{ext}}(\widehat{X}_Z; (\gamma_1, n_1), (\gamma_2, n_2))_{\sigma_k}^{\geqslant n_1} \longrightarrow \mathfrak{Y}_k((\gamma_1, n_1), (\gamma_2, n_2))$$

such that $y_k \circ \psi_k^{\geqslant n_1} = \eta^{\geqslant n_1}$, and a canonical equivalence

$$\psi_k^{n_1} \colon \mathfrak{Coh}_{2,1}^{\mathsf{ext}}(\widehat{X}_Z; (\gamma_1, n_1), (\gamma_2, n_2))_{\sigma_k}^{n_1} \longrightarrow \mathfrak{Y}_k((\gamma_1, n_1), (\gamma_2, n_2))^\circ$$

such that $y_k^\circ \circ \psi_k^{n_1} = \psi_k^{\geqslant n_1} \circ \eta^{n_1}$.

⊘

Remark 3.3. In §5, we shall construct an explicit example of an open exhaustion $\{\mathfrak{S}_k(\gamma, n)\}_{k \in I}$ of $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma, n)$ satisfying the above assumption. △

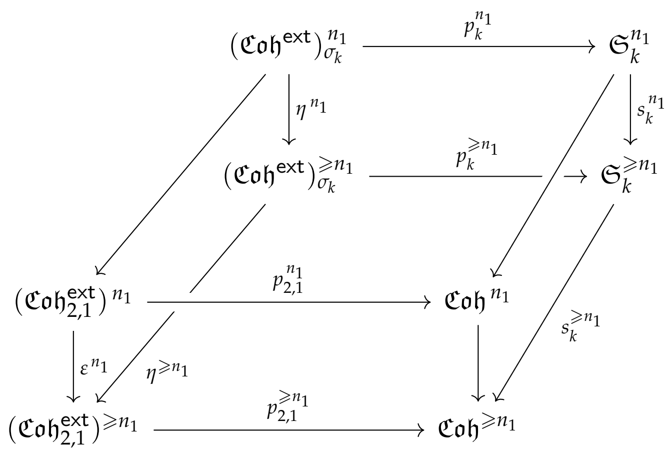
From now on, we impose that Assumption 0.2 holds. In order to simplify the notation, below we will use the abbreviations:

$$\mathfrak{Coh}_i \coloneqq \mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_i}; \gamma_i, n_i) \ \text{ for } \ 1 \leq i \leq 2\,,$$

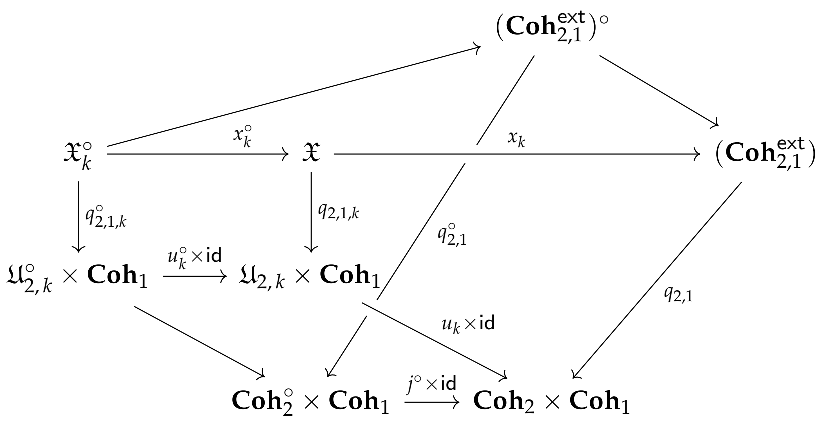
$$\mathfrak{Coh}_2^\circ \coloneqq \mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_2}; \gamma_2, n_2)^\circ\,,\ \mathfrak{Coh} \coloneqq \mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma, n)$$

$$\mathfrak{Coh}^{\mathsf{ext}} \coloneqq \mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{ext}}(\widehat{X}_Z; (\gamma_1, n_1), (\gamma_2, n_2))\,,\ \mathfrak{Coh}_{2,1}^{\mathsf{ext}} \coloneqq \mathfrak{Coh}_{2,1}^{\mathsf{ext}}(\widehat{X}_Z; (\gamma_1, n_1), (\gamma_2, n_2))\,,$$

and similarly for their derived enhancements. Consider the following commutative diagrams, where the squares are all pullbacks:

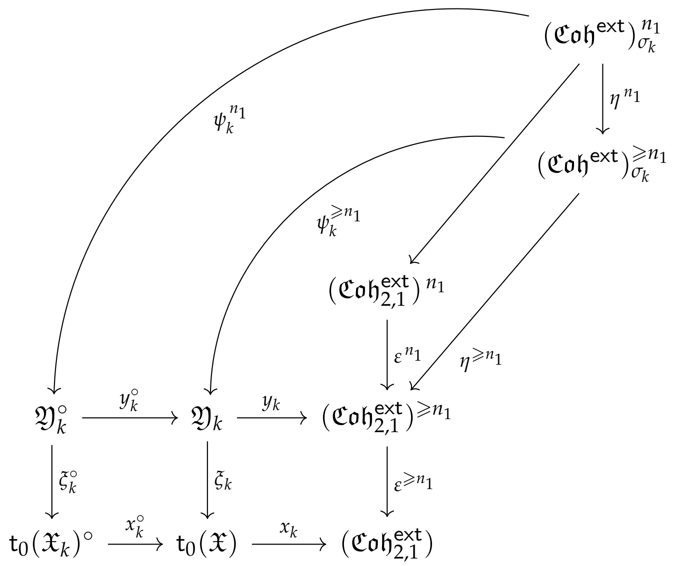


,



, (3.7)

and



. (3.8)

Lemma 3.4. *Under Assumptions 0.1 and 0.2, the following hold:*

(i) The maps x_k, x_k° and y_k, y_k°, as well as $\eta^{\geqslant n_1}, \eta^{n_1}$ are open immersions.

(ii) ξ_k and ξ_k° are canonical equivalences.

(iii) The map $p_k^{\geqslant n_1}$ is a proper representable map and $p_k^{n_1}$ is a canonical equivalence.

(iv) $q_{2,1,k}$ is derived lci and $q^\circ_{2,1,k}$ is a vector bundle stack of rank $\gamma_1 \cdot \gamma_2$.

(v) The following relations hold

$$(s_k^{\geqslant n_1})^* \circ (p_{2,1}^{\geqslant n_1})_* \circ (\varepsilon^{\geqslant n_1})^* \circ q_{2,1}^! = (p_k^{\geqslant n_1})_* \circ (\psi_k^{\geqslant n_1})^* \circ \xi_k^* \circ (q_{2,1,k})^! \circ (u_k \times \mathrm{id})^* ,$$

$$(s_k^{n_1})^* \circ (s_k^{\geqslant n_1})^* \circ (p_{2,1}^{\geqslant n_1})_* \circ (\varepsilon^{\geqslant n_1})^* \circ q_{2,1}^! = (p_k^{n_1})_* \circ (\psi_k^{n_1})^* \circ (\xi_k^\circ)^* \circ (q_{2,1,k}^\circ)^! \circ (u_k^\circ \times \mathrm{id})^* \circ (u_k \times \mathrm{id})^*$$

in T-equivariant (motivic) Borel-Moore homology.

Proof. Claim (i) follows by base change since the maps $u_k \times \mathrm{id}$, $u_k^\circ \times \mathrm{id}$, $s_k^{\geqslant n_1}$, and $s_k^{n_1}$ are all open immersions. Since open immersions are representable, Claims (ii) and (iii) follow from Lemma 3.1 by base change. The first part of Claim (iv) follows base change, while the second part follows from Corollary 2.7. Finally, Claim (v) follows from (i)–(iv) through standard diagram manipulations. □

4. AMALGAMATION MAP

In this section, we introduce the *amalgamation map*

$$\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_s}, \gamma_s)) \widehat{\otimes}_{\mathbf{HA}^T_{X,Z_s^{s-1}}} \cdots \widehat{\otimes}_{\mathbf{HA}^T_{X,Z_2^1}} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}, \gamma_1)) \longrightarrow \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma)) .$$

Here, to ease the notation we write $\mathbf{HA}^T_{X,Z_i^{i-1}}$ instead of $\mathbf{HA}^T_{X,{}^{\mathsf{red}}Z_i^{i-1}}$.

Moreover, we prove that the map is *topologically surjective*, i.e., the image is dense with respect to the quasi-compact topology on the target, under a list of Assumptions (cf. Theorem 4.21).

The results presented below are stated for (constructible) Borel-Moore homology but also hold for motivic Borel-Moore homology.

4.1. **Preliminary definitions.** In this section, we shall continue to consider a one-step stratification

$$Z^\bullet\colon\quad \emptyset =: Z^{-1} = Z^0 \subset Z^1 \subset Z^2 := Z$$

as in §2 satisfying the Assumptions 0.1. Using the notation in Lemma 3.1, we introduce the following definition.

Definition 4.1.

(i) Let

$$\mu_n\colon \bigoplus_{n_1\in\mathbb{Z}} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2, n-n_1)\times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1, n_1)) \longrightarrow \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z};\gamma, n))$$

be the unique homomorphism of H_T-modules which restricts to

$$f_*^{\geqslant n_1}\circ (p_{2,1}^{\geqslant n_1})_*\circ(\varepsilon^{\geqslant n_1})^*\circ q_{2,1}^!$$

on the associated direct summand of the domain.

(ii) For any $a\in\mathbb{Z}$ let

$$\mu_n^{\geqslant a} := \bigoplus_{n_1\geqslant a} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2, n-n_1)\times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1, n_1)) \longrightarrow \mathsf{H}^T_\bullet(\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z};\gamma, n)^{\geqslant a})$$

be the unique homomorphism of H_T-modules which restricts to

$$f_*^{a,n_1}\circ (p_{2,1}^{\geqslant n_1})_*\circ(\varepsilon^{\geqslant n_1})^*\circ q_{2,1}^!$$

on the associated direct summand of the domain, where

$$f^{a,n_1}\colon \mathsf{H}^T_\bullet(\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z};\gamma, n)^{\geqslant n_1}) \longrightarrow \mathsf{H}^T_\bullet(\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z};\gamma, n)^{\geqslant a})$$

is the canonical closed immersion.

⊘

Note the following immediate consequence of Lemma 3.1.

Corollary 4.2. *For any $a\in\mathbb{Z}$, the restriction of μ_n to*

$$\bigoplus_{n_1\geqslant a} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2, n-n_1)\times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1, n_1))$$

factors as $f_*^{\geqslant a}\circ\mu_n^{\geqslant a}$. Moreover, for any pair $a,b\in\mathbb{Z}$, with $a\leq b$, there is a commutative diagram

$$\begin{array}{ccc}
\displaystyle\bigoplus_{n_1\geqslant b} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2, n-n_1)\times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1, n_1)) & \xrightarrow{\mu_n^{\geqslant b}} & \mathsf{H}^T_\bullet(\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z};\gamma, n)^{\geqslant b}) \\
\downarrow & & \downarrow{\scriptstyle f_*^{a,b}} \\
\displaystyle\bigoplus_{n_1\geqslant a} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2, n-n_1)\times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1, n_1)) & \xrightarrow{\mu_n^{\geqslant a}} & \mathsf{H}^T_\bullet(\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z};\gamma, n)^{\geqslant a})
\end{array},$$

where the left vertical arrow is the canonical inclusion.

Using the notation in Lemma 3.4, we introduce the following definition.

Definition 4.3. For any $a\in\mathbb{Z}$ and any $k\in I$, let

$$\nu_k^{\geqslant a}\colon \bigoplus_{n_1\geqslant a} \mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2, n-n_1)\times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1, n_1)) \longrightarrow \mathsf{H}^T_\bullet(\mathfrak{S}_k(\gamma, n)^{\geqslant a})$$

be the unique homomorphism of H_T-modules which restricts to

$$(s_k^{a,n_1})_*\circ(p_k^{\geqslant n_1})_*\circ(\psi_k^{\geqslant n_1})^*\circ\xi_k^*\circ q_{2,1,k}^!$$

on each direct summand of the domain, where

$$s_k^{a,b}\colon \mathfrak{S}_k(\gamma,n)^{\geqslant b} \longrightarrow \mathfrak{S}_k(\gamma,n)^{\geqslant a}\ ,$$

for $a \leq b$, is the canonical closed immersion. ⊘

Lemma 3.4 yields the following.

Corollary 4.4.

(i) For any $a,k\in\mathbb{Z}$, with $k\geq 0$, there is a commutative diagram

$$\begin{array}{ccc} \bigoplus_{n_1\geqslant a} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2,n-n_1)\times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1,n_1)) & \xrightarrow{\mu_n^{\geqslant a}} & \mathsf{H}^T_\bullet(\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)^{\geqslant a}) \\ \downarrow & & \downarrow (s_k^{\geqslant a})^* \\ \bigoplus_{n_1\geqslant a} \mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2,n-n_1)\times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1,n_1)) & \xrightarrow{\nu_k^{\geqslant a}} & \mathsf{H}^T_\bullet(\mathfrak{S}_k(\gamma,n)^{\geqslant a}) \end{array}\ ,$$

where the left vertical arrow is the unique homomorphism of H_T-moduleswhich restricts to $(\mathsf{id}\times u_k)^*$ on each direct summand of the domain.

(ii) For any for any $a,b,k\in\mathbb{Z}$, with $a\leq b$ and $k\geq 0$, there is a commutative diagram

$$\begin{array}{ccc} \bigoplus_{n_1\geqslant b} \mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2,n-n_1)\times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1,n_1)) & \xrightarrow{\nu_k^{\geqslant b}} & \mathsf{H}^T_\bullet(\mathfrak{S}_k(\gamma,n)^{\geqslant b}) \\ \downarrow & & \downarrow (s_k^{a,b})_* \\ \bigoplus_{n_1\geqslant a} \mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2,n-n_1)\times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1,n_1)) & \xrightarrow{\nu_k^{\geqslant a}} & \mathsf{H}^T_\bullet(\mathfrak{S}_k(\gamma,n)^{\geqslant a}) \end{array}\ ,$$

where the left vertical arrow is the canonical inclusion.

4.2. **Conditional surjectivity results.** For any $a,b\in\mathbb{Z}$, with $a<b$, recall the locally closed substack $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)^{a,b}$ of $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)$ defined in the pullback diagram (3.1), which is displayed below for convenience:

$$\begin{array}{ccc} \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)^{a,b} & \longrightarrow & \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)^{\geqslant a} \\ \downarrow & & \downarrow f^{\geqslant a} \\ \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)^{<b} & \longrightarrow & \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n) \end{array}\ .$$

For any $k\in I$, let $\mathfrak{S}_k(\gamma,n)^{a,b}$ be the classical stack defined by the pullback diagram

$$\begin{array}{ccc} \mathfrak{S}_k(\gamma,n)^{a,b} & \longrightarrow & \mathfrak{S}_k(\gamma,n)^{\geqslant a} \\ \downarrow & & \downarrow s_k^{\geqslant a} \\ \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)^{a,b} & \longrightarrow & \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)^{\geqslant a} \end{array}\ .$$

Let $j_k^{a,b}\colon \mathfrak{S}_k(\gamma,n)^{a,b}\to\mathfrak{S}_k(\gamma,n)^{\geqslant a}$ be the canonical open immersion. Note that the open immersion $s_k^a\colon \mathfrak{S}_k(\gamma,n)^a\to\mathfrak{S}_k(\gamma,n)^{\geqslant a}$ factors naturally through $j_k^{a,b}$.

Assumption 0.3 (Surjectivity of restriction maps I)**.**

(1) (Constructible version). The restriction maps $(u_k^\circ\times\mathsf{id})^*$ and $(u_k\times\mathsf{id})^*$ in the diagram (3.7) are surjective in T-equivariant Borel-Moore homology for all $k\in I$ and for all values of the topological invariants (γ,n).

(2) (Motivic version). The restriction maps $(u_k^\circ \times \mathsf{id})^*$ and $(u_k \times \mathsf{id})^*$ in the diagram (3.7) are surjective in weight zero T-equivariant motivic Borel-Moore homology for all $k \in I$ and for all values of the topological invariants (γ, n).

⊘

We next show that Theorem B.3 and Propositions B.21 and B.24 provide two instances where the above assumption is satisfied. First note that the open substacks

$$\mathfrak{U}_{2,h}(\gamma_2, n_2) \times \mathfrak{U}_{1,h}(\gamma_1, n_1) \subset \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2, n_2) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}; \gamma_1, n_1)\,, \tag{4.1}$$

with $h \geq 0$, form a T-invariant open exhaustion.

Theorem B.3 yields the following:

Proposition 4.5. *Assume that the open exhaustion* (4.1) *is T-equivariantly indstratifiable, i.e, each product*

$$^{\mathsf{red}}\mathfrak{U}_{2,h}(\gamma_2, n_2) \times {}^{\mathsf{red}}\mathfrak{U}_{1,h}(\gamma_1, n_1)$$

is T-equivariantly stratifiable by global quotients for all $h \geq 0$ and for all values of the topological invariants. Then, Assumption 0.3–(2) holds.

Proof. Under the stated assumptions, surjectivity of the restriction map

$$(u_k \times \mathsf{id})^*\colon \mathsf{H}^{\mathsf{mot},T}_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2, n_2) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}; \gamma_1, n_1); 0) \longrightarrow \mathsf{H}^{\mathsf{mot},T}_\bullet(\mathfrak{U}_{2,k}(\gamma_2, n_2) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}; \gamma_1, n_1); 0)$$

follows from Lemma B.6. Furthermore, the same result shows that the restriction map associated to the open immersion

$$(u_k \times \mathsf{id}) \circ (u_k^\circ \times \mathsf{id})\colon \mathfrak{U}_2^\circ(\gamma_2, n_2) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}; \gamma_1, n_1) \longrightarrow \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2, n_2) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}; \gamma_1, n_1)$$

is also surjective. Since $(u_k \times \mathsf{id})^*$ is surjective, this implies that $(u_k^\circ \times \mathsf{id})^*$ is surjective as well. □

Remark 4.6. Note that the explicit admissible open exhaustion of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n)$ constructed in [DPS+25b, §4.6] is T-equivariantly indstratifiable (cf. Remark A.3). △

Next, using the construction in §3, recall that the closed substack

$$\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2, n_2)^{\geqslant 1} \subset \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2, n_2)$$

is a natural closed complement for the open substack $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2, n_2)^\circ$. For any $k \in I$ let $\mathfrak{U}_{2,k}(\gamma_2, n_2)^{\geqslant 1} \subset \mathfrak{U}_{2,k}(\gamma_2, n_2)$ be the closed substack defined by the pull-back square

$$\begin{array}{ccc} \mathfrak{U}_{2,k}(\gamma_2, n_2)^{\geqslant 1} & \longrightarrow & \mathfrak{U}_k(\gamma_1, n_1) \\ \downarrow & & \downarrow u_k \\ \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2, n_2)^{\geq 1} & \longrightarrow & \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2, n_2) \end{array}\;.$$

By construction, the open substacks

$$\mathfrak{U}_{2,h}(\gamma_2, n_2)^{\geqslant 1} \times \mathfrak{U}_{1,h}(\gamma_1, n_1) \subset \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2, n_2)^{\geqslant 1} \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}; \gamma_1, n_1)\,,$$

for $h \in J$, also form a T-invariant open exhaustion. Then, Propositions B.21 and B.24 yield the following.

Proposition 4.7. *Assume that each of the open exhaustions*

$$\{\mathfrak{U}_{i,k}(\gamma_i, n_i)\}_{k \in I}$$

for $(\gamma_i, n_i) \in \langle Z_i \rangle \times \mathbb{Z}$ and $1 \leq i \leq 2$, and

$$\{\mathfrak{U}_{2,k}(\gamma_2, n_2)^\circ\}_{k \in I} \quad \textit{and} \quad \{\mathfrak{U}_{2,k}(\gamma_2, n_2)^{\geqslant 1}\}_{k \in I}$$

for $(\gamma_2, n_2) \in \langle Z_2\rangle \times \mathbb{Z}$, has a T-equivariant ℓ-cellular structure in the sense of Definition B.22, for some $\ell \geq 0$. Then, Assumption 0.3–(1) and –(2) hold.

Proof. Under the stated conditions, surjectivity of the restriction map $(u_k \times \mathsf{id})^*$ follows from Proposition B.24. More precisely, from the surjectivity of the right vertical map in diagram (B.6). Furthermore, Proposition B.24 also yields the commutative diagram

$$\begin{array}{ccc} \mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2,n_2))\widehat{\otimes}_{\mathsf{H}_T}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1)) & \longrightarrow & \mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2,n_2)\times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1)) \\ \big\downarrow{\scriptstyle (u_k^\circ)^*\otimes \mathsf{id}} & & \big\downarrow{\scriptstyle (u_k^\circ\times \mathsf{id})^*} \\ \mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2,n_2)^\circ)\widehat{\otimes}_{\mathsf{H}_T}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1)) & \longrightarrow & \mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2,n_2)^\circ\times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1)) \end{array},$$

where the horizontal arrows are isomorphisms. Since the closed substacks ${}^{\mathsf{red}}\mathfrak{U}_{2,k}(\gamma_2,n_2)^{\geqslant 1}$ are assumed to be T-equivariantly ℓ-cellular, the negative weights of the motivic homology groups of ${}^{\mathsf{red}}\mathfrak{U}_{2,k}(\gamma_2,n_2)^{\geqslant 1}$ vanish, hence the restriction map $(u_k^\circ)^*$ is surjective for any $k \in I$ by Theorem B.3 and Proposition B.24. This implies that the left vertical arrow in the above diagram is surjective. Hence, the right vertical arrow is also surjective.

Furthermore, under the stated assumptions, by Proposition B.21, the T-equivariant Borel-Moore homology of each of the reduced classical stacks

$$\{{}^{\mathsf{red}}\mathfrak{U}_{i,k}(\gamma_i,n_i)\}_{k\in I}$$

for $(\gamma_i, n_i) \in \langle Z_i\rangle \times \mathbb{Z}$ and $1 \leq i \leq 2$, and

$$\{{}^{\mathsf{red}}\mathfrak{U}_{2,k}(\gamma_2,n_2)^\circ\}_{k\in I} \quad \text{and} \quad \{{}^{\mathsf{red}}\mathfrak{U}_{2,k}(\gamma_2,n_2)^{\geqslant 1}\}_{k\in I}$$

for $(\gamma_2, n_2) \in \langle Z_2\rangle \times \mathbb{Z}$, is strongly generated by algebraic cycles (in the sense of Definition B.7). Hence, Assumption 0.3–(2) also holds. □

Now note the following consequence of Lemma 3.4.

Corollary 4.8. *Suppose that Assumptions 0.1, 0.2, and 0.3 hold. Then, the restriction map $(s_k^a)^*$ is surjective both for (weight zero) motivic as well as constructible T-equivariant Borel-Moore homology for all $a \in \mathbb{Z}$, $k \in I$, and all values of the topological invariants.*

Proof. By Lemma 3.4–(v), it suffices to prove that the composition

$$(p_k^a)_*\circ(\psi_k^a)^*\circ(\xi_k^\circ)^*\circ(q^\circ_{2,1,k})^!\circ(u_k^\circ\times\mathsf{id})^*\circ(u_k\times\mathsf{id})^*\colon$$
$$\mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2,n-a)\times\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1,a))\longrightarrow \mathsf{H}^T_\bullet(\mathfrak{S}_k(\gamma,n)^a)$$

is surjective. By Assumption 0.2–(2) ψ_k^a is an equivalence, while Lemma 3.4 shows that p_k^a and ξ° are equivalences, and $(q^\circ_{1,2,p})^!$ is a degree $\gamma_1\cdot\gamma_2$ isomorphism. Since $(u_k^\circ\times\mathsf{id})^*$ and $(u_k\times\mathsf{id})^*$ are surjective by Assumption 0.3, the claim follows. □

Now, note that for any $a < b$ one has a closed stratification

$$\varnothing\subset\mathfrak{S}_k(\gamma,n)^{b-1,b}\subset\cdots\subset\mathfrak{S}_k(\gamma,n)^{a,b}$$

with locally closed strata $\mathfrak{S}_k(\gamma,n)^c$, with $a \leq c \leq b-1$. For any $a \leq c \leq b-1$ let

$$I^{a,c}\subset\mathsf{H}^T_\bullet(\mathfrak{S}_k(\gamma,n)^{a,b})$$

denote the image of the push-forward map associated to the closed immersion

$$\mathfrak{S}_k(\gamma,n)^{c,b}\hookrightarrow\mathfrak{S}_k(\gamma,n)^{a,b}\quad .$$

Note that one obtains a filtration

$$0 =: I^{b,b}\subset I^{b-1,b}\subset\cdots\subset I^{c,b}\subset\cdots\subset I^{a,b} := \mathsf{H}^T_\bullet(\mathfrak{S}_k(\gamma,n)^{a,b})\ . \tag{4.2}$$

Moreover, each complementary pair

$$\mathfrak{S}_k(\gamma,n)^{c+1,b} \xhookrightarrow{\mathsf{cl}} \mathfrak{S}_k(\gamma,n)^{c,b} \xhookleftarrow{\mathsf{op}} \mathfrak{S}_k(\gamma,n)^{c}$$

yields a commutative diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathsf{H}^T_\bullet(\mathfrak{S}_k(\gamma,n)^{c+1,b}) & \longrightarrow & \mathsf{H}^T_\bullet(\mathfrak{S}_k(\gamma,n)^{c,b}) & \longrightarrow & \mathsf{H}^T_\bullet(\mathfrak{S}^c_k(\gamma,n)) & \longrightarrow \cdots \\ & \big\downarrow r^{c+1,b} & & \big\downarrow r^{c,b} & & & \\ & I^{c+1,b} & \longrightarrow & I^{c,b} & & & \end{array} \tag{4.3}$$

where the top row is the localization exact sequence, the vertical maps are the canonical surjections, and the bottom horizontal arrow is the canonical inclusion. This yields in particular an induced map

$$r^c \colon \mathsf{H}^T_\bullet(\mathfrak{S}_k(\gamma,n)^c) \longrightarrow I^{c,a}/I^{c+1,a}$$

for any $a \leq c \leq b-1$. By straightforward diagram chasing, one shows:

Lemma 4.9. *r^c is surjective.*

Remark 4.10. For equivariant constructible Borel-Moore homology, Corollary 4.8 shows that the restriction map $\mathsf{H}^T_\bullet(\mathfrak{S}_k(\gamma,n)^{c,b}) \to \mathsf{H}^T_\bullet(\mathfrak{S}_k(\gamma,n)^c)$ is surjective for all $a \leq c \leq b$. Therefore, in this case the top sequence in diagram (4.3) splits into three term short exact sequences. This implies that the maps $r^{c,b}$ and r^c are isomorphisms for all $a \leq c \leq b$. △

Using Lemmas 3.4 and 4.9. one obtains the following.

Proposition 4.11. *Suppose that Assumptions 0.1, 0.2, and 0.3 hold. For any given (γ,n) and $k \in I$, let b be the constant in Assumption 0.2–(1). Then, the convolution map*

$$\bigoplus_{n_1 \geqslant a} \mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2, n-n_1) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1,n_1)) \xrightarrow{v_k^{\geqslant a}} \mathsf{H}^T_\bullet(\mathfrak{S}_k(\gamma,n)^{\geqslant a})$$

is surjective for any $a \in \mathbb{Z}$.

Proof. By Assumption 0.2–(1), we have $\mathfrak{S}_k(\gamma,n)^{a,b} = \mathfrak{S}_k(\gamma,n)^{\geqslant a}$. In particular, $\mathsf{H}^T_\bullet(\mathfrak{S}_k(\gamma,n)^{\geqslant a})$ admits a filtration as in (4.2). Given Definition 4.1–(ii), the composition map $(\jmath_k^{a,b})^* \circ v_k^{\geqslant a}$ factors through the canonical projection

$$\bigoplus_{n_1 \geqslant a} \mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2, n-n_1) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1,n_1)) \longrightarrow \bigoplus_{n_1 = a}^{b-1} \mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2, n-n_1) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1,n_1)) \, .$$

Moreover, by Corollary 4.4–(ii), the resulting map

$$\bigoplus_{n_1 = a}^{b-1} \mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2, n-n_1) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1,n_1)) \longrightarrow \mathsf{H}^T_\bullet(\mathfrak{S}_k(\gamma,n)^{a,b}) \tag{4.4}$$

maps the canonical subspace

$$\bigoplus_{n_1 = c}^{b-1} \mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2, n-n_1) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1,n_1))$$

to $I^{c,b}$ for any $a \leq c \leq b-1$. Hence the map (4.4) is a map of filtered H_T-modules, where we use the filtration (4.2) for the target.

Using the second equation in Lemma 3.4–(v), it follows that the induced map of associated graded spaces is diagonal, with components

$$r^c \circ (p^c_k)_* \circ (\psi^c_k)^* \circ (\xi^\circ_k)^* \circ (q^\circ_{2,1,k})^! \circ (u^\circ_k \times \mathsf{id})^* \circ (u_k \times \mathsf{id})^* :$$

$$\mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2, n-c) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}; \gamma_1, c)) \longrightarrow I^{c,b}/I^{c+1,b}$$

Since r^c is surjective by Lemma 4.9, it suffices to show that the map

$$(p^c_k)_* \circ (\psi^c_k)^* \circ (\xi^\circ_k)^* \circ (q^\circ_{2,1,k})^! \circ (u^\circ_k \times \mathsf{id})^* \circ (u_k \times \mathsf{id})^* \colon \mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2, n-c) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}; \gamma_1, c)) \longrightarrow \mathsf{H}^T_\bullet(\mathfrak{S}^c_k(\gamma, n))$$

is surjective for all $a \leq c \leq b-1$. By Lemma 3.4–(iii) and Assumption 0.2–(2), each stratum $\mathfrak{S}_k(\gamma, n)^c$ is identified via $p^c_k \circ (\psi^c_k)^{-1}$ to the the upper left corner of the pullback diagram

$$\begin{array}{ccc} \mathfrak{X}^\circ & \longrightarrow & \mathbf{Coh}^{\mathsf{ext}}_{2,1}(\widehat{X}_Z; (\gamma_1, n_1), (\gamma_2, n_2)) \\ \downarrow{\scriptstyle q^\circ_{2,1,k}} & & \downarrow \\ \mathfrak{U}_{2,k}(\gamma_2, n-c)^\circ \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}; \gamma_1, c) & \longrightarrow & \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2, n-c) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}; \gamma_1, c) \end{array} .$$

Moreover, by Lemma 3.4–(iv), the left column is a vector bundle stack. By Assumption 0.2–(2), ψ^c_k is an equivalence, while Lemma 3.4 shows that p^c_k and ξ°_k are equivalences, and $(q^\circ_{2,1,k})^!$ is a degree $\gamma_1 \cdot \gamma_2$ isomorphism. Since $(u^\circ_k \times \mathsf{id})^*$ and $(u_k \times \mathsf{id})^*$ are surjective by Assumption 0.3, the claim follows. □

In order to obtain a second "conditional" surjectivity result, we need to impose the following assumption.

Assumption 0.4 (Surjectivity of restriction maps II)**.** For any $k \in I$ and any invariant (γ, n),

(1) there exists an open embedding $\mathfrak{U}_k(\gamma, n) \to \mathfrak{S}_k(\gamma, n)$;

(2) there exists a constant $a \in \mathbb{Z}$ so that:
   (i) the open embedding $\mathfrak{U}_k(\gamma, n) \to \mathfrak{S}_k(\gamma, n)$ factors via an open immersion
   $$\rho_k \colon \mathfrak{U}_k(\gamma, n) \longrightarrow \mathfrak{S}_k(\gamma, n)^{\geqslant b}$$
   for all $b \leq a$;
   (ii) a is maximal with this property.

(3) (Constructible version). For any $b \leq a$, where a is the constant in (2), the open restriction
$$\rho^*_k \colon \mathsf{H}^T_\bullet(\mathfrak{S}_k(\gamma, n)^{\geqslant b}) \longrightarrow \mathsf{H}^T_\bullet(\mathfrak{U}_k(\gamma, n))$$
is surjective.

(4) (Motivic version). For any $b \leq a$, where a is the constant in (2), the open restriction
$$\rho^*_k \colon \mathsf{H}^{\mathsf{mot},T}_\bullet(\mathfrak{S}_k(\gamma, n)^{\geqslant b}; 0) \longrightarrow \mathsf{H}^{\mathsf{mot},T}_\bullet(\mathfrak{U}_k(\gamma, n); 0)$$
is surjective.

⊘

Remark 4.12. For any $k \geq 0$, let $\sigma^{\geqslant a}_k \colon \mathfrak{S}_k(\gamma, n)^{\geqslant a} \to \mathfrak{S}_k(\gamma, n)$ be the canonical closed immersion. Then, $\rho^*_k = \mathsf{res}'_k \circ (\sigma^{\geqslant a}_k)_*$, where res'_k is restriction map $\mathsf{res}'_k \colon \mathsf{H}^T_\bullet(\mathfrak{S}_k(\gamma, n)) \to \mathsf{H}^T_\bullet(\mathfrak{U}_k(\gamma, n))$. Under the above assumption, the res'_k is surjective as well. △

Fix $k \in I$, for any $\ell \in I$, let $\mathfrak{V}_{k,\ell}(\gamma, n)^{\geqslant b} \subset \mathfrak{S}_k(\gamma, n)^{\geqslant b}$ be the open substack defined by the pull-back square

$$\begin{array}{ccc} \mathfrak{V}_{k,\ell}(\gamma, n)^{\geqslant b} & \longrightarrow & \mathfrak{S}_k(\gamma, n)^{\geqslant b} \\ \downarrow & & \downarrow \\ \mathfrak{U}_\ell(\gamma, n) & \longrightarrow & \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n) \end{array} ,$$

Note that the collection $\{\mathfrak{V}_{k,\ell}(\gamma,n)^{\geqslant b}\}_{\ell\in I}$ is a T-invariant open exhaustion of $\mathfrak{S}_k(\gamma,n)^{\geqslant b}$. Then Lemma B.6 and Proposition B.8 yield:

Proposition 4.13. *The following hold for arbitrary $k\in I$, $a\in\mathbb{Z}$ and $(\gamma,n)\in\langle Z\rangle\times\mathbb{Z}$.*

(i) If the open exhaustion $\{{}^{\mathsf{red}}\mathfrak{V}_{k,\ell}(\gamma,n)^{\geqslant b}\}_{\ell\in I}$ is T-equivariantly ind stratifiable, then Assumption 0.4–(4) holds. More explicitly, the restriction map

$$\rho_k^*\colon \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathfrak{S}_k(\gamma,n)^{\geqslant b};0)\longrightarrow \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathfrak{U}_k(\gamma,n);0)$$

is surjective for any $b\leq a$.

(ii) In addition, if the T-equivariant Borel-Moore homology of the reduced classical stack ${}^{\mathsf{red}}\mathfrak{U}_k(\gamma,n)$ is generated by algebraic cycles in the sense of Definition B.7, then Assumption 0.4–(3) also holds.

Remark 4.14. By Proposition B.21, Proposition 4.13–(ii) holds in particular if the open exhaustion $\{\mathfrak{U}_k(\gamma,n)\}_{k\in I}$ has a T-equivariant ℓ-cellular structure in the sense of Definition B.22, for some $\ell\geq 0$. △

Now let

$$\sigma_{k,n}\colon \bigoplus_{n_1}\mathsf{H}_\bullet^T(\mathfrak{U}_{2,k}(\gamma_2,n-n_1)\times\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_1};\gamma_1,n_1))\longrightarrow\mathsf{H}_\bullet^T(\mathfrak{U}_k(\gamma,n)) \tag{4.5}$$

be the unique homomorphism of H_T-modules which restricts to $\rho_k^*\circ\nu_k^{\geqslant a}$ on each subspace of the domain defined by $n_1\geq a$, where $a\in\mathbb{Z}$ is smaller or equal to the constant appearing in Assumption 0.4.

In order to conclude this section, note that Proposition 4.11 yields:

Theorem 4.15. *Suppose that Assumptions 0.1, 0.2, 0.3, and 0.4 hold. Then, for any values of the topological invariants and for any $k\in I$, one has a canonical commutative diagram*

$$\begin{array}{ccc}
\displaystyle\bigoplus_{n_1\in\mathbb{Z}}\mathsf{H}_\bullet^T(\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_2};\gamma_2,n_2)\times\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_1};\gamma_1,n_1)) & \xrightarrow{\mu_n} & \mathsf{H}_\bullet^T(\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z;\gamma,n))\\
\big\downarrow{\scriptstyle (u_k\times\mathsf{id})^*} & & \big\downarrow{\scriptstyle \mathsf{res}_k}\\
\displaystyle\bigoplus_{n_1\in\mathbb{Z}}\mathsf{H}_\bullet^T(\mathfrak{U}_{2,k}(\gamma_2,n-n_1)\times\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_1};\gamma_1,n_1)) & \xrightarrow{\sigma_{k,n}} & \mathsf{H}_\bullet^T(\mathfrak{U}_k(\gamma,n))
\end{array},$$

where the vertical arrows are restriction maps induced by the canonical open immersions, and the bottom horizontal arrow is surjective. In particular, $\mathsf{res}_k\circ\mu_n$ is surjective.

4.3. **Composition with the Künneth map.** For any values of the topological invariants, one has a natural Künneth map

$$\mathsf{H}_\bullet^T(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_2};\gamma_2,n_2))\widehat{\otimes}_{\mathsf{H}_T^\bullet}\mathsf{H}_\bullet^T(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_1};\gamma_1,n_1))\longrightarrow\mathsf{H}_\bullet^T(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_2};\gamma_2,n_2)\times\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_1};\gamma_1,n_1))\ .$$

By composition, this yields a multiplication map

$$\begin{array}{ccc}
\displaystyle\bigoplus_{n_1\in\mathbb{Z}}\mathsf{H}_\bullet^T(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_2};\gamma_2,n_2))\widehat{\otimes}_{\mathsf{H}_T^\bullet}\mathsf{H}_\bullet^T(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_1};\gamma_1,n_1)) & & \\
\big\downarrow & \searrow{\scriptstyle \mathfrak{m}_n} & \\
\displaystyle\bigoplus_{n_1\in\mathbb{Z}}\mathsf{H}_\bullet^T(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_2};\gamma_2,n_2)\times\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_1};\gamma_1,n_1)) & \xrightarrow{\mu_n} & \mathsf{H}_\bullet^T(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z;\gamma,n))
\end{array},$$

for any decomposition $(\gamma,n)=(\gamma_1,n_1)+(\gamma_2,n_2)$.

Now recall that $Z_{1,2}$ denotes the scheme-theoretic intersection of Z_1 and Z_2 in X. Let $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_{1,2}};m)$ denote the derived moduli stack of length m zero-dimensional coherent sheaves on X with set-theoretic support contained in $(Z_{1,2})_{\mathsf{red}}$.

For any $n_1, n_2, m \in \mathbb{Z}$, $m \geq 1$, one obtains analogously the multiplication maps

$$\kappa^1_{n_1,m}\colon \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_{1,2}}; m))\widehat{\otimes}_{\mathsf{H}^\bullet_T}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}; \gamma_1, n_1)) \longrightarrow \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}; \gamma_1, n_1+m)) ,$$

$$\kappa^2_{n_2,m}\colon \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2, n_2-m))\widehat{\otimes}_{\mathsf{H}^\bullet_T}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_{1,2}}; m)) \longrightarrow \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2, n_2)) .$$

For any $\gamma_i \in \langle Z_i\rangle$ and $n_i \in \mathbb{Z}$, with $1 \leq i \leq 2$, let

$$\kappa_n\colon \bigoplus_{n_1+n_2=n}\bigoplus_{m\geq 1} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2, n_2-m))\widehat{\otimes}_{\mathsf{H}^\bullet_T}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_{1,2}}; m))\widehat{\otimes}_{\mathsf{H}^\bullet_T}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}; \gamma_1, n_1))$$
$$\longrightarrow \bigoplus_{n_1+n_2=n} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}; \gamma_1, n_1))\widehat{\otimes}_{\mathsf{H}^\bullet_T}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2, n_2))$$

be the unique homomorphism of H_T-modules which restricts to $\kappa^2_{n_2,m}\otimes \mathsf{id} - \mathsf{id}\otimes\kappa^1_{n_1,m}$ on each direct summand on the left-hand-side.

The following holds from associativity of the Hall multiplication.

Lemma 4.16. *One has* $\mathfrak{m}_n \circ \kappa_n = 0$ *for any* $n \in \mathbb{Z}$.

Now, for any $\gamma \in \langle Z\rangle$ and $n \in \mathbb{Z}$, let $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n)$ be the derived moduli stack of coherent sheaves on X, set-theoretically supported on Z, with first Chern class γ and Euler characteristic n, and similarly for $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_i}; \gamma_i, n_i)$ with $i = 1, 2$. Set

$$\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma) \coloneqq \bigsqcup_{n\in\mathbb{Z}} \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n) ,$$

and similarly for $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_i}; \gamma_i)$. Then, for any $\gamma \in \langle Z\rangle$, we have

$$\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma)) \coloneqq \bigoplus_{n\in\mathbb{Z}} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n)) .$$

Moreover, fix $1 \leq i \leq 2$. Then, for any $\gamma_i \in \langle Z_i\rangle$, we have

$$\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_i}; \gamma_i)) \coloneqq \bigoplus_{n_i\in\mathbb{Z}} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_i}; \gamma_i, n_i)) .$$

Also, let $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_{1,2}}; m)$ be the derived moduli stack of coherent sheaves on X, set-theoretically supported on $(Z_{1,2})_{\mathsf{red}}$, with Euler characteristic m. Then, recall that $\mathbf{HA}^T_{X,Z_{1,2}}$ denote the cohomological Hall algebra whose underlying vector space is

$$\bigoplus_{m\geq 0} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_{1,2}}; m)) .$$

Recall that under the current assumptions, any numerical class $\gamma \in \langle Z\rangle$ has a unique decomposition $\gamma = \gamma_1 + \gamma_2$ with $\gamma_i \in \langle Z_i\rangle$ and $1 \leq i \leq 2$. Then, let

$$\mathfrak{a}_\gamma\colon \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2))\widehat{\otimes}_{\mathsf{H}^\bullet_T}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}; \gamma_1)) \longrightarrow \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma))$$

be the unique homomorphism of H_T-modules which restricts to $\mathfrak{m}_n$ on the degree n summand of the graded tensor product

$$\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2))\widehat{\otimes}_{\mathsf{H}^\bullet_T}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}; \gamma_1)) =$$
$$\bigoplus_{n\in\mathbb{Z}}\bigoplus_{n_1+n_2=n} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2, n_2))\widehat{\otimes}_{\mathsf{H}^\bullet_T}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}; \gamma_1, n_1)) .$$

Note the natural left/right actions

$$\mathbf{HA}^T_{X,Z_{1,2}}\widehat{\otimes}_{\mathsf{H}^\bullet_T}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}; \gamma_1)) \longrightarrow \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}; \gamma_1)) ,$$

$$\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2))\widehat{\otimes}_{\mathsf{H}^\bullet_T}\mathbf{HA}^T_{X,Z_{1,2}} \longrightarrow \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2)) ,$$

defined by the Hall multiplication maps $\kappa^1_{n_1,m}$ and $\kappa^2_{n_2,m}$. Then, Lemma 4.16 implies that the map $\mathfrak{a}_\gamma$ descends to a map

$$\overline{\mathfrak{a}}_\gamma\colon \bigoplus_{\gamma_1+\gamma_2=\gamma} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2))\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_{1,2}}} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1)) \longrightarrow \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma)) .$$

We will formulate below a "conditional" surjectivity result for this map, depending also on the following assumption.

Assumption 0.5 (Künneth isomorphism for open exhaustions)**.** The Künneth map

$$\mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2,n_2))\widehat{\otimes}_{\mathsf{H}^\bullet_T}\mathsf{H}^T_\bullet(\mathfrak{U}_{1,k}(\gamma_1,n_1)) \longrightarrow \mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2,n_2)\times\mathfrak{U}_{1,k}(\gamma_1,n_1))$$

is an isomorphism for all $k\in I$ and for all values of the topological invariants (γ_1,n_1) and (γ_2,n_2). ⊘

Remark 4.17. We denote by $\mathfrak{U}_k(\gamma)$ the disjoint union of the stacks $\mathfrak{U}_k(\gamma,n)$ by varying of $n\in\mathbb{Z}$, and similarly for $\mathfrak{U}_{i,k}(\gamma_i)$ for $i=1,2$.

We also denote by

$$\sigma_k\colon \mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2)\times\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1)) \longrightarrow \mathsf{H}^T_\bullet(\mathfrak{U}_k(\gamma))$$

the unique homomorphism of H_T-modules with components $\sigma_{k,n}$ defined in Equation (4.5). Moreover, by composition with the Künneth map, $\sigma_{k,n}$ determines an H_T-module homomorphism

$$\tau_{k,n}\colon \bigoplus_{n_1+n_2=n} \mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2,n_2))\widehat{\otimes}_{\mathsf{H}^\bullet_T}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1,n_1)) \longrightarrow \mathsf{H}^T_\bullet(\mathfrak{U}_k(\gamma,n))$$

for each $n\in\mathbb{Z}$. Let

$$\tau_k\colon \mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2))\widehat{\otimes}_{\mathsf{H}^\bullet_T}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1)) \longrightarrow \mathsf{H}^T_\bullet(\mathfrak{U}_k(\gamma))$$

be the unique homomorphism of H_T-modules with components τ_n. △

Proposition B.21 implies the following.

Proposition 4.18. *If both open exhaustions $\{\mathfrak{U}_{i,k}(\gamma_i,n_i)\}_{k\in I}$ with $i=1,2$ are T-equivariantly ℓ-cellular in the sense of Definition B.22 for some $\ell\geq 0$, then Assumption 0.5 holds.*

Assumption 0.6 (Induced action on the homology modules of quasi-compact opens)**.** For each $k\in I$ and each $\gamma_2\in\langle Z_2\rangle$, one has an action

$$\mathsf{H}^T_\bullet(\mathfrak{U}_k(\gamma_2))\widehat{\otimes}_{\mathsf{H}^\bullet_T}\mathbf{HA}^T_{X,Z_{1,2}} \longrightarrow \mathsf{H}^T_\bullet(\mathfrak{U}_k(\gamma_2)) ,$$

which is compatible with the action

$$\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2))\widehat{\otimes}_{\mathsf{H}^\bullet_T}\mathbf{HA}^T_{X,Z_{1,2}} \longrightarrow \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2)) .$$

⊘

Theorem 4.19. *Suppose that Assumptions 0.1, 0.2, 0.3–(1), 0.4–(1), 0.4–(2), 0.4–(3), 0.5, and 0.6 hold. Then, for any $\gamma\in\langle Z\rangle$ there is a natural commutative diagram*

$$\begin{array}{ccc}
\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2))\ \widehat{\otimes}_{\mathbf{HA}^T_{X,Z_{1,2}}}\ \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1)) & \xrightarrow{\overline{\mathfrak{a}}_\gamma} & \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma)) \\
\big\downarrow{\scriptstyle (u_k\times\mathsf{id})^\ast} & & \big\downarrow{\scriptstyle \mathsf{res}_k} \\
\mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2))\ \widehat{\otimes}_{\mathbf{HA}^T_{X,Z_{1,2}}}\ \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1)) & \xrightarrow{\overline{\tau}_k} & \mathsf{H}^T_\bullet(\mathfrak{U}_k(\gamma))
\end{array} \tag{4.6}$$

where the vertical arrows, as well as the bottom horizontal arrow are surjective. In particular, the composition $\mathsf{res}_k\circ\overline{\mathfrak{a}}_\gamma$ is surjective.

Proof. Under the stated conditions, Theorem 4.15 and Proposition B.24 yield a commutative diagram

$$\begin{array}{ccc} \displaystyle\bigoplus_{n_1+n_2=n} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2,n_2))\widehat{\otimes}_{\mathsf{H}^\bullet_T}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1,n_1)) & \xrightarrow{\mathfrak{m}_n} & \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)) \\ \downarrow{\scriptstyle (u_k\times \mathsf{id})^*} & & \downarrow{\scriptstyle \mathsf{res}_k} \\ \mathsf{H}^T_\bullet(\mathfrak{U}_{2,k}(\gamma_2))\,\widehat{\otimes}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1)) & \xrightarrow{\tau_{k,n}} & \mathsf{H}^T_\bullet(\mathfrak{U}_k(\gamma,n)) \end{array}$$

for any $\gamma_i \in \langle Z_i\rangle$, with $1 \leq i \leq 2$. The top horizontal arrow factors through the projection

$$\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2))\,\widehat{\otimes}_{\mathsf{H}^\bullet_T}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1)) \longrightarrow \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2};\gamma_2))\,\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_{1,2}}}\mathsf{H}^T_\bullet(\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1};\gamma_1))$$

by Lemma 4.16. The same argument proves that the analogous factorization holds for the bottom horizontal arrow. This yields the diagram (4.6) with the required properties. □

In view of Propositions 4.7, 4.13, and 4.18, we have the following.

Corollary 4.20. *Suppose that Assumptions 0.1, 0.4–(1), and 0.4–(2) hold. If the admissible open exhaustions $\{\mathfrak{U}_{i,k}(\gamma_i,n_i)\}_{k\in I}$ for $1 \leq i \leq 2$, $\{\mathfrak{U}_{2,k}(\gamma_2,n_2)^\circ\}_{k\in I}$, $\{\mathfrak{U}^{\geq 1}_{2,k}(\gamma_2,n_2)^\circ\}_{k\in I}$ and $\{\mathfrak{U}_k(\gamma,n)\}_{k\in I}$ are T-equivariantly ℓ-cellular in the sense of Definition B.22, then Theorem 4.19 holds.*

4.4. **Iteration.** Let

$$\emptyset =: Z^{-1} = Z^0 \subset Z^1 \subset \cdots \subset Z^s := Z$$

be a stratification as in §2 satisfying Assumption 0.1. Recall that the scheme-theoretic closure of the complement $Z_i^\circ := Z^i \smallsetminus Z^{i-1}$ is denoted by Z_i, for each $1 \leq i \leq s$. Let Z_i^{i-1} denote the scheme-theoretic intersection of Z^{i-1} and Z_i, which is zero-dimensional. Let $\mathbf{HA}^T_{X,Z_i^{i-1}}$ denote the T-equivariant cohomological Hall algebra structure on the Borel-Moore homology of the derived moduli stack of zero-dimensional sheaves on X with set-theoretic support contained in $(Z_i^{i-1})_{\mathsf{red}}$.

For each $1 \leq i \leq s$, let $W^i \subset Z$ be the scheme theoretic closure of the complement $Z \smallsetminus Z^{i-1}$. Hence W^i is a reduced closed subscheme of Z, containing Z_i as a closed subscheme and the scheme-theoretic closure of the complement $W^i \smallsetminus Z_i$ coincides with W^{i+1}. Moreover, the scheme-theoretic intersection of W^i and Z_{i-1} coincides with Z_i^{i-1}. In particular, note that $W^1 = Z$ and $W^s = Z_s$.

Let $\gamma \in \langle Z\rangle$ and note that γ has a unique decomposition $\gamma = \sum_{i=1}^s \gamma_i$ with $\gamma_i \in \langle Z_i\rangle$, for $1 \leq i \leq s$. Now, set $\gamma^i := \sum_{j=i}^s \gamma_i$. In particular, $\gamma^1 = \gamma$. Denote by $\mathfrak{U}_{i,k}(\gamma^i)$ the quasi-compact quasi-separated indgeometric stack defined by

$$\begin{array}{ccc} \mathfrak{U}_{i,k}(\gamma^i) & \longrightarrow & \mathfrak{U}_k(\gamma^i) \\ \downarrow & & \downarrow \\ \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{W^i};\gamma^i) & \longrightarrow & \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma_i) \end{array}.$$

Let

$$\mathsf{res}^i_k \colon \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{W^i};\gamma^i)) \longrightarrow \mathsf{H}^T_\bullet(\mathfrak{U}_{i,k}(\gamma^i))$$

be the associated restriction map.

Applying Theorem 4.19 to the one-step stratification $\emptyset \subset Z_{i-1} \subset W^{i-1}$, one obtains an amalgamation map

$$\overline{\mathfrak{a}}_{\gamma^{i-1}} \colon \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{W^i};\gamma^i))\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_i^{i-1}}}\mathsf{H}^T(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_{i-1}};\gamma_{i-1})) \longrightarrow \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{W^{i-1}};\gamma^{i-1}))\,.$$

By iteration, this yields an amalgamation map

$$\overline{\mathfrak{a}}_\gamma\colon \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_s},\gamma_s))\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_s^{s-1}}}\cdots\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_2^1}}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1},\gamma_1))\longrightarrow \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z,\gamma))\ .$$

Furthermore, for any fixed $k\in I$, an analogous construction using Theorem 4.19 yields a map

$$\overline{\tau}_{k,\gamma}\colon \mathsf{H}^T_\bullet(\mathfrak{U}_{s,k}(\gamma_s))\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_s^{s-1}}}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_{s-1}},\gamma_{s-1}))\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_{s-1}^{s-2}}}\cdots\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_2^1}}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1},\gamma_1))\longrightarrow \mathsf{H}^T_\bullet(\mathfrak{U}_k(\gamma))\ .$$

Theorem 4.21. *Suppose that Assumptions 0.1, 0.2, 0.3–(1), 0.4–(1), 0.4–(2), 0.4–(3), 0.5, and 0.6 hold for each one-step stratification $Z_i\subset W^i$ for $1\leq i\leq s$. Then, for any $\gamma\in\langle Z\rangle$ the composition*

$$\begin{array}{ccc}
\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_s},\gamma_s))\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_s^{s-1}}}\cdots\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_2^1}}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1},\gamma_1)) & \xrightarrow{\overline{\mathfrak{a}}_\gamma} & \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma)) \\
\downarrow & & \downarrow{\scriptstyle \mathsf{res}_k} \\
\mathsf{H}^T_\bullet(\mathfrak{U}_{s,k}(\gamma_s))\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_s^{s-1}}}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_{s-1}},\gamma_{s-1}))\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_{s-1}^{s-2}}}\cdots\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_2^1}}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1},\gamma_1)) & \xrightarrow{\overline{\tau}_{k,\gamma}} & \mathsf{H}^T_\bullet(\mathfrak{U}_k(\gamma))
\end{array}$$

where the vertical arrows are determined by the restriction maps associated to the open immersions $\mathfrak{U}_{s,k}(\gamma_i)\to\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_s};\gamma_s)$ and $\mathfrak{U}_k(\gamma)\to\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma)$, respectively. Moreover, both horizontal arrows as well as the bottom horizontal arrow are surjective. In particular the composition

$$\begin{array}{ccc}
\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_s},\gamma_s))\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_s^{s-1}}}\cdots\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_2^1}}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1},\gamma_1)) & \xrightarrow{\overline{\mathfrak{a}}_\gamma} & \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma)) \\
 & & \downarrow{\scriptstyle \mathsf{res}_k} \\
 & & \mathsf{H}^T_\bullet(\mathfrak{U}_k(\gamma))
\end{array}$$

is surjective for all $k\in I$ and for all $\gamma\in\langle Z\rangle$.

Proof. By Theorem 4.19, for each $1\leq i\leq s$, one has the commutative diagram

$$\begin{array}{ccc}
\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{W^i};\gamma^i))\ \widehat{\otimes}_{\mathbf{HA}^T_{X,Z_i^{i-1}}}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_{i-1}};\gamma_{i-1})) & \xrightarrow{\overline{\mathfrak{a}}_{\gamma^i}} & \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{W^{i-1}};\gamma^{i-1})) \\
\downarrow{\scriptstyle \mathsf{res}^i_k\otimes\mathsf{id}} & & \downarrow{\scriptstyle \mathsf{res}^{i-1}_k} \\
\mathsf{H}^T_\bullet(\mathfrak{U}_{i,k}(\gamma^i))\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_i^{i-1}}}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_{i-1}};\gamma_{i-1})) & \xrightarrow{\overline{\tau}^i_k} & \mathsf{H}^T_\bullet(\mathfrak{U}_{i-1,k}(\gamma^{i-1}))
\end{array}\ .$$

Here, the map $\overline{\tau}^i_k$ is introduced in Remark 4.17. As shown in *loc. cit.* under the current assumptions, the maps $\mathsf{res}^i_k\otimes\mathsf{id}$, σ^i_k, and res^{i-1}_k in the above diagram are surjective for any $k\in I$.

Iterating this construction, and using the abbreviated notation

$$\mathsf{H}^T_\bullet(\gamma^i)\coloneqq\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{W^i};\gamma^i))\ ,$$

$$\mathsf{H}^T_\bullet(\gamma_i)\coloneqq\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_i};\gamma_i))\ ,$$

$$\mathsf{H}^T_\bullet(\gamma^i,\gamma_{i-1},\gamma_{i-2})\coloneqq\mathsf{H}^T_\bullet(\gamma^i)\ \widehat{\otimes}_{\mathbf{HA}^T_{X,Z_i^{i-1}}}\mathsf{H}^T_\bullet(\gamma_{i-1})\ \widehat{\otimes}_{\mathbf{HA}^T_{X,Z_{i-1}^{i-2}}}\mathsf{H}^T_\bullet(\gamma_{i-2})\ ,$$

one obtains the following commutative diagram

$$\begin{array}{ccc}
\mathsf{H}^T_\bullet(\gamma^i,\gamma_{i-1},\gamma_{i-2}) & \xrightarrow{\mathsf{res}^i_k\otimes\mathsf{id}\otimes\mathsf{id}} & \mathsf{H}^{\mathsf{BM}}_\bullet(\mathfrak{U}_{i,k}(\gamma^i))\,\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_i^{i-1}}}\mathsf{H}^T_\bullet(\gamma_{i-1})\,\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_{i-1}^{i-2}}}\mathsf{H}^T_\bullet(\gamma_{i-2}) \\
\downarrow{\scriptstyle \overline{\mathfrak{a}}_{\gamma^i}\otimes\mathsf{id}} & & \downarrow{\scriptstyle \overline{\tau}^i_k\otimes\mathsf{id}} \\
\mathsf{H}^T_\bullet(\gamma^{i-1})\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_{i-1}^{i-2}}}\mathsf{H}^T_\bullet(\gamma_{i-2}) & \xrightarrow{\mathsf{res}^{i-1}_k\otimes\mathsf{id}} & \mathsf{H}^T_\bullet(\mathfrak{U}_{i-1,k}(\gamma^{i-1}))\widehat{\otimes}_{\mathbf{HA}^T_{X,Z_{i-1}^{i-2}}}\mathsf{H}^T_\bullet(\gamma_{i-2}) \\
\downarrow{\scriptstyle \overline{\mathfrak{a}}_{\gamma^{i-1}}} & & \downarrow{\scriptstyle \overline{\tau}^{i-1}_k} \\
\mathsf{H}^T_\bullet(\gamma^{i-2}) & \xrightarrow{\mathsf{res}^{i-2}_k} & \mathsf{H}^T_\bullet(\mathfrak{U}_{i-2,k}(\gamma^{i-2}))
\end{array}\ .$$

Theorem 4.19 yields that the vertical arrows on the left-hand-side, as well as the horizontal arrows are surjective. Hence the composition

$$\mathsf{res}^{i-2}_k\circ\overline{\mathfrak{a}}_{\gamma^{i-1}}\circ\overline{\mathfrak{a}}_{\gamma^i}\otimes\mathsf{id}$$

is surjective. □

We finish this section by providing a motivic variant of the previous theorem. For $1\leq i\leq s$, set

$$\mathsf{H}^{\mathsf{mot},T}_\bullet(\gamma_i,\dots,\gamma_1)\coloneqq\mathsf{H}^{\mathsf{mot},T}_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_i},\gamma_i);0)\widehat{\otimes}_{\mathbf{HA}^{\mathsf{mot},T}_{X,Z_i^{i-1}}}\cdots\widehat{\otimes}_{\mathbf{HA}^{\mathsf{mot},T}_{X,Z_2^1}}\mathsf{H}^{\mathsf{mot},T}_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1},\gamma_1);0)\ .$$

Theorem 4.22. *Suppose that Assumptions 0.1, 0.2, 0.3–(2), 0.4–(1), 0.4–(2), 0.4–(4), 0.5, and 0.6 hold for each one-step stratification $Z_i\subset W^i$ for $1\leq i\leq s$. Then, for any $\gamma\in\langle Z\rangle$ the composition*

$$\begin{array}{ccc}
\mathsf{H}^{\mathsf{mot},T}_\bullet(\gamma_s,\dots,\gamma_1) & \xrightarrow{\overline{\mathfrak{a}}_\gamma} & \mathsf{H}^{\mathsf{mot},T}_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma);0) \\
\downarrow & & \downarrow{\scriptstyle \mathsf{res}_k} \\
\mathsf{H}^{\mathsf{mot},T}_\bullet(\mathfrak{U}_{s,k}(\gamma_s);0)\widehat{\otimes}_{\mathbf{HA}^{\mathsf{mot},T}_{X,Z_s^{s-1}}}\mathsf{H}^{\mathsf{mot},T}_\bullet(\gamma_{s-1},\dots,\gamma_1;0) & \xrightarrow{\overline{\tau}_{k,\gamma}} & \mathsf{H}^{\mathsf{mot},T}_\bullet(\mathfrak{U}_k(\gamma);0)
\end{array}$$

where the vertical maps are determined by the restriction maps associated to the open immersions $\mathfrak{U}_{s,k}(\gamma_s)\to\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_s};\gamma_s)$ and $\mathfrak{U}_k(\gamma)\to\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma)$, respectively. Moreover, both horizontal arrows as well as the bottom horizontal arrow are surjective. In particular the composition

$$\begin{array}{ccc}
\mathsf{H}^{\mathsf{mot},T}_\bullet(\gamma_s,\dots,\gamma_1) & \xrightarrow{\overline{\mathfrak{a}}_\gamma} & \mathsf{H}^{\mathsf{mot},T}_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma);0) \\
 & & \downarrow{\scriptstyle \mathsf{res}_k} \\
 & & \mathsf{H}^{\mathsf{mot},T}_\bullet(\mathfrak{U}_k(\gamma);0)
\end{array}$$

is surjective for all $k\in I$ and for all $\gamma\in\langle Z\rangle$.

5. Explicit construction of open exhaustions

In this section, we construct explicit examples of the open exhaustions $\{\mathfrak{U}_k(\gamma,n)\}_{k\in\mathbb{N}}$ and $\{\mathfrak{S}_k(\gamma,n)\}_{k\in\mathbb{N}}$ used in the previous sections.

We shall recall a notion of semistability for properly supported torsion sheaves on X. Let H denote an ample divisor[10] on X and let $B\in N_1(X)_\mathbb{R}$.

[10] When X is only quasi-projective, we fix an ample divisor on a smooth projective compactification of X.

Definition 5.1. Given a pure one-dimensional sheaf $\mathcal{E}$ on X with $\mathsf{ch}_1(\mathcal{E}) \neq 0$, the *$(H, B)$-slope of* $\mathcal{E}$ is defined as

$$\mu_{(H,B)}(\mathcal{E}) \coloneqq \frac{\chi(\mathcal{E}) + B \cdot \mathsf{ch}_1(\mathcal{E})}{H \cdot \mathsf{ch}_1(\mathcal{E})} . \tag{5.1}$$

⊘

This yields a natural notion of (H, B)-stability. In particular, any one-dimensional sheaf $\mathcal{E}$ with $\mathsf{ch}_1(\mathcal{E}) \neq 0$ admits a unique Harder-Narasimhan filtration

$$0 \subset \mathcal{H}_0 \subset \mathcal{H}_1 \subset \cdots \subset \mathcal{H}_\ell \coloneqq \mathcal{E}$$

where $\mathcal{H}_0 \subset \mathcal{E}$ is the maximal zero-dimensional subsheaf. Employing standard terminology, the *minimal (H, B)-slope of* $\mathcal{E}$ is defined by

$$\mu_{(H,B)\text{-}\min}(\mathcal{E}) \coloneqq \mu_{(H,B)}(\mathcal{H}_\ell / \mathcal{H}_{\ell-1}) .$$

If $\mathcal{H}_0 = 0$, *the maximal slope of E* is defined by

$$\mu_{(H,B)\text{-}\max}(\mathcal{E}) = \mu_{(H,B)}(\mathcal{H}_1) .$$

Let us recall the notion of *Harder-Narasimhan polygons*. In the present context, a Harder-Narasimhan polygon is an ordered sequence $v_0 = (x_0, y_0), \ldots, v_k = (x_k, y_k)$ of points in $\mathbb{R}^2$, with $k \geq 1$, so that

$$0 = x_0 < x_1 < \cdots < x_k , \qquad y_0 \geq 0 ,$$

and

$$\frac{y_i - y_{i-1}}{x_i - x_{i-1}} > \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

for all $1 \leq i \leq k-1$. As usual, we will identify such a sequence with the union of line segments

$$\Gamma \coloneqq \bigcup_{i=-1}^{k-1} \overline{v_i, v_{i+1}} .$$

where $v_{-1} = (0, 0)$. For any $r \in \mathbb{Z}$, $r > 0$, let $\mathcal{P}_r$ be the set of all Harder-Narasimhan polygons with $x_k = r$. Note that any polygon $\Gamma \in \mathcal{P}$ is contained in the subset $[0,\ r] \times \mathbb{R}$ and the complement $([0,\ r] \times \mathbb{R}) \smallsetminus \Gamma$ splits naturally into a union of two disjoint components, $\Gamma_+ \cup \Gamma_-$. Here, Γ_+ consists of all points above Γ, while Γ_- consists of all points under Γ. Let $\overline{\Gamma}_\pm \coloneqq \Gamma_\pm \cup \Gamma$. There is a standard partial order relation on $\mathcal{P}_r$ defined by

$$\Gamma \preceq \Gamma' \ \Leftrightarrow \ \Gamma \subset \overline{\Gamma'}_- .$$

Definition 5.2. The *Harder-Narasimhan polygon associated to a one dimensional sheaf $\mathcal{E}$ with respect to (H, B)-stability* is the polygon $\Gamma(\mathcal{E}) \subset \mathbb{R}^2$ determined by the vertices

$$(0, \chi(\mathcal{H}_0)),\ (H \cdot \mathsf{ch}_1(\mathcal{H}_1), \chi(\mathcal{H}_1) + B \cdot \mathsf{ch}_1(\mathcal{H}_1)),\ \ldots,\ (H \cdot \mathsf{ch}_1(\mathcal{H}_\ell), \chi(\mathcal{H}_\ell) + B \cdot \mathsf{ch}_1(\mathcal{H}_\ell)) .$$

⊘

By analogy to [Sha77, Theorem 3], one has the following.

Proposition 5.3. *For any $(\gamma, n) \in \mathsf{NS}(X) \times \mathbb{Z}$, with $\gamma \neq 0$, the assignment $\mathcal{E} \mapsto \Gamma_{(H,B)}(\mathcal{E})$ determines a constructible, upper semicontinuous function on the moduli stack of sheaves on X with invariants (γ, n). In particular, let $\mathcal{E}$ be a flat family of one-dimensional sheaves on X parametrized by a connected, locally noetherian scheme T so that $\mathsf{ch}_1(\mathcal{E}_t) \neq 0$ for $t \in T$. Let $t, t_0 \in T$ so that t_0 is a specialization of t. Then $\Gamma_{(H,B)}(\mathcal{E}_t) \preceq \Gamma_{(H,B)}(\mathcal{E}_{t_0})$.*

Fix $\alpha \in \mathbb{R}$ and let $\mathfrak{U}_\alpha(\widehat{X}_Z; \gamma, n)$ be the corresponding qcqs indgeometric derived stack introduced in [DPS+25b, §4.6] with respect to the twisted slope (5.1) (cf. §A). Recall that, if $\gamma = 0$, we have

$$\mathfrak{U}_\alpha(\widehat{X}_Z; n) = \mathfrak{U}_\alpha(\widehat{X}_Z; 0, n) \coloneqq \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; 0, n)$$

for any $\alpha \in \mathbb{R}$. In what follows, we shall use also the same notation for the classical truncation of $\mathfrak{U}_\alpha(\widehat{X}_Z;\gamma,n)$. By fixing a sequence $\alpha_0 > \alpha_1 > \alpha_2 > \cdots$ of real numbers, we obtain an admissible open exhaustion of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)$ formed by

$$\mathfrak{U}_k(\gamma,n) = \mathfrak{U}_k(\widehat{X}_Z;\gamma,n) := \mathfrak{U}_{\alpha_k}(\widehat{X}_Z;\gamma,n)\ . \tag{5.2}$$

Let

$$Z^\bullet\colon \quad \emptyset =: Z^{-1} = Z^0 \subset Z^1 \subset Z^2 := Z$$

be a stratification as in §2 satisfying the Assumptions 0.1. Recall that Z_i is the scheme-theoretic closure of $Z_i^\circ := Z^i \smallsetminus Z^{i-1}$, and the scheme-theoretic intersection $Z_i^{i-1} := Z^{i-1} \cap Z_i$ is zero-dimensional for $i=1,2$. In particular $Z_1 = Z^1$, hence we will set $Z_{1,2} := Z_2^1$. Furthermore, since Z and Z_i, for $i=1,2$, are reduced, all thickenings $Z_{\mathsf{cl}}^{(k)}$ and $(Z_i)_{\mathsf{cl}}^{(k)}$, for $i=1,2$, are effective divisors on X for all $k \geq 1$.

The support $Z^\bullet$-filtration of any coherent sheaf sheaf $\mathcal{E}$ with set-theoretic support contained in Z reduces to $\mathcal{E}^1 \subset \mathcal{E}$. Note that $\mathcal{E}_1 = \mathcal{E}^1$ and set $\mathcal{E}_2 := \mathcal{E}/\mathcal{E}_1$.

Fix topological invariants (γ,n). In the following, we shall consider the admissible open exhaustion $\{\mathfrak{U}_k(\gamma,n)\}_{k\in\mathbb{N}}$ of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)$ introduced above (cf. Formula (5.2)). Similarly, we shall consider the admissible open exhaustion $\{\mathfrak{U}_{i,k}(\gamma_i,n_i)\}_{k\in\mathbb{N}}$ of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_i};\gamma_i,n_i)$ given by the same construction, for $i=1,2$.

By Proposition A.2, for fixed (γ,n) and $k \geq 0$, ${}^{\mathsf{red}}\mathfrak{U}_k(\gamma,n)$ is T-equivariantly equivalent to a quotient stack. In particular, the open exhaustion $\{\mathfrak{U}_k(\gamma,n)\}_{k\in\mathbb{N}}$ is T-equivariantly ind stratifiable. Furthermore, Proposition A.2 yields the following.

Corollary 5.4. *For any decomposition $(\gamma,n) = (\gamma_1,n_1)+(\gamma_2,n_2)$, the product*

$${}^{\mathsf{red}}\mathfrak{U}_k(\gamma_1,n_1) \times {}^{\mathsf{red}}\mathfrak{U}_k(\gamma_2,n_2)$$

is T-equivariantly equivalent to a quotient stack. In particular, the hypothesis of Proposition 4.5 is satisfied, hence Assumption 0.3–(2) holds in this case.

Now, we prove that Assumption 0.6 is also satisfied by this choice of quasi-compact quasi-separated indgeometric stacks.

Proposition 5.5. *Let W be a reduced pure one-dimensional subscheme of X and let $W^0 \subset W$ be a zero-dimensional subscheme. Let $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{W^0})$ be the classical moduli stack of zero-dimensional sheaves on X with set-theoretic support contained in W^0. For any $\gamma \in \langle W\rangle$ and any $k \in \mathbb{N}$, let $\{\mathfrak{U}_k(\gamma)\}_{k\in\mathbb{N}}$ be the admissible open exhaustion of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_W;\gamma)$ defined in Formula (5.2). Let $\mathfrak{X}$ be the stack of extensions defined by the pull-back diagram*

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \mathbf{Coh}^{\mathsf{ext}}_{\mathsf{ps}}(\widehat{X}_W) \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathsf{ev}_2\times\mathsf{ev}_0} \\ \mathfrak{U}_k(\gamma)\times\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{W^0}) & \longrightarrow & \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_W;\gamma)\times\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_W) \end{array} .$$

Then the central projection $\mathsf{ev}_1\colon \mathfrak{X} \to \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_W;\gamma)$ factors through the open immersion $\mathfrak{U}_k(\gamma) \to \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_W;\gamma)$. Moreover, the resulting convolution diagram

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \mathfrak{U}_k(\gamma) \\ \big\downarrow & & \\ \mathfrak{U}_k(\gamma)\times\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{W^0}) & & \end{array}$$

defines an action

$$\mathsf{H}^T_\bullet(\mathfrak{U}_k(\gamma))\widehat{\otimes}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{W^0})) \longrightarrow \mathsf{H}^T_\bullet(\mathfrak{U}_k(\gamma))\ ,$$

which is naturally compatible with the action

$$\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_W;\gamma))\widehat{\otimes}\mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{W^0})) \longrightarrow \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_W;\gamma))\ .$$

Proof. It suffices to prove the stated factorization property for the central projection at the level of the underlying classical stacks. If $\gamma = 0$, by definition $\mathfrak{U}_k(\gamma)$ coincides with the stack of zero-dimensional sheaves with set-theoretic support contained in Z. Therefore, there is nothing to prove.

Assume that $\gamma \neq 0$. Let T be a parameter scheme, locally of finite type, and let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0$$

be the exact sequence associated to an arbitrary morphism $T \to \mathfrak{X}$. It suffices to prove that

$$\mu_{(H,B)\text{-}\min}(\mathcal{E}_t) = \mu_{(H,B)\text{-}\min}(\mathcal{G}_t)$$

for any $t \in T$. Since $\mathcal{F}$ is zero dimensional, there is a natural one-to-one correspondence between pure one-dimensional quotients of $\mathcal{E}_t$ and pure one-dimensional quotients of $\mathcal{G}_t$. Hence the claim is clear. □

Corollary 5.6. *Assumption 0.6 holds.*

Our next goal is the construction of the open exhaustion $\{\mathfrak{S}_k(\gamma, n)\}_{k\in\mathbb{N}}$ for any invariant (γ, n). We start with a preliminary result.

Lemma 5.7. *Let $k(\gamma, n; \alpha)$ be the integer as in* [DPS+25b, Proposition 4.64] *and set $D_2 := Z_2^{k(\gamma,n;\alpha)}$. For any coherent sheaf $\mathcal{E}$ on X, set-theoretically supported on Z, with invariants (γ, n), such that $\mu_{(H,B)\text{-}\min}(\mathcal{E}) > \alpha$, let $\mathcal{T}_{1,2} \subset \mathcal{E} \otimes \mathcal{O}_{D_2}$ be the maximal zero dimensional subsheaf with set-theoretic support contained in $Z_{1,2}$. Then, there exists an isomorphism $\mathcal{E} \otimes \mathcal{O}_{D_2}/\mathcal{T}_{1,2} \to \mathcal{E}_2$ which fits in the commutative diagram*

$$\begin{array}{ccc} \mathcal{E} & \twoheadrightarrow & \mathcal{E} \otimes \mathcal{O}_{D_2}/\mathcal{T}_{1,2} \\ & \searrow & \downarrow \\ & & \mathcal{E}_2 \end{array} ,$$

where the other two arrows are the canonical epimorphisms.

Proof. Set $D_1 := Z_1^{k(\gamma,n;\alpha)}$ and let $\zeta_i \in H^0(X, \mathcal{O}_X(D_i))$ be the defining sections of D_i for $i = 1, 2$, respectively. Note that the set-theoretic intersection of D_1 and D_2 coincides with $Z_{1,2}$ and $\zeta = \zeta_1\zeta_2$ is a defining section of $D := Z^{k(\gamma,n;\alpha)}$. Moreover, $\mathcal{E}$ is annihilated by ζ by [DPS+25b, Proposition 4.64]. Hence the same holds for $\mathcal{E}_1$ and $\mathcal{E}_2$.

Since $\mathcal{E}_2$ is pure at $Z_{1,2}$ by Proposition 2.6, the multiplication map

$$\zeta_1 \colon \mathcal{E}_2 \longrightarrow \mathcal{E}_2 \otimes \mathcal{O}_X(D_1)$$

is injective. Since $\mathcal{E}_2$ is annihilated by ζ, this implies that it must be annihilated by ζ_2. Hence $\mathcal{E}_2$ is scheme-theoretically supported on D_2. This implies that the canonical morphism $\mathcal{E}_2 \to \mathcal{E}_2 \otimes \mathcal{O}_{D_2}$ is an isomorphism. Then, applying $- \otimes \mathcal{O}_{D_2}$ to the short exact sequence

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0\ ,$$

one obtains a second exact sequence

$$\mathcal{E}_1 \otimes \mathcal{O}_{D_2} \longrightarrow \mathcal{E} \otimes \mathcal{O}_{D_2} \longrightarrow \mathcal{E}_2 \longrightarrow 0\ .$$

Let $\mathcal{T}'_{1,2}$ be the image of the morphism $\mathcal{E}_1 \otimes \mathcal{O}_{D_2} \to \mathcal{E} \otimes \mathcal{O}_{D_2}$. Note that $\mathcal{E}_1 \otimes \mathcal{O}_{D_2}$ is zero-dimensional, with set-theoretic support contained in $Z_{1,2}$, since the set-theoretic support of $\mathcal{E}_1$

is contained in Z_1. Therefore, the same holds for $\mathcal{T}'_{1,2}$. This implies that $\mathcal{T}'_{1,2} \subset \mathcal{T}_{1,2}$. One then obtains a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}'_{1,2} & \longrightarrow & \mathcal{T}_{1,2} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\ & & \big\downarrow \mathsf{id} & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \mathcal{T}'_{1,2} & \longrightarrow & \mathcal{E}\otimes\mathcal{O}_{D_2} & \longrightarrow & \mathcal{E}_2 & \longrightarrow & 0 \end{array},$$

where $\mathcal{Q} := \mathcal{T}_{1,2}/\mathcal{T}'_{1,2}$, and central vertical arrow is the natural inclusion. Then, the snake lemma implies that the morphism $\mathcal{Q} \to \mathcal{E}_2$ must be injective. Since $\mathcal{E}_2$ is pure at $Z_{1,2}$, this further implies that $\mathcal{Q} = 0$. Applying the snake lemma again, this yields an isomorphism

$$\mathcal{E}\otimes\mathcal{O}_{D_2}/\mathcal{T}_{1,2} \longrightarrow \mathcal{E}_2$$

as claimed. □

Lemma 5.8. *There exists an integer $c(\gamma, n; \alpha) \in \mathbb{Z}$ so that*

$$|\chi(\mathcal{E}_1)| \leq c(\gamma, n; \alpha)$$

for any coherent sheaf $\mathcal{E}$ on X, set-theoretically supported on Z, with invariants (γ, n) such that $\mu_{(H,B)\text{-}\min}(\mathcal{E}) > \alpha$

Proof. Let $k(\gamma, n; \alpha)$ be the integer as in [DPS+25b, Proposition 4.64]. Using the notation in Lemma 5.7, one has an isomorphism $\mathcal{E}\otimes\mathcal{O}_{D_2}/\mathcal{T}_{1,2} \simeq \mathcal{E}_2$. Applying $\mathcal{E}\otimes -$ to the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D_2) \longrightarrow \mathcal{O}_{D_2}\otimes\mathcal{O}_X(D_2) \longrightarrow 0\,,$$

one obtains the exact sequence

$$0 \longrightarrow \mathsf{Tor}_1(\mathcal{E}, \mathcal{O}_{D_2}\otimes\mathcal{O}_X(D_2)) \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}\otimes\mathcal{O}_X(D_2) \longrightarrow \mathcal{E}\otimes\mathcal{O}_{D_2}\otimes\mathcal{O}_X(D_2) \longrightarrow 0\,.$$

Set $\mathcal{E}'_2 := \mathsf{Tor}_1(\mathcal{E}, \mathcal{O}_{D_2}\otimes\mathcal{O}_X(D_2))$. The above exact sequence yields

$$\chi(\mathcal{E}\otimes\mathcal{O}_{D_2}\otimes\mathcal{O}_X(D_2)) = \chi(\mathcal{E}'_2) + \chi(\mathcal{E}\otimes\mathcal{O}_X(D_2)) - \chi(\mathcal{E})\,.$$

Using the Grothendieck-Riemann-Roch theorem, this further yields

$$\chi(\mathcal{E}\otimes\mathcal{O}_{D_2}) + D_2\cdot\mathsf{ch}_2(\mathcal{E}\otimes\mathcal{O}_{D_2}) = \chi(\mathcal{E}'_2) + D_2\cdot\mathsf{ch}_2(\mathcal{E})\,.$$

By Lemma 5.7, one has

$$\mathsf{ch}_1(\mathcal{E}\otimes\mathcal{O}_{D_2}) = \mathsf{ch}_1(\mathcal{E}_2) = \mathsf{ch}_1(\mathcal{E}) - \mathsf{ch}_1(\mathcal{E}_1)\,.$$

Hence one obtains

$$\chi(\mathcal{E}\otimes\mathcal{O}_{D_2}) = \chi(\mathcal{E}'_2) + D_2\cdot\mathsf{ch}_1(\mathcal{E}_1)\,. \tag{5.3}$$

Note also that

$$\mathsf{ch}_1(\mathcal{E}'_2) = \mathsf{ch}_1(\mathcal{E}) + \mathsf{ch}_1(\mathcal{E}\otimes\mathcal{O}_{D_2}) - \mathsf{ch}_1(\mathcal{E}\otimes\mathcal{O}_X(D_2)) = \mathsf{ch}_1(\mathcal{E}_2)\,.$$

Recall that $\gamma_i := \mathsf{ch}_1(\mathcal{E}_i)$ are uniquely determined by γ by Corollary 2.7. Several cases will be considered separately below, depending on γ, γ_2.

Case (a). Assume that $\gamma \neq 0$ and $\gamma_2 \neq 0$. Let $\mathcal{T} \subset \mathcal{E}$ be the maximal zero-dimensional subsheaf of $\mathcal{E}$ and set $\mathcal{F} := \mathcal{E}/\mathcal{T}$. Note that the set of isomorphism classes of coherent sheaves $\mathcal{E}$ on X, set-theoretically supported on Z, with invariants (γ, n), such that $\mu_{(H,B)\text{-}\min}(\mathcal{E}) > \alpha$, is bounded since the set of associated (H, B)-Harder-Narasimhan polygons is finite. This implies that there exists a constant $c_1(\gamma, n; \alpha)$ so that

$$\chi(\mathcal{T}) < c_1(\gamma, n; \alpha) \tag{5.4}$$

and a second constant $c_2(\gamma, n; \alpha)$ so that

$$\mu_{(H,B)\text{-}\max}(\mathcal{F}) < c_2(\gamma, n; \alpha)\,. \tag{5.5}$$

Moreover, note that $\mathcal{T}_2' \coloneqq \mathcal{T} \cap \mathcal{E}_2' \subset \mathcal{E}_2'$ is the maximal zero-dimensional subsheaf of $\mathcal{E}_2$. Set $\mathcal{F}_2' \coloneqq \mathcal{E}_2'/\mathcal{T}_2'$. Then, one obtains a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}_2' & \longrightarrow & \mathcal{E}_2' & \longrightarrow & \mathcal{F}_2' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array},$$

where the first two vertical arrows from the left-hand-side are the natural inclusions. Then, the snake lemma yields an exact sequence

$$0 \longrightarrow \ker(\mathcal{F}_2' \to \mathcal{F}_2) \longrightarrow \mathcal{T}/\mathcal{T}_2' \longrightarrow \cdots$$

Since $\mathcal{F}_2'$ is purely one-dimensional, we get that $\ker(\mathcal{F}_2' \to \mathcal{F}_2) = 0$, i.e., the morphism $\mathcal{F}_2' \to \mathcal{F}_2$ is injective. This yields

$$\mu_{(H,B)}(\mathcal{F}_2') \leq \mu_{(H,B)\text{-}\max}(\mathcal{F}) .$$

Using inequalities (5.4) and (5.5), this implies

$$\chi(\mathcal{E}_2') < c_1(\gamma, n; \alpha) + (H \cdot \gamma_2) c_2(\gamma, n; \alpha) - B \cdot \gamma_2 .$$

Using Formula (5.3), this further yields

$$\chi(\mathcal{E} \otimes \mathcal{O}_{D_2}) < c_1(\gamma, n; \alpha) + (H \cdot \gamma_2) c_2(\gamma, n; \alpha) - B \cdot \gamma_2 + k(\gamma, n; \alpha) Z_2 \cdot \gamma_1 .$$

Finally, since $\mathcal{E}_2 \simeq \mathcal{E} \otimes \mathcal{O}_{D_2}/\mathcal{T}_{1,2}$, one obtains

$$\chi(\mathcal{E}_2) < c_1(\gamma, n; \alpha) + (H \cdot \gamma_2) c_2(\gamma, n; \alpha) - B \cdot \gamma_2 + k(\gamma, n; \alpha) Z_2 \cdot \gamma_1 . \tag{5.6}$$

At the same time, since $\mathcal{E}_2$ is a quotient of $\mathcal{E}$, one also has $\mu(\mathcal{E}_2) > \alpha$. This implies

$$\chi(\mathcal{E}_2) > (H \cdot \gamma_2)\alpha - H \cdot \gamma_2 . \tag{5.7}$$

Since $\chi(\mathcal{E}_1) = n - \chi(\mathcal{E}_2)$, the claim follows from inequalities (5.6) and (5.7).

Case (b). Now, assume that $\gamma \neq 0$, while $\gamma_2 = 0$. Then, clearly, $\chi(\mathcal{E}_2) \geq 0$. Moreover, in this case, $\mathcal{E}_2' = \mathcal{T}_2'$, hence $\mathcal{E}_2'$ is a subsheaf of $\mathcal{T}$. Using identity (5.3), this implies

$$\chi(\mathcal{E} \otimes \mathcal{O}_{D_2}) < c_1(\gamma, n; \alpha) + k(\gamma, n; \alpha) Z_2 \cdot \gamma_1 .$$

Since $\mathcal{E}_2 \simeq \mathcal{E} \otimes \mathcal{O}_{D_2}/\mathcal{T}_{1,2}$, one further obtains

$$\chi(\mathcal{E}_2) < c_1(\gamma, n; \alpha) + k(\gamma, n; \alpha) Z_2 \cdot \gamma_1 .$$

Since $\chi(\mathcal{E}_1) = n - \chi(\mathcal{E}_2)$, this implies the claim.

Case (c). Finally, assume that $\gamma = 0$ and $\gamma_2 = 0$. Then, both $\mathcal{E}_1$ and $\mathcal{E}_2$ are zero-dimensional and $\chi(\mathcal{E}_2) + \chi(\mathcal{E}_2) = n$. In this case the claim is obvious. □

We now explain the construction of the open exhaustion $\{\mathfrak{S}_k(\gamma, n)\}_{k \in \mathbb{N}}$. Fix a coherent sheaf $\mathcal{E}$ on X, set-theoretically supported on Z. Recall that the first Chern classes

$$\gamma_i \coloneqq \mathsf{ch}_1(\mathcal{E}_i) ,$$

for $i = 1, 2$, are uniquely determined by $\gamma = \mathsf{ch}_1(\mathcal{E})$ via Assumption 0.1–(3) and Corollary 2.7.

Suppose $\gamma_2 \neq 0$. For any $\alpha \in \mathbb{R}$, let S_α be the subspace of $\left|\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma, n)\right|$ defined by the condition

$$\mu_{(H,B)\text{-}\min}(\mathcal{E}_2) > \alpha$$

In the remaining part of this section it will be proven that S_α is an open subspace.

Lemma 5.9. *Assuming $\gamma_2 \neq 0$, let*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}_2 \longrightarrow 0$$

be a short exact sequence of coherent sheaves on X so that the set-theoretic support of $\mathcal{F}_1$ is contained in $Z_1 = Z^1$, while the set-theoretic support of $\mathcal{F}_2$ is contained in Z_2. Then, $\mathsf{ch}_1(\mathcal{F}_2) = \gamma_2$ and

$$\mu_{(H,B)\text{-}\min}(\mathcal{F}_2) = \mu_{(H,B)\text{-}\min}(\mathcal{E}_2)\ .$$

Proof. By Proposition 2.10 , one has $\mathcal{F}_1 \subset \mathcal{E}_1$ and $\mathcal{E}_1/\mathcal{F}_1$ is zero-dimensional. Moreover, there is an exact sequence

$$0 \longrightarrow \mathcal{E}_1/\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{E}_2 \longrightarrow 0\ .$$

which yields a one-to-one correspondence between pure one-dimensional quotients $\mathcal{F}_2 \to \mathcal{G}$ and $\mathcal{E}_2 \to \mathcal{G}$. This implies the claim. □

Lemma 5.10. *Assume that $\gamma_2 \neq 0$. Let T be a scheme locally of finite type over $\mathbb{C}$. Let $\mathcal{E}$ be a flat family of sheaves on X, set-theoretically supported on Z, parametrized by T, with invariants (γ, n). For any point $t \in T$, let $\mathcal{E}_t^0 \subset \mathcal{E}_t$ be the support filtration of $\mathcal{E}_t$ and let $\mathcal{E}_{t,2} := \mathcal{E}_t/\mathcal{E}_t^1$. Set also $\mathcal{E}_{t,1} := \mathcal{E}_t^1$. Then, the following inequality holds for any points $t, t_0 \in T$ so that t_0 is a specialization of t:*

$$\mu_{(H,B)\text{-}\min}(\mathcal{E}_{t,2}) \geq \mu_{(H,B)\text{-}\min}(\mathcal{E}_{t_0,2})\ .$$

Proof. It suffices to prove the claim for $T = \mathsf{Spec}(R)$, where R is a DVR over $\mathbb{C}$. Let t and t_0 denote the generic and the closed point, respectively. By [Gro95, Lemma 3.7], there exists a unique T-flat quotient $\mathcal{E} \twoheadrightarrow \mathcal{G}$ so that $\mathcal{G}_t = \mathcal{E}_{t,2}$. Let $\mathcal{F} := \ker(\mathcal{E} \twoheadrightarrow \mathcal{G})$, which is also flat over T. Note also that $\mathcal{F}_t = \mathcal{E}_{t,1}$ by construction. By restriction to t_0 one obtains an exact sequence

$$0 \longrightarrow \mathcal{F}_{t_0} \longrightarrow \mathcal{E}_{t_0} \longrightarrow \mathcal{G}_{t_0} \longrightarrow 0\ .$$

Corollary 2.16 shows that $\mathcal{F}_{t_0}$ and $\mathcal{G}_{t_0}$ are set-theoretically supported on $Z_1 \times \{t_0\}$ and $Z_2 \times \{t_0\}$ respectively. Then, Lemma 5.9 yields the identity

$$\mu_{(H,B)\text{-}\min}(\mathcal{G}_{t_0}) = \mu_{(H,B)\text{-}\min}(\mathcal{E}_{t_0,2})\ .$$

Moreover, Proposition 5.3 shows that

$$\Gamma_{(H,B)}(\mathcal{G}_t) \preceq \Gamma_{(H,B)}(\mathcal{G}_{t_0})\ ,$$

which implies

$$\mu_{(H,B)\text{-}\min}(\mathcal{G}_t) \geq \mu_{(H,B)\text{-}\min}(\mathcal{G}_{t_0})\ .$$

This proves the claim. □

Using Lemma 5.10, one obtains the following.

Lemma 5.11. *Assume that $\gamma_2 \neq 0$. Then, the subset of $\left|\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma, n)\right|$ consisting of those points*[11] *$[\mathcal{E}]$ defined by the condition*

$$\mu_{(H,B)\text{-}\min}(\mathcal{E}_2) > \alpha$$

is open.

Proof. We shall show that the locus we are considering is constructible and stable under generization. The latter follows from Lemma 5.10, while for the former we need to prove that the function

$$[\mathcal{E}] \longmapsto \mu_{(H,B)\text{-}\min}(\mathcal{E}_2) \tag{5.8}$$

is constructible on $\left|\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma, n)\right|$. As in [AB13, Example A.4], it suffices to prove the following claim for reduced irreducible schemes T of finite type over $\mathbb{C}$: let $\mathcal{E}$ be a flat family of coherent sheaves on X, parametrized by T, set-theoretically supported on Z, and let $\xi \in T$ be the generic

[11] The notion of a point of the underlying topological space of an algebraic stack is given e.g. in [Sta25, Tag 04XE, Definition 100.4.2].

point. Then, there exists an open subscheme $U \subset T$ so that $\mu_{(H,B)\text{-}\min}(\mathcal{E}_{u,2}) = \mu_{(H,B)\text{-}\min}(\mathcal{E}_{\xi,2})$ for any $u \in U$.

The claim follows from Lemma 2.19 and Proposition 5.3. Thus, the constructibility of the function (5.8) follows. □

Lemma 5.11 yields the following.

Corollary 5.12. *Under the same assumptions as in Lemma 5.11, for any $\alpha \in \mathbb{R}$ the condition*

$$\mu_{(H,B)\text{-}\min}(\mathcal{E}_2) > \alpha$$

determines an open substack $\mathfrak{S}_\alpha(\gamma, n) \subset \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n)$. Moreover, for any $\alpha' \in \mathbb{R}$, with $\alpha' \leq \alpha$, there is a canonical open immersion $\mathfrak{S}_\alpha(\gamma, n) \hookrightarrow \mathfrak{S}_{\alpha'}(\gamma, n)$.

For future reference, we also note the following.

Lemma 5.13. *For fixed (γ, n) and $\alpha \in \mathbb{R}$ there exists a constant c_1 so that $\chi(\mathcal{E}_1) < c_1$ for all $[\mathcal{E}] \in \left|\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n)\right|$ so that $\mu_{(H,B)\text{-}\min}(\mathcal{E}) > \alpha$.*

Proof. Since $\chi(\mathcal{E}_1) + \chi(\mathcal{E}_2) = n$, it suffices to prove that $\chi(\mathcal{E}_2)$ is bounded below. For $\gamma_2 = 0$, this is clear. For $\gamma_2 \neq 0$, let $\mathcal{T}_2 \subset \mathcal{E}_2$ be the maximal zero-dimensional subsheaf of $\mathcal{E}_2$ and let $\mathcal{E}'_2 := \mathcal{E}_2/\mathcal{T}_2$. Then, the claim follows from the inequalities

$$\chi(\mathcal{E}_2) \geq \chi(\mathcal{E}'_2) \quad \text{and} \quad \mu_{(H,B)}(\mathcal{E}'_2) \geq \alpha\ .$$

□

Now, let $\alpha_0 > \alpha_1 > \cdots$ be a fixed descending sequence of real numbers $\alpha_k \in \mathbb{R}$, with $k \in \mathbb{N}$. We define the following two families of substacks $\{\mathfrak{U}_k(\gamma, n)\}_{k\in\mathbb{N}}$ and $\{\mathfrak{S}_k(\gamma, n)\}_{k\in\mathbb{N}}$ of the stack $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n)$, respectively, as follows.

Definition 5.14. Fix invariants (γ, n).

(1) Assume that $\gamma, \gamma_2 \neq 0$. For each $k \in \mathbb{N}$, let $\mathfrak{U}_k(\gamma, n)$ be the open substack of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n)$ consisting of those sheaves $\mathcal{E}$ satisfying the condition

$$\mu_{(H,B)\text{-}\min}(\mathcal{E}) > \alpha_k\ .$$

Similarly, let $\mathfrak{S}_k(\gamma, n)$ be the open substack of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n)$ consisting of those sheaves $\mathcal{E}$ satisfying the condition

$$\mu_{(H,B)\text{-}\min}(\mathcal{E}_2) > \alpha_k\ .$$

(2) Assume that $\gamma = 0$ (which implies that $\gamma_2 = 0$ as well). Set $\mathfrak{U}_k(0, n) := \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; n)$ and $\mathfrak{S}_k(0, n) := \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; n)$ for any $k \in \mathbb{N}$.

(3) Assume that $\gamma \neq 0$, while $\gamma_2 = 0$. For each $k \in \mathbb{N}$, let $\mathfrak{U}_k(\gamma, n)$ be the open substack of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n)$ consisting of those sheaves $\mathcal{E}$ satisfying the condition

$$\mu_{(H,B)\text{-}\min}(\mathcal{E}) > \alpha_k\ .$$

Set $\mathfrak{S}_k(0, n) := \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n)$.

⊘

Theorem 5.15. *For any given (γ, n), we have:*

(1) the family $\{\mathfrak{U}_k(\gamma, n)\}_{k\in\mathbb{N}}$ is an open exhaustion of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n)$ formed by quasi-compact quasi-separated indgeometric stacks;

(2) the family $\{\mathfrak{S}_k(\gamma, n)\}_{k\in\mathbb{N}}$ is open exhaustion of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n)$ formed by indgeometric stacks satisfying Assumption 0.2.

Moreover, for any $k \in \mathbb{N}$ and invariant (γ, n), there is a canonical open immersion $\mathfrak{U}_k(\gamma, n) \to \mathfrak{S}_k(\gamma, n)$; in addition, there exists $a \in \mathbb{Z}$, so that $\mathfrak{U}_k(\gamma, n)$ is contained as an open substack in $\mathfrak{S}_k(\gamma, n)^{\geqslant b}$ for any $b \leq a$, and a is maximal with this property. In particular, Assumptions 0.4–(1) and –(2) hold.

Proof. Claim (1) follows from [DPS+25b, §4.6], while the fact that $\mathfrak{S}_k(\gamma, n)$ is an open substack of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n)$ follows from Lemma 5.11. Moreover, Lemma 5.13 yields Assumption 0.2–(1).

For any $k \in \mathbb{N}$ and invariant (γ, n), by construction there is a canonical open immersion $\mathfrak{U}_k(\gamma, n) \to \mathfrak{S}_k(\gamma, n)$. In addition, Lemma 5.8 shows that there exists $a \in \mathbb{Z}$, so that $\mathfrak{U}_k(\gamma, n)$ is contained as an open substack in $\mathfrak{S}_k(\gamma, n)^{\geqslant b}$ for any $b \leq a$ and a is maximal with this property.

Now, we are left to prove that $\mathfrak{S}_k(\gamma, n)$ satisfies Assumption 0.2–(2).

Since $\eta_k^{\geqslant n_1}$ and y_k are open immersions in the diagram (3.8), to construct the open immersion $\psi_k^{\geqslant n_1}$ it suffices to check that

$$|(\mathfrak{Coh}^{\mathsf{ext}})_k^{\geqslant n_1}| \subset |\mathfrak{Y}_k| \ .$$

For a field K, a K-valued point of $(\mathfrak{Coh}^{\mathsf{ext}})_k^{\geqslant n_1}$ is a short exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}_2 \longrightarrow 0 \tag{5.9}$$

of $\mathcal{O}_{X_K}$-modules so that $[\mathcal{F}_i] \in |\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_i}; \gamma_i, n_i)|$ and $[\mathcal{E}] \in |\mathfrak{S}_k^{\geqslant n_1}|$. Let

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0 \tag{5.10}$$

be the $Z_{1,K}$-filtration of $\mathcal{E}$.

If $\gamma_2 = 0$, by definition, $\mathfrak{S}_k(\gamma, n)^{\geqslant 1} = \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n)^{\geqslant n_1}$ and $\mathfrak{U}_{2,k}(\gamma_2, n_2) = \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_2}; \gamma_2, n_2)$. Then both $\eta_k^{\geqslant n_1}$ and y_k are canonical equivalences, and the claim trivially follows.

Assume that $\gamma_2 \neq 0$. Then Lemma 5.9 shows that

$$\mu_{(H,B)\text{-}\min}(\mathcal{E}_2) = \mu_{(H,B)\text{-}\min}(\mathcal{F}_2) \ .$$

Since the defining condition of $\mathfrak{S}_k^{\geqslant n_1}$ is

$$\mu_{(H,B)\text{-}\min}(\mathcal{E}_2) > \alpha_k \ ,$$

this further implies that the exact sequence (5.9) determines a K-valued point of $\mathfrak{Y}_k$. Moreover, Proposition 2.10 shows that $\mathcal{E}_2 = \mathcal{F}_2$ if $\chi(\mathcal{E}_1) = \chi(\mathcal{F}_1) = n_1$. Therefore, if this is the case, the exact sequences (5.9) and (5.10) coincide. In conclusion, one obtains a canonical open immersion $\psi_k^{\geqslant n_1}$ such that its restriction $\psi_k^{n_1}$ to $(\mathfrak{Coh}^{\mathsf{ext}})_k^{n_1}$ is a canonical equivalence. Therefore, the commutativity of the enhanced diagram is completely canonical. □

We conclude this section by showing that the open exhaustions $\{\mathfrak{S}_k(\gamma, n)\}_{k\in\mathbb{N}}$ and $\{\mathfrak{U}_k(\gamma, n)\}_{k\in\mathbb{N}}$ satisfy the hypothesis of Proposition 4.13. As in loc. cit., for any fixed $a \in \mathbb{Z}$ and $k \in I$, let $\mathfrak{V}_{k,\ell}(\gamma, n)^{\geqslant a}$ be the open substack of $\mathfrak{S}_k(\gamma, n)^{\geqslant a}$ defined in Formula 4.2. The collection $\{\mathfrak{V}_{k,\ell}(\gamma, n)^{\geqslant a}\}_{\ell\in I}$ is an open exhaustion of $\mathfrak{S}_k(\gamma, n)^{\geqslant a}$. The next result shows that this open exhaustion is T-equivariantly ind stratifiable.

Proposition 5.16. *Given a, k as above, for any $\ell \in I$, the reduced stack ${}^{\mathsf{red}}\mathfrak{V}_{k,\ell}(\gamma, n)^{\geqslant a}$ is T-equivariantly equivalent to the quotient stack $[\mathsf{T}^{\geqslant a}/G]$, where $\mathsf{T}^{\geqslant a}$ is a reduced quasi-projective $T \times G$-scheme, and G is a general linear group. In particular, the hypothesis of Proposition 4.13 holds in this case, hence Assumption 0.4–(4) is satisfied.*

Proof. As shown in Proposition A.2, for any $\ell \in I$, the reduced stack ${}^{\mathsf{red}}\mathfrak{U}_\ell(\gamma, n)$ is T-equivariantly equivalent to the global quotient $[\mathsf{Q}/G]$, where Q is a reduced quasi-projective scheme and G is a general linear group. By construction, Q is a locally closed subscheme of a Quot scheme parametrizing quotients $V \otimes \mathcal{O}_X(-NH) \to \mathcal{E}$, with V a finite-dimensional vector space and N a positive integer. Moreover, $G := \mathsf{GL}(V)$ acting naturally on Q.

By Corollary 5.12, there exists an open subscheme $\mathsf{T} \subset \mathsf{Q}$ parametrizing quotients $V \otimes \mathcal{O}_X(-NH) \to \mathcal{E}$ so that $\mu_{(H,B)\text{-}\min}(\mathcal{E}_2) > \alpha_k$. Furthermore, by Proposition 2.20, there exists a reduced closed subscheme $\mathsf{T}^{\geqslant a} \subset \mathsf{T}$ parametrizing quotients $V \otimes \mathcal{O}_X(-NH) \to \mathcal{E}$ satisfying the additional condition $\chi(\mathcal{E}_1) \geq a$. Both T and $\mathsf{T}^{\geqslant a}$ are clearly preserved by the G-action, and the reduced stack ${}^{\mathsf{red}}\mathfrak{Y}_{k,\ell}(\gamma, n)^{\geqslant a}$ is T-equivariantly naturally equivalent to the quotient stack $[\mathsf{T}^{\geqslant a}/G]$. □

6. Conditional results for substacks defined by slope conditions

In this section, we shall show the cellular structure on the moduli stack of zero-dimensional sheaves on X set-theoretically supported on Z. Moreover, we discuss the cellular structure of the admissible open exhaustion of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n)$ given by Harder-Narasimhan strata.

6.1. **Cellular structure for the stacks of zero-dimensional coherent sheaves.** Let $C \subset X$ be a smooth connected rational (-2) curve on X and let $p \in C$ be a closed point. Set $C^\circ := C \smallsetminus \{p\}$. For any $\ell \in \mathbb{N}$, with $\ell \geq 1$, let $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_C; \ell)$ be the moduli stack parametrizing length ℓ zero-dimensional sheaves on X with set-theoretic support on C. Let $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{C^\circ}; \ell) \subset \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_C; \ell)$ be the open substack consisting of zero-dimensional sheaves with set-theoretic support contained in C°. Similarly, let $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_p; \ell)$ be the moduli stack of length ℓ zero-dimensional sheaves with set-theoretic support at p. These are quasi-compact quasi-separated indgeometric derived stacks. In particular, the corresponding reduced stacks are quasi-compact geometric classical stacks. In addition, they are of finite type.

Lemma 6.1. *The stack ${}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{C^\circ}; \ell)$ is 1-cellular and the stack ${}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_p; \ell)$ is 2-cellular for any $\ell \geq 1$.*

Proof. Let T be the total space of the canonical line bundle on C. Abusing notation, we will identify C with its image via the zero section $C \to T$. By [Bal08, Lemma 3.14], the infinitesimal neighborhood $\widehat{X}_C$ is isomorphic as a formal scheme to the infinitesimal neighborhood $\widehat{T}_C$. Therefore one has an equivalence of indgeometric derived stacks

$$\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_C; \ell) \simeq \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{T}_C; \ell)$$

for any $\ell \geq 1$. Moreover, $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{T}_C; \ell)$ is equivalent to the moduli stack $\mathbf{Higgs}_{\mathsf{nilp}}(C; \ell)$ of length ℓ nilpotent zero-dimensional Higgs sheaves on C. This equivalence identifies $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{C^\circ}; \ell)$ to the open substack $\mathbf{Higgs}^\circ_{\mathsf{nilp}}(C; \ell)$ of zero-dimensional Higgs sheaves with set-theoretic support contained on $C^\circ \simeq \mathbb{A}^1$.

By [SS20, Proposition 2.5], the moduli stack ${}^{\mathsf{red}}\mathfrak{Higgs}^\circ_{\mathsf{nilp}}(C; \ell)$ admits a stratification, where each stratum is equivalent to an iterated vector bundle stack over a finite product

$$\bigtimes_k {}^{\mathsf{red}}\mathfrak{Coh}(C^\circ, \ell_k) ,$$

where the k-the factor is the reduced moduli stack of length ℓ_k zero-dimensional sheaves on C°. Since $C^\circ \simeq \mathbb{A}^1$, each factor ${}^{\mathsf{red}}\mathfrak{Coh}(C^\circ; \ell_k)$ is equivalent to the moduli space of ℓ_k-dimensional representations of the one-loop quiver. In particular, each factor is a linear quotient stack, which proves the claim for the moduli stack ${}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{C^\circ}; \ell)$.

Next, note that the moduli stack $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_p; \ell)$ is equivalent to the moduli stack of nilpotent zero-dimensional Higgs sheaves with set-theoretic support at p. The construction of [SS20, Proposition 2.5] provides again a stratification of ${}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_p; \ell)$, where each stratum an iterated vector bundle stack over a finite product

$$\bigtimes_k {}^{\mathsf{red}}\mathfrak{Coh}(\widehat{C}_p, \ell_k) .$$

Furthermore, each stack ${}^{\mathsf{red}}\mathfrak{Coh}(\widehat{C}_p, \ell_k)$ is equivalent to the reduced moduli stack of ℓ_k-dimensional nilpotent representations of the one-loop quiver. The latter can be again stratified by Jordan type, so that each Jordan stratum is an iterated vector bundle stack over a product of linear stacks. In conclusion, each factor ${}^{\mathsf{red}}\mathfrak{Coh}(\widehat{C}_p, \ell_k)$ is 1-cellular, hence ${}^{\mathsf{red}}\mathfrak{Coh}(\widehat{C}_p, \ell)$ has a 2-cellular structure. □

Lemma 6.2. *Let T be a scheme locally of finite type, and let $\mathcal{F}$ and $\mathcal{G}$ be T-flat coherent sheaves on $T \times X$. Assume that the restrictions $\mathcal{F}_t$ and $\mathcal{G}_t$ are set-theoretically supported on disjoint subsets of X_t for all $t \in T$. Then, $\mathcal{F}$ and $\mathcal{G}$ have disjoint set-theoretic support as sheaves on $X \times T$.*

Proof. Assume that the intersection of the set-theoretic supports of $\mathcal{F}$ and $\mathcal{G}$ is non-empty, hence it contains at least one point $\wp$. In particular the restrictions $\mathcal{F} \otimes \kappa(\wp)$ and $\mathcal{G} \otimes \kappa(\wp)$ are non-zero. Let t be the image of $\wp$ via the projection $X \times T \to T$. Then $\wp$ belongs to the fiber X_t (cf. e.g. [Sta25, Tag 01JT, Lemma 26.17.5]). By assumption, this implies that both restrictions have to be zero, which leads to a contradiction. □

Corollary 6.3. *Under the same assumptions of Lemma 6.1, the stack ${}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_C; , \ell)$ has a 3-cellular structure for all $\ell \geq 1$.*

Proof. First, note that $\{p\} \subset C$ is a stratification of C satisfying Assumptions 0.1–(1) and –(2), and 0.7. Let $[\mathcal{E}]$ be a K-valued point of $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_C, \ell)$, where K is an arbitrary $\mathbb{C}$-field. Let

$$\mathcal{E}_0 \subset \mathcal{E} \tag{6.1}$$

be the support filtration associated to the stratification $\{p_K\} \subset C_K$. Note that one has in fact a canonical decomposition

$$\mathcal{E} \simeq \mathcal{E}_0 \oplus \mathcal{F} \tag{6.2}$$

where $\mathcal{E}_0$ is set-theoretically supported at p_K while the set-theoretic support of $\mathcal{F}$ is contained in C°_K.

By Corollary 2.24, the stack $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_C; \ell)$ admits a locally closed stratification $\{\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_C; \ell)^m\}$, with $m \geq 0$, where the ℓ-th stratum is defined by the condition $\chi(\mathcal{E}_0) = m$. Let $\mathcal{E}^{(m)}$ be the restriction of the universal family to $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_C; \ell)^m \times X$. By construction, $\mathcal{E}^{(m)}$ admits a one-step relative support filtration

$$0 \subset \mathcal{E}^{(m)}_0 \subset \mathcal{E}^{(m)}$$

which restricts to the filtration (6.1) on each fiber $[\mathcal{E}] \times X$. Let $\mathcal{F}^{(m)} := \mathcal{E}^{(m)} / \mathcal{E}^{(m)}_0$ and note that the restrictions of $\mathcal{E}^{(m)}_0$ and $\mathcal{F}^{(m)}$ to any fiber $[\mathcal{E}] \times X$ are isomorphic to the summands $\mathcal{E}_0$ and $\mathcal{F}$, respectively, in Formula (6.2). By Lemma 6.2, this implies that the set-theoretic support of $\mathcal{F}^{(m)}$ is disjoint from the set-theoretic support of $\mathcal{E}^{(m)}_0$. Therefore one has a canonical decomposition

$$\mathcal{E}^{(m)} \simeq \mathcal{E}^{(m)}_0 \oplus \mathcal{F}^{(m)} .$$

This yields a canonical equivalence

$$\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_C; \ell)^m \simeq \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_p; m) \times \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{C^\circ}; \ell - m) \tag{6.3}$$

for any $0 \leq m \leq \ell$.

In conclusion, note that one has a stratification

$$\emptyset \subset {}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_C; \ell)^{\geqslant \ell} \subset \cdots \subset {}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_C; \ell)^{\geqslant 0} = {}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_C; \ell)$$

where ${}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_C; \ell)^{\geqslant m}$ is the reduced closed substack consisting of zero-dimensional sheaves such that $\chi(\mathcal{E}_0) \geq m$. In particular, the m-th locally closed stratum is equivalent to ${}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_C; \ell)^m$ for all $0 \leq m \leq \ell$. Given the equivalence (6.3), Lemma 6.1 shows that all strata are 2-cellular. Hence ${}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_C; \ell)$ admits a 3-cellular structure. □

Let $Z \subset X$ be a reduced effective divisor and let

$$\emptyset =: Z^{-1} \subset Z^0 \subset Z^1 \subset \cdots \subset Z^s := Z \tag{6.4}$$

be a stratification of Z by reduced closed subschemes satisfying satisfying Assumptions 0.1–(1) and –(2) in §2. Recall that the locally closed strata are denoted by $Z_i^\circ := Z^i \smallsetminus Z^{i-1}$, and Z_i denotes the scheme theoretic closure of Z_i° for all $0 \leq i \leq s$. Moreover, the scheme-theoretic intersection $Z_i^{i-1} := Z^{i-1} \cap Z_i$ is zero-dimensional. In particular, $Z_i = Z^i$ for $0 \leq i \leq 1$, and $Z_1^0 = Z_0$ is a zero-dimensional subscheme of Z_1.

In addition, suppose the following holds.

Assumption 0.7.

(1) For all $1 \leq i \leq s$, each connected component $Z_{i,\alpha}$ of Z_i is a smooth rational (-2)-curve on X for $1 \leq \alpha \leq c_i$.

(2) For all $2 \leq i \leq s$ and $1 \leq \alpha \leq c_i$, the set-theoretic intersection of Z_i^{i-1} and $Z_{i,\alpha}$ consists of exactly one closed point of $Z_{i,\alpha}$.

(3) For $1 \leq \alpha \leq c_1$, the set theoretic intersection of Z_0 and $Z_{1,\alpha}$ is either empty or consists of a exactly one closed point of $Z_{1,\alpha}$.

⊘

Remark 6.4. Assumption 0.7 implies that the scheme-theoretic intersection $Z_i^\circ \cap Z_{i,\alpha}$ is isomorphic to $\mathbb{A}^1$ for all $2 \leq i \leq s$ and $1 \leq \alpha \leq c_i$.

For $1 \leq \alpha \leq c_1$, the scheme-theoretic intersection $Z_1^\circ \cap Z_{1,\alpha}$ is isomorphic to $Z_{1,\alpha}$ if the set-theoretic intersection $Z_0 \cap Z_{1,\alpha}$ is empty, or it is isomorphic to $\mathbb{A}^1$ if the set-theoretic intersection $Z_0 \cap Z_{1,\alpha}$ is non-empty. △

Now, let $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \ell)$ be the derived moduli stack of length ℓ zero-dimensional sheaves on X with set-theoretic support on Z, with $\ell \in \mathbb{N}$ and $n \geq 1$. Let $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z^\circ}; \ell) \subset \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \ell)$ be the open substack consisting of zero-dimensional sheaves with set-theoretic support contained in $Z^\circ := Z \smallsetminus Z^0$. Using the construction in §3.1, we also have a closed substack $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \ell)^{\geqslant 1} \subset \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \ell)$ consisting of zero dimensional sheaves whose support contains at least one point of Z_0. These are quasi-compact quasi-separated indgeometric derived stacks. In particular, the corresponding reduced stacks are quasi-compact quasi-separated geometric classical stacks. Moreover, they are of finite type.

Proposition 6.5. *The stacks* ${}^{\mathsf{red}}\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \ell)$*,* ${}^{\mathsf{red}}\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z^\circ}; \ell)$*, and* $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \ell)^{\geqslant 1}$ *are T-equivariantly* 4*-cellular for all* $\ell \geq 1$.

Proof. We will provide the details only for ${}^{\mathsf{red}}\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \ell)$, the remaining two cases being analogous. Given a $\mathbb{C}$-field K and a K-valued point $[\mathcal{E}]$ of $\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \ell)$, let

$$0 =: \mathcal{E}^{-1} \subset \mathcal{E}^0 \subset \cdots \subset \mathcal{E}^s := \mathcal{E}$$

be the support filtration associated to the stratification (6.4), i.e., each $\mathcal{E}^{i-1}$ is the maximal subsheaf of $\mathcal{E}^i$ with set-theoretic support contained in Z_K^{i-1}, for $1 \leq i \leq s$. By construction, each quotient $\mathcal{E}^i/\mathcal{E}^{i-1}$ is a zero-dimensional sheaf with set-theoretic support contained in the stratum $(Z_i^\circ)_K$.

Applying Corollary 2.24, for any $\mathsf{m} = (m_0, m_1, \ldots, m_s) \in \mathbb{Z}^{s+1}$, with $m_0, m_1, \ldots, m_s \geq 0$ and $m_s = \ell$, let $\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \ell)^{\preceq \mathsf{m}} \subset \mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \ell)$ be the open substack defined by the conditions

$$\chi(\mathcal{E}^i) \leq m_i \quad \text{for } 0 \leq i \leq s\,.$$

Let also $\mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \ell)^{\mathsf{m}} \subset \mathfrak{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \ell)^{\preceq \mathsf{m}}$ be the closed substack where

$$\chi(\mathcal{E}^i) = m_i \quad \text{for } 0 \leq i \leq s\,.$$

All the above stacks are quasi-compact and quasi-separated.

By construction, the restriction $\mathcal{E}^{(\mathsf{m})}$ of the universal family $\mathcal{E}$ to $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\ell)^{\mathsf{m}} \times X$ admits a relative $Z^\bullet$-filtration

$$0 =: (\mathcal{E}^{(\mathsf{m})})^{-1} \subset (\mathcal{E}^{(\mathsf{m})})^{0} \subset \cdots \subset \cdots \subset (\mathcal{E}^{(\mathsf{m})})^{s} := \mathcal{E}^{(\mathsf{m})} .$$

Furthermore, each subquotient $\mathcal{E}_i^{(\mathsf{m})} := (\mathcal{E}^{(\mathsf{m})})^i/(\mathcal{E}^{(\mathsf{m})})^{i-1}$ is a flat family of length $\ell_i := m_i - m_{i-1}$ zero-dimensional sheaves with set-theoretic support contained in Z_i°. Then, Lemma 6.2 shows that one has a canonical decomposition

$$\mathcal{E}^{(\mathsf{m})} \simeq \bigoplus_{i=0}^{s} \mathcal{E}_i^{(\mathsf{m})} .$$

This yields a canonical equivalence

$$\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\ell)^{\mathsf{m}} \simeq \bigtimes_{i=0}^{s} \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_i^\circ};\ell_i) .$$

Using the notation in Assumption 0.7, one further obtains an equivalence

$$\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\ell)^{\mathsf{m}} \simeq \bigtimes_{i=0}^{s} \bigtimes_{\alpha=1}^{c_i} \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_{i,\alpha}^\circ};\ell_i) . \tag{6.5}$$

As noted in Remark 6.4, one has $Z_{i,\alpha}^\circ \simeq \mathbb{A}^1$ for all $2 \leq i \leq s$ and $1 \leq \alpha \leq c_i$. Moreover, $Z_{1,\alpha}^\circ$ is either isomorphic to $\mathbb{A}^1$ or it coincides with its scheme-theoretic closure $Z_{1,\alpha}$. Moreover, for each $1 \leq \alpha \leq c_0$, $Z_{0,\alpha}^\circ$ is a zero-dimensional scheme supported at a single closed point.

Now let

$$\emptyset =: \mathfrak{X}_{-1} \subset \mathfrak{X}_0 \subset \cdots \subset \mathfrak{X}_n := {}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\ell) \tag{6.6}$$

be the stratification constructed inductively as follows. First, let $\mathfrak{X}_0 \subset {}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\ell)$ be the scheme theoretic-union of the *closed* strata ${}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\ell)^{\mathsf{m}}$. Since ${}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z,\ell)$ is quasi-compact, there are finitely many such strata and $\mathfrak{X}_0$ is a reduced quasi-compact geometric classical stack.

Next, for any $j \geq 1$, let $\mathfrak{X}_j \subset {}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\ell)$ be the scheme-theoretic union of all strata ${}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\ell)^{\mathsf{m}}$ such that the scheme-theoretic closure of ${}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\ell)^{\mathsf{m}}$ is contained in the scheme theoretic union

$$\mathfrak{X}_{j-1} \cup {}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\ell)^{\mathsf{m}} .$$

Since ${}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\ell)$ is quasi-compact, this yields a finite stratification as above. Furthermore, each stratum $\mathfrak{X}_j^\circ := \mathfrak{X}_j \smallsetminus \mathfrak{X}_{j-1}$ is a disjoint union of locally closed substacks ${}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\ell)^{\mathsf{m}}$.

Given the equivalence (6.5), Lemma 6.1, and Corollary 6.3 show that the stratification (6.6) defines a 4-cellular structure on ${}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\ell)$. □

6.2. **Cellular structure for open substacks defined by slope conditions.** In the framework of §6.1, let (H, B) be a twisted stability condition for sheaves on X. Fix $\gamma \neq 0$ and $\alpha \in \mathbb{R}$, and let

$$\mathfrak{U}_\alpha(\gamma, n) \subset \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma, n)$$

be a quasi-compact quasi-separated open substack parametrizing those coherent sheaves $\mathcal{E}$ for which

$$\mu_{(H,B)\text{-}\min}(\mathcal{E}) > \alpha .$$

As shown in [DPS+25b, §4.6] and §A, this is a quasi-compact quasi-separated indgeometric derived stack. In particular, the corresponding reduced stack is a quasi-compact quasi-separated geometric classical stack. Moreover, it is of finite type.

We will assume that Z admits a stratification (6.4) satisfying Assumptions 0.1–(1) and –(2), and 0.7. Note that $Z^0 \subset Z$ defines a coarser stratification of Z satisfying Assumptions 0.1–(1) and –(2).

Let $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z^\circ};\gamma,n)^\circ \subset \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z^\circ};\gamma,n)$ be the substack parametrizing those coherent sheaves $\mathcal{E}$ such that the maximal zero-dimensional subsheaf of $\mathcal{E}$ with set-theoretic support contained in Z^0_{red} is identically zero. This is an open substack by Corollary 2.24. Set

$$\mathfrak{U}_\alpha(\gamma,n)^\circ := \mathfrak{U}_\alpha(\gamma,n) \times_{\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)} \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)^\circ \ .$$

Note that $\mathfrak{U}_\alpha(\gamma,n)^\circ$ is an open substack of $\mathfrak{U}_\alpha(\gamma,n)$. Similarly, the construction in §3 provides a closed substack $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z^\circ};\gamma,n)^{\geqslant 1} \subset \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z^\circ};\gamma,n)$ consisting of those coherent sheaves $\mathcal{E}$ such that the maximal zero-dimensional subsheaf of $\mathcal{E}$ with set-theoretic support contained in Z^0_{red} is nonzero. Let

$$\mathfrak{U}_\alpha(\gamma,n)^{\geqslant 1} := \mathfrak{U}_\alpha(\gamma,n) \times_{\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)} \mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)^{\geqslant 1} \ .$$

Moreover, for any pair $(\gamma',m) \in \langle Z\rangle \times \mathbb{Z}$, with $\gamma' \neq 0$, let $\mathbf{Coh}^{\mathsf{ss}}(\widehat{X}_Z;\gamma',m)$ denote the derived moduli stack of (H,B)-semistable sheaves on X with set-theoretic support contained in Z. Again, this is a quasi-compact quasi separated indgeometric derived stack. Its reduced stack ${}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{ss}}(\widehat{X}_Z;\gamma',m)$ is a quasi-compact quasi-separated geometric classical stack. In particular, it is of finite type.

In addition to Assumptions 0.1–(1) and –(2), and 0.7, suppose the following conditions hold.

Assumption 0.8. There exists a twisted stability condition (H,B) and an integer $\ell \geq 1$ so that for any $(\gamma',m) \in \langle Z\rangle \times \mathbb{Z}$, $\gamma' \neq 0$, the reduced moduli stack ${}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{ss}}(\widehat{X}_Z;\gamma',m)$, is ℓ-cellular for some $\ell \in \mathbb{N}$. ⊘

Assumption 0.9. There exists a Zariski open subscheme $U \subset X$ containing Z as a closed subscheme, so that $\omega_X|_U \simeq \mathcal{O}_U$. ⊘

Proposition 6.6. *Suppose Assumptions 0.1–(1) and –(2), 0.7, 0.8, and 0.9 hold. Let (H,B) and $\ell \geq 1$ be as in Assumption 0.8. Let $\ell' := \max\{5, \ell+1\}$. Then, for any $(\gamma,n) \in \langle Z\rangle \times \mathbb{Z}$, and for any $\alpha \in \mathbb{R}$, the stacks*

$$\mathfrak{U}_\alpha(\gamma,n)\ , \quad \mathfrak{U}_\alpha(\gamma,n)^\circ\ , \quad \mathfrak{U}_\alpha(\gamma,n)^{\geqslant 1}$$

is ℓ'-cellular with respect to the Definition B.16. In particular, for any sequence $\alpha_0 > \alpha_1 > \cdots$ of real numbers, the open exhaustions

$$\{\mathfrak{U}_{\alpha_k}(\gamma,n)\}_{k\in\mathbb{N}}\ , \quad \{\mathfrak{U}_{\alpha_k}(\gamma,n)^\circ\}_{k\in\mathbb{N}}\ , \quad \{\mathfrak{U}_{\alpha_k}(\gamma,n)^{\geqslant 1}\}_{k\in\mathbb{N}}$$

are T-equivariantly ind ℓ'-cellular as in Definition B.22.

Proof. We will show that $\mathfrak{U}_\alpha(\gamma,n)$ is ℓ'-cellular. The remaining two cases are analogous.

We will fist show that one has a stratification

$$\emptyset =: \mathfrak{X}_0 \subset \cdots \subset \mathfrak{X}_s := {}^{\mathsf{red}}\mathfrak{U}_\alpha(\gamma,n) \tag{6.7}$$

so that each stratum $\mathfrak{X}_i^\circ := \mathfrak{X}_i \smallsetminus \mathfrak{X}_{i-1}$ is T-equivariantly equivalent to an iterated vector bundle stack over a product of 4-cellular stacks and ℓ-cellular stacks. In particular, ${}^{\mathsf{red}}\mathfrak{U}_\alpha(\gamma,n)$ is ℓ'-cellular.

For each $m \in \mathbb{Z}$, let $\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;m)$ be the moduli stack of length m zero-dimensional sheaves with set-theoretic support contained in Z. By construction, $\mathfrak{U}_\alpha(\gamma,n)$ has a Harder-Narasimhan stratification where each stratum $\mathfrak{U}^\lambda_\alpha(\gamma,n)$ maps naturally to a product

$$\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;m_0) \times \bigtimes_{j=1}^{h_\lambda} \mathfrak{Coh}^{\mathsf{ss}}(\widehat{X}_Z;\gamma_j,m_j)$$

with $\gamma_j \neq 0$, for $1 \leq j \leq h_\lambda$. Using Serre duality, Assumption 0.9 implies that the above map has an iterated vector bundle stack structure. This implies that ${}^{\mathsf{red}}\mathfrak{U}_\alpha(\gamma,n)$ has a stratification where each stratum is an iterated vector bundle stack over the product

$$ {}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; m_0) \times \bigtimes_{j=1}^{h_\lambda} {}^{\mathsf{red}}\mathfrak{Coh}^{\mathsf{ss}}(\widehat{X}_Z; \gamma_j, m_j) \,. $$

By Proposition 6.5, the first factor above is 4-cellular, while the remaining factors are ℓ-cellular by Assumption 0.8.

The stratification (6.7) will be constructed inductively as in the proof of Proposition 6.5. Let $\mathcal{X}_1$ be the scheme-theoretic union of all closed Harder-Narasimhan strata in ${}^{\mathsf{red}}\mathfrak{U}_\alpha(\gamma, n)$. Then, for each $j \geq 2$, let $\mathcal{X}_j$ be the scheme-theoretic union of all strata ${}^{\mathsf{red}}\mathfrak{U}^\lambda_\alpha(\gamma, n)$, whose closure is contained in $\mathcal{X}_{j-1} \cup {}^{\mathsf{red}}\mathfrak{U}^\lambda_\alpha(\gamma, n)$. By construction, each locally closed stratum is a union of disjoint reduced Harder-Narasimhan strata ${}^{\mathsf{red}}\mathfrak{U}^\lambda_\alpha(\gamma, n)$ as above. This proves the claim.

To prove the second claim, for any $k, h \in \mathbb{N}$, $k < h$, let $\mathcal{Z}_{k,h}$ be the reduced closed complement of the open substack ${}^{\mathsf{red}}\mathfrak{U}_{\alpha_k}(\gamma, n) \subset {}^{\mathsf{red}}\mathfrak{U}_{\alpha_h}(\gamma, n)$. Note that the Harder-Narasimhan stratification is naturally compatible with the open and closed immersions, respectively,

$$ {}^{\mathsf{red}}\mathfrak{U}_{\alpha_k}(\gamma, n) \longrightarrow {}^{\mathsf{red}}\mathfrak{U}_{\alpha_h}(\gamma, n) \quad \text{and} \quad \mathcal{Z}_{k,h} \longrightarrow {}^{\mathsf{red}}\mathfrak{U}_{\alpha_h}(\gamma, n) \,. $$

Repeating the argument in the above paragraph, this implies that the induced stratifications

$$ \mathcal{X}_i \times_{{}^{\mathsf{red}}\mathfrak{U}_{\alpha_h}(\gamma,n)} {}^{\mathsf{red}}\mathfrak{U}_{\alpha_k}(\gamma, n) \quad \text{and} \quad \mathcal{X}_i \times_{{}^{\mathsf{red}}\mathfrak{U}_{\alpha_h}(\gamma,n)} \mathcal{Z}_{k,h} $$

are T-equivariantly ℓ'-cellular. Therefore the conditions (1) and (2) in Definition B.22 are satisfied. □

7. First example: ADE quiver

Starting with this section, we will present several explicit examples where Theorem 4.21 holds. In all these examples, Z will be a reduced tree of rational (-2)-curves on X so that the equivalence classes of the irreducible components of Z generate a free abelian subgroup $\langle Z \rangle \subset \mathsf{NS}(X)$. Moreover, we will impose that Assumption 0.9 holds, i.e., Z is contained in a Zariski open subscheme $U \subset X$ so that $\omega_X|_U \simeq \mathcal{O}_U$. We first introduce the notion of generic twisted stability conditions.

7.1. **Generic stability conditions.** As in §5, let H be an ample divisor on X and let $B \in \mathsf{NS}(X)_\mathbb{R}$. Recall that the (H, B)-slope of a pure one-dimensional coherent sheaf $\mathcal{E}$ on X, with $\mathsf{ch}_1(\mathcal{E}) \neq 0$, is defined as

$$ \mu_{(H,B)}(\mathcal{E}) = \frac{\chi(\mathcal{E}) + B \cdot \mathsf{ch}_1(\mathcal{E})}{H \cdot \mathsf{ch}_1(\mathcal{E})} \,. $$

Given any one-dimensional coherent sheaf $\mathcal{E}$, the pair $v(E) := (\mathsf{ch}_1(\mathcal{E}), \chi(\mathcal{E})) \in \mathsf{NS}(X) \times \mathbb{Z}$ is by definition the Mukai vector of E. For any vector $v = (\gamma, n) \in \mathsf{NS}(X) \oplus \mathbb{Z}$, with $\gamma \neq 0$, let

$$ \mu_{(H,B)}(v) := \frac{n + B \cdot \gamma}{H \cdot \gamma} \,. $$

Definition 7.1. We say that the twisted stability parameter $B \in \mathsf{NS}(X)_\mathbb{R}$ is *generic* if given any element $\gamma \in \langle Z \rangle$, the condition $\gamma \cdot B \in \mathbb{Q}$ implies $\gamma = 0$. ⊘

Remark 7.2. It is straightforward to show that B is generic if the intersection numbers of B with the irreducible components of Z are sufficiently generic irrational numbers. △

Note that Definition 7.1 implies the following.

Lemma 7.3. *Let $B \in \mathsf{NS}(X)_\mathbb{R}$ be generic. Let $H \in \mathsf{NS}(X)$ be an ample class and let $v = (\gamma, n), v' = (\gamma', n') \in \langle Z \rangle \times \mathbb{Z}$, with $\gamma, \gamma' \neq 0$. Then*

$$ \mu_{(H,B)}(v) = \mu_{(H,B)}(v') $$

if and only if

$$v = \frac{H \cdot \gamma'}{H \cdot \gamma} v' \,.$$

7.2. **Preliminary results on (H, B)-semistable sheaves.** For future reference we record some basic results on (H, B)-semistable sheaves on X with set theoretic support on Z.

Lemma 7.4. *Let $\mathcal{E}$ be a nonzero purely one dimensional (H, B)-stable sheaf on X. Then, the following hold.*

(i) If $\mathsf{ch}_1^2(\mathcal{E}) = 0$, then $\dim \mathsf{Ext}^1_X(\mathcal{E}, \mathcal{E}) = 2$.

(ii) If $\mathsf{ch}_1(\mathcal{E})^2 \leq -2$, then $\mathsf{ch}_1(\mathcal{E})^2 = -2$ and $\dim \mathsf{Ext}^1_X(\mathcal{E}, \mathcal{E}) = 0$.

Proof. Given that Z satisfies Assumption 0.9, one has an isomorphism $\omega_X \otimes \mathcal{E} \simeq \mathcal{E}$. In particular, $K_X \cdot \mathsf{ch}_1(\mathcal{E}) = 0$. Then the Grothendieck-Riemann-Roch theorem yields

$$\chi(\mathcal{E}, \mathcal{E}) = -\mathsf{ch}_1(E)^2 \,.$$

Since $\mathcal{E}$ is assumed to be (H, B)-stable, $\mathsf{Ext}^0_X(\mathcal{E}, \mathcal{E}) \simeq \mathbb{C}$. Hence, Serre duality, one obtains

$$\chi(\mathcal{E}, \mathcal{E}) = 2 - \dim \mathsf{Ext}^1_X(\mathcal{E}, \mathcal{E}) \,.$$

Therefore one obtains

$$\dim \mathsf{Ext}^1_X(\mathcal{E}, \mathcal{E}) = 2 + \mathsf{ch}_1(\mathcal{E})^2 \,.$$

Keeping in mind that $\dim \mathsf{Ext}^1_X(\mathcal{E}, \mathcal{E}) \geq 0$, this implies the claim. □

For any $v = (\gamma, n) \in \langle Z \rangle \times \mathbb{Z}$, with $\gamma \neq 0$, let $\mathcal{S}^{\mathsf{ss}}_{(H,B)}(v)$ denote the set of isomorphism classes of (H, B)-semistable sheaves on X with Mukai vector v, which are set theoretically supported on Z. Let $\mathcal{S}^{s}_{(H,B)}(v)$ be the subset of isomorphism classes of stable objects.

Lemma 7.5. *Let $v = (\gamma, n) \in \langle Z \rangle \times \mathbb{Z}$, with $\gamma^2 = -2$. Then, for any twisted stability condition (H, B), the set $\mathcal{S}^{s}_{(H,B)}(v)$ is either empty or it has one element.*

Proof. Assume that $\mathcal{E}, \mathcal{F}$ are (H, B)-stable sheaves with $v(\mathcal{E}) = v(\mathcal{F}) = v$ so that $\mathcal{E} \not\simeq \mathcal{F}$. Then

$$\mathsf{Ext}^0_X(\mathcal{E}, \mathcal{F}) = 0 \quad \text{and} \quad \mathsf{Ext}^0_X(\mathcal{F}, \mathcal{E}) = 0 \,.$$

Moreover, since Assumption 0.9 holds, one has

$$\omega_X \otimes \mathcal{E} \simeq \mathcal{E} \quad \text{and} \quad \omega_X \otimes \mathcal{F} \simeq \mathcal{F} \,,$$

as well as

$$K_X \cdot \mathsf{ch}_1(\mathcal{E}) = K_X \cdot \mathsf{ch}_1(\mathcal{F}) = 0 \,.$$

Then, the Grothendieck-Riemann-Roch theorem yields $\chi(\mathcal{E}, \mathcal{F}) = -\gamma^2 = 2$. Hence, by Serre duality, one obtains

$$\dim \mathsf{Ext}^1_X(\mathcal{E}, \mathcal{F}) = -2 \,,$$

leading to a contradiction. □

7.3. **ADE trees.** In this section, X will be the minimal resolution $\pi\colon X \to X_{\mathsf{con}}$ of a canonical ADE singularity, and Z will be the reduced exceptional divisor $\pi^{-1}(0)_{\mathsf{red}}$. We will assume that Assumption 0.9 holds, i.e., Z is contained in a Zariski open subscheme $U \subset X$ so that $\omega_X|_U \simeq \mathcal{O}_U$. Moreover, we will also assume that there is a torus action $T \times X \to X$ preserving Z, allowing the case $T = \{1\}$.

First note the following.

Lemma 7.6. *The subgroup $\langle Z \rangle$ is naturally isomorphic to the associated ADE root lattice, the intersection pairing being identified with the negative of the Cartan matrix.*

Moreover, if D is a nonzero effective divisor on X so that $[D] \in \langle Z \rangle$, then the set-theoretic support of D is contained in Z.

Proof. It is well known that Z is a reduced ADE tree of rational (-2)-curves. By [Han14, Proposition 1.5], the Chow group $\mathsf{CH}_1(Z)$ is freely generated by its irreducible components. Moreover, by [Han14, Proposition 2.2], one has an exact sequence

$$0 \longrightarrow \mathsf{CH}_1(Z) \longrightarrow \mathsf{CH}_1(X) \xrightarrow{\pi_*} \mathsf{CH}_1(X_{\mathsf{con}}) \longrightarrow 0 \ , \tag{7.1}$$

where the first arrow from the left-hand-side is the natural push-forward map. In particular, this induces an isomorphism $\mathsf{CH}_1(Z) \to \langle Z \rangle$. This proves the first part.

Given a nonzero effective divisor D so that $[D] \in \langle Z \rangle$, suppose the set theoretic support of D is not contained in Z. This implies that the restriction of D to $X \smallsetminus Z$ is nonzero in $\mathsf{CH}_1(X \smallsetminus Z)$. Since the projection $X \to X_{\mathsf{con}}$ identifies $X \smallsetminus Z$ to the complement of the singular point, it follows that $\pi_*([D]) \neq 0$. This is in contradiction with the exact sequence (7.1). □

Lemmas 7.4 and 7.6 yield the following.

Corollary 7.7. *Let $\mathcal{E}$ be a nonzero (H, B)-stable sheaf with $\mathsf{ch}_1(\mathcal{E}) \in \langle Z \rangle$. Then $\mathsf{ch}_1^2(\mathcal{E}) = -2$ and $\mathsf{Ext}^1_X(\mathcal{E}, \mathcal{E}) = 0$. Moreover, the set-theoretic support of $\mathcal{E}$ is contained in Z.*

Furthermore, using Lemma 7.5 and Corollary 7.7 we next prove the following.

Lemma 7.8. *Assume that B is generic. Then, for any $v = (\gamma, n) \in \langle Z \rangle \times \mathbb{Z}$, with $\gamma \neq 0$, the set $\mathcal{S}^{\mathsf{ss}}_{(H,B)}(v)$ is either empty or it consists of a unique polystable sheaf with set-theoretic support contained in Z.*

Proof. Suppose $\mathcal{S}^{\mathsf{ss}}_{(H,B)}(v)$ is not empty. Let $\mathcal{E}$ be a purely one dimensional (H, B)-semistable sheaf on X with $v(\mathcal{E}) = v$, and let $\mathcal{E}_i$ be its Jordan-Hölder subquotients with $1 \leq i \leq \ell$. Since $\mathsf{ch}_1(\mathcal{E}) \in \langle Z \rangle$, the set theoretic support of $\mathcal{E}$ is contained in Z by Lemma 7.6. Thus, the same holds for each $\mathcal{E}_i$, with $1 \leq i \leq \ell$. Then, Corollary 7.10 shows that $\mathsf{ch}_1(\mathcal{E}_i)^2 = -2$ for all $1 \leq i \leq \ell$. Since B is generic, for any $1 \leq i, j \leq \ell$, with $i \neq j$, one has

$$\mathsf{ch}_1(\mathcal{E}_i) = \lambda \mathsf{ch}_1(\mathcal{E}_j) \quad \text{and} \quad \chi(\mathcal{E}_i) = \lambda \chi(\mathcal{E}_j) \ ,$$

for some $\lambda \in \mathbb{Q}$, with $\lambda > 0$. Then $\lambda^2 = 1$, hence $\lambda = 1$. Therefore

$$\mathsf{ch}_1(\mathcal{E}) = \ell \beta \quad \text{and} \quad \chi(\mathcal{E}) = \ell m \ ,$$

for some $(\beta, m) \in \langle Z \rangle \times \mathbb{Z}$, with $\beta^2 = -2$, so that $v(\mathcal{E}_i) = (\beta, m)$ for all $1 \leq i \leq \ell$. Then, Lemma 7.5 shows that all $\mathcal{E}_i$ are isomorphic to some (H, B)-stable purely one-dimensional sheaf $\mathcal{E}_v$ with Mukai vector (β, m). Moreover, as shown in loc. cit., $\mathcal{E}_v$ is uniquely determined by (β, m). By Lemma 8.20, one has $\mathsf{Ext}^1_X(\mathcal{E}_v, \mathcal{E}_v) = 0$, hence, indeed $\mathcal{E} \simeq \mathcal{E}_v^{\oplus \ell}$.

In order to complete the proof, note that given any two (H, B)-semistable sheaves $\mathcal{E}, \mathcal{E}'$ with Mukai vectors $v(\mathcal{E}) = v(\mathcal{E}')$, one has $\ell = \ell'$. This follows from the above argument, which implies that γ can be simultaneously written as

$$\gamma = \ell \beta = \ell' \beta'$$

with $\beta^2 = (\beta')^2 = -2$. This implies $\ell^2 = (\ell')^2$, hence $\ell = \ell'$ since they are both positive. In conclusion, ℓ and (β, m) are uniquely determined by v. This proves the claim. $\square$

In conclusion, we obtain the following.

Proposition 7.9. *Assume that B is generic. Then, for any $v = (\gamma, n) \in \langle Z \rangle \times \mathbb{Z}$, with $\gamma \neq 0$, the natural map $\mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(\widehat{X}_Z; v) \to \mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(X; v)$ is an isomorphism. Moreover, the moduli stack $\mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(X; v)$ is either empty or it is isomorphic to the classifying stack of a general linear group. In particular, Assumption 0.8 holds with $\ell = 0$.*

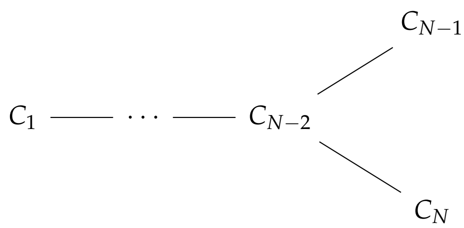
Fix $\alpha \in \mathbb{R}$ and $\gamma \in \langle Z \rangle \smallsetminus 0\}$. Now, let $\mathfrak{U}_\alpha(\gamma, n) \subset \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n)$ be the open substack parametrizing those coherent sheaves $\mathcal{E}$ such that $\mu_{(H,B)\text{-}\min}(\mathcal{E}) \geq \alpha$. Then, Propositions 6.6 and 7.9 yield the following.

Corollary 7.10. *Let (H, B) be a twisted stability condition with B generic. Then, ${}^{\mathsf{red}}\mathfrak{U}_\alpha(\gamma, n)$ has a 5-cellular structure for all $\gamma \in \langle Z \rangle \smallsetminus \{0\}$ and $n \in \mathbb{Z}$.*

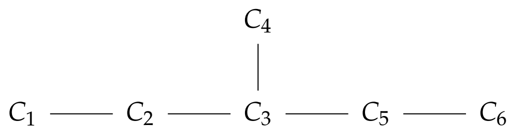
Next we show that Z admits a stratification satisfying Assumptions 0.1 and 0.7. We first recall the dual intersection graphs associated to such fibers on a case by case basis. We shall denote by C_i the irreducible components of Z, with $1 \leq i \leq N$.

- Type A_N, with $N \geq 1$:

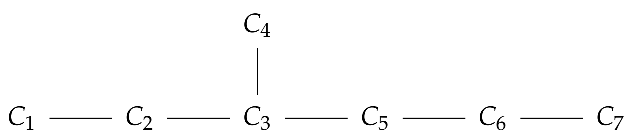
- Type D_N, with $N \geq 4$:



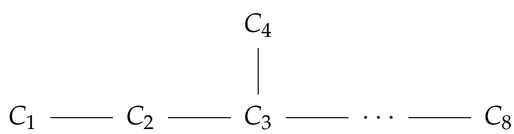
- Type E_6:



- Type E_7:



- Type E_8:



For the A series we will assume $N \geq 2$ below, since in the case $N = 1$ amalgamation is trivial.

Set $Z^N := Z$, and let Z^{i-1} be the scheme-theoretic closure of the complement $Z^i \smallsetminus C_i$ for any $1 \le i \le N$. Note that $Z^0 = \emptyset$. We get the following stratification of Z:

$$\emptyset = Z^0 \subset Z^1 \subset \cdots \subset Z^N := Z\,. \tag{7.2}$$

Lemma 7.11. *The following hold.*

(i) For any $1 \le i \le N$ the stratification

$$\emptyset = Z^0 \subset Z^1 \subset \cdots \subset Z^{i-1} \subset Z^i$$

of Z^i satisfies Assumption 0.1 and Assumption 0.7.

(ii) Each stratum $Z_i^\circ := Z^i \smallsetminus Z^{i-1}$ is isomorphic to $\mathbb{A}^1$, while its scheme-theoretic closure $Z_i := \overline{Z^i \smallsetminus Z^{i-1}}$ coincides with C_i, for $2 \le i \le N$. Moreover, $Z_1^\circ = Z^1 = C_1$.

(iii) The scheme-theoretic intersection $Z_{i-1}^i := Z^{i-1} \cap Z_i$ is a length one zero-dimensional subscheme contained in the smooth loci of Z^{i-1} and Z_i for any $2 \le i \le N$, while $Z^0 = \emptyset$. In particular, the one-step stratification

$$Z_i^{i-1} \subset Z_i$$

satisfies Assumptions 0.1 and 0.7 for any $2 \le i \le N$.

Proof. In order to prove (i), note that conditions (1) and (2) in Assumption 0.1 follow immediately by construction, while condition (3) in Assumption 0.1 follows from Lemma 7.6. Statements (ii) and (iii) listed above are also are clear by construction. □

Using the construction in §4.4, let $W^i \subset Z$ be the scheme-theoretic closure of the complement $Z \smallsetminus Z^{i-1}$. As observed in *loc. cit.*, W^i is a reduced closed subscheme of Z, containing Z_i, and the scheme-theoretic closure of the complement $W^i \smallsetminus Z_i$ coincides with W^{i+1}. Moreover, the scheme-theoretic intersection of W^i and Z_{i-1} coincides with Z_i^{i-1}.

Given topological invariants $\gamma_{i-1} \in \langle Z_{i-1} \rangle$, $\gamma^i \in \langle W^i \rangle$, and $n_1, n_2 \in \mathbb{Z}$, let

$$\{\mathfrak{U}_{i-1,k}(\gamma_{i-1}, n_1)\}_{k\in\mathbb{N}} \quad \text{and} \quad \{\mathfrak{U}_{i,k}(\gamma^i, n_2)\}_{k\in\mathbb{N}}$$

be the admissible open exhaustions of $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_{i-1}}; \gamma_{i-1}, n_1)$ and $\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{W^i}; \gamma^i, n_2)$, respectively, formed by quasi-compact quasi-separated indgeometric derived stacks, which are introduced in Definition 5.14. Let also $\mathfrak{U}_{i,k}(\gamma^i, n_2)^\circ \subset \mathfrak{U}_{i,k}(\gamma^i, n_2)$ be the open substack consisting of sheaves pure at Z_i^{i-1} for $k \in \mathbb{N}$, and let $\mathfrak{U}_{i,k}(\gamma^i, n_2)^{\geqslant 1} \subset \mathfrak{U}_{i,k}(\gamma^i, n_2)$ be its natural closed complement constructed in §3.1.

We now check that the one step stratification

$$\emptyset \subset Z_i \subset W^i \tag{7.3}$$

satisfies all the condition required for Theorem 4.21 to hold for all $1 \le i \le N$.

First note that lemma Lemma 7.11 yields:

Corollary 7.12. *The one step stratification $Z_i \subset W^i$ satisfies Assumption 0.1 for all $1 \le i \le N$.*

Next note that the following:

Remark 7.13.

(i) Theorem 5.15 implies Assumption 0.2, as well as Conditions (1) and (2) in Assumption 0.4 for all stratifications (7.3).

(ii) Corollary 7.10 yields condition (3) in Assumption 0.4 for all stratifications (7.3).

(iii) Assumption 0.6 for all stratifications (7.3) holds thanks to Corollary 5.6.

△

Finally, we note that Assumptions 0.3 and 0.5 are also satisfied for each one-step stratification (7.3) thanks to Corollary 7.10. More precisely, we have:

Proposition 7.14. *Let (H, B) be a twisted stability condition with B generic. Then, the open exhaustions $\{\mathfrak{U}_{i-1,k}(\gamma_{i-1}, n_1)\}_{k\in\mathbb{N}}$, $\{\mathfrak{U}_{i,k}(\gamma^i, n_2)\}_{k\in\mathbb{N}}$, $\{\mathfrak{U}_{i,k}(\gamma^i, n_2)^\circ\}_{k\in\mathbb{N}}$, and $\{\mathfrak{U}_{i,k}(\gamma^i, n_2)^{\geq 1}\}_{k\in\mathbb{N}}$ are T-equivariantly ind 5-cellular for any $1 \leq i \leq N$.*

Using Propositions 4.7 and 4.18, the above proposition yields:

Corollary 7.15.

(i) *For any $1 \leq i \leq N$, the restriction maps in T-equivariant (motivic) Borel-Moore homology associated to the natural open immersions*

$$\mathfrak{U}_{i,k}(\gamma^i, n_2) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_{i-1}}; \gamma_{i-1}, n_1) \longrightarrow \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{W^i}; \gamma^i, n_2) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_{i-1}}; \gamma_{i-1}, n_1)$$

and

$$\mathfrak{U}_{i,k}(\gamma^i, n_2)^\circ \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_{i-1}}; \gamma_{i-1}, n_1) \longrightarrow \mathfrak{U}_{i,k}(\gamma^i, n_2) \times \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_{i-1}}; \gamma_{i-1}, n_1)$$

are surjective for any $k \in \mathbb{N}$ and any values of the topological invariants. In particular, Assumption 0.3 holds in this case.

(ii) *For any $1 \leq i \leq N$ the Künneth maps*

$$\mathsf{H}^T_\bullet(\mathfrak{U}_{i,k}(\gamma^i, n_2)) \widehat{\otimes} \mathsf{H}^T_\bullet(\mathfrak{U}_{i-1,k}(\gamma_{i-1}, n_1)) \longrightarrow \mathsf{H}^T_\bullet(\mathfrak{U}_{i,k}(\gamma^i, n_2) \times \mathfrak{U}_{i-1,k}(\gamma_{i-1}, n_1))$$

and

$$\mathsf{H}^T_\bullet(\mathfrak{U}_{i,k}(\gamma^i, n_2)^\circ) \widehat{\otimes} \mathsf{H}^T_\bullet(\mathfrak{U}_{i-1,k}(\gamma_{i-1}, n_1)) \longrightarrow \mathsf{H}^T_\bullet(\mathfrak{U}_{i,k}(\gamma^i, n_2)^\circ \times \mathfrak{U}_{i-1,k}(\gamma_{i-1}, n_1))$$

are isomorphisms for any $k \in \mathbb{N}$ and any values of the topological invariants. In particular, Assumption 0.5 holds.

Using the construction in §4.4, the stratification (7.2) yields an amalgamation map

$$\overline{\mathfrak{a}}_\gamma \colon \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_N}, \gamma_N)) \widehat{\otimes}_{\mathbf{HA}^T_{X, Z_N^{N-1}}} \cdots \widehat{\otimes}_{\mathbf{HA}^T_{X, Z_2^1}} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}, \gamma_1)) \longrightarrow \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z, \gamma))$$

for any $\gamma \in \langle Z \rangle$, where $\gamma_i \in \langle Z_i \rangle$ are uniquely determined by the relation $\gamma = \sum_{i=1}^N \gamma_i$. For the sake of exposition, set:

$$\mathsf{H}^T_\bullet(\gamma_i, \ldots, \gamma_1) \coloneqq \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_i}, \gamma_i)) \widehat{\otimes}_{\mathbf{HA}^T_{X, Z_i^{i-1}}} \cdots \widehat{\otimes}_{\mathbf{HA}^T_{X, Z_2^1}} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}, \gamma_1))$$

for all $1 \leq i \leq N$.

In conclusion, since the assumptions of Theorem 4.21 hold by Remark 7.13 and Corollaries 7.12 and 7.15, we obtain the following.

Theorem 7.16. *For any $\gamma \in \langle Z \rangle$ there is a commutative diagram*

$$\begin{array}{ccc} \mathsf{H}^T_\bullet(\gamma_N, \ldots, \gamma_1) & \xrightarrow{\overline{\mathfrak{a}}_\gamma} & \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma)) \\ \downarrow & & \downarrow \\ \mathsf{H}^T_\bullet(\mathfrak{U}_{N,k}) \widehat{\otimes}_{\mathbf{HA}^T_{X, Z_N^{N-1}}} \mathsf{H}^T_\bullet(\gamma_{N_1}, \ldots, \gamma_1) & \xrightarrow{\overline{\tau}_{k,\gamma}} & \mathsf{H}^T_\bullet(\mathfrak{U}_k(\gamma)) \end{array} \tag{7.4}$$

where the vertical maps are determined by the restriction maps associated to the open immersions $\mathfrak{U}_{N,k}(\gamma_N) \to \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_N}; \gamma_N)$ and $\mathfrak{U}_k(\gamma) \to \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma)$, respectively.

Moreover, both horizontal arrows as well as the bottom horizontal arrow in the diagram (7.4) *are surjective. In particular the composition*

$$\begin{array}{ccc} \mathsf{H}^T_\bullet(\gamma_N, \ldots, \gamma_1) & \xrightarrow{\overline{\alpha}_\gamma} & \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma)) \\ & & \big\downarrow \\ & & \mathsf{H}^T_\bullet(\mathfrak{U}_k(\gamma)) \end{array}$$

is surjective for any $k \in \mathbb{N}$ and for all $\gamma \in \langle Z \rangle$.

In addition, note the following consequence of the proof of Theorem 7.16.

Corollary 7.17. *The T-equivariant Borel-Moore homology of the stack $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma)$ is strongly generated by algebraic cycles for all values of the topological invariants.*

8. Second example: Singular elliptic fibers

In this section, we construct a second explicit example for which Theorem 4.21 holds. Here Z will be a reduced tree of rational (-2)-curves of affine DE type.

8.1. **Elliptic surfaces.**

Let $\pi\colon X \to C$ be a smooth projective elliptic surface with a section $\sigma\colon C \to X$ over a smooth projective curve C. The scheme-theoretic image of C via σ will be denoted by C_0.

Definition 8.1. We say that X is called *relatively minimal* over C if there are no (-1)-rational curves on X whose scheme-theoretic image via π is zero-dimensional. ⊘

From now on, we assume that X is relatively minimal.

Recall that the *multiplicity* of any irreducible effective divisor D on X is defined as the unique positive integer m so that $[D] = m[D_{\mathsf{red}}]$ in the Picard group of X. Note that the existence of a section implies that all irreducible fibers of π have multiplicity one as divisors on X. In particular, any irreducible fiber is reduced. Furthermore, one has:

Lemma 8.2. *Suppose that E is a reducible fiber of π. Then*

(i) Any irreducible component of E_{red} is a rational (-2)-curve on X.

(ii) The section C_0 intersects nontrivially a unique reduced irreducible component E_0 of E, which has multiplicity one. Moreover the scheme-theoretic intersection of C_0 and E_0 is a single closed point contained in the smooth locus of E.

(iii) There exists a contraction $\xi\colon X \to \overline{X}$, where $\overline{X}$ is a Weierstrass model over C with a rational singular point $\nu \subset \overline{X}$ of type ADE so that $\xi^{-1}(\nu)$ coincides with the scheme-theoretic closure of the complement $E \smallsetminus E_0$ as subschemes of X.

Proof. Claim (i) is proven in [Fri98, Chapter 7, Lemma 11]. Claim (ii) follows from the fact that C_0 is a section of π, and Claim (iii) follows from [Fri98, Chapter 7, Lemma 12]. □

By [Fri98, Chapter 7, Theorem 15], one also has the following.

Lemma 8.3. *The dualizing sheaf ω_X is isomorphic to the pull-back of a line bundle on C via π.*

Assumption 0.10. There exists at least one point $o \in C$ so that the fiber $E := \pi^{-1}(o)$ is singular and reducible. ⊘

Remark 8.4. Note that a surface satisfying Assumption 0.10 can be constructed as a minimal resolution of a singular Weierstrass model. See for example [Fri98, Chapter 7] or [SS10, §4]. In addition, one of the isotrivial rational elliptic surfaces listed in [SS10, §9.7, Case (3)] provides an example admitting a nontrivial $\mathbb{C}^\times$-action. The associated singular Weierstrass model is the hypersurface

$$y^2z = x^3 + u^2v^2xz^2 + u^3v^3z^3$$

in the total space P of the projective bundle $\mathsf{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-4) \oplus \mathcal{O}_{\mathbb{P}^1}(-6))$. Here (x, y, z) are the canonical relative homogeneous coordinates with respect to the canonical projection $P \to \mathbb{P}^1$, and (u, v) are homogeneous coordinates on $\mathbb{P}^1$. The torus action is defined by the assignment

$$t \times (u, v, x, y, z) \mapsto (t^2u, v, t^2x, t^3y, z) .$$

It is straightforward to check that this action lifts to a $\mathbb{C}^*$-action on the minimal resolution X of the above Weierstrass model. Moreover, X is a rational elliptic surface over $\mathbb{P}^1$ with singular fibers of type affine D_4 at $(u, v) = (0, 1)$ and $(u, v) = (1, 0)$, all other fibers being smooth.

For completeness, note that one construct more examples of Weiestrass models over $\mathbb{A}^1$ which admit non-trivial $\mathbb{C}^*$-actions using for example the classification of isotrivial elliptic fibrations in [VA11]. However, in all cases except the one listed above, the projective completions of these models exhibit non-rational singularities at infinity. This makes the use of Fourier-Mukai functors in §8.2.3 problematic. △

Let $\mathsf{NS}(X)$ be the Neron-Severi group of X. By [SS10, Theorem 6.2], $\mathsf{NS}(X)$ is torsion-free and finitely generated. Furthermore, by [SS10, Theorem 6.5], it coincides with the group $N_1(X)$ of numerical equivalence classes of divisors on X. Let E_i denote the irreducible components of E_{red} for $0 \leq i \leq N-1$, and let m_i be their multiplicities, for $0 \leq i \leq N-1$, i.e.,

$$E = \sum_{i=0}^{N-1} m_i E_i$$

as divisors on X. Set $Z := E_{\mathsf{red}}$. Under the current assumptions one has the following result, which follows from [SS10, Theorem 6.3 and Proposition 6.6].

Lemma 8.5.

(i) *The subgroup* $\langle C_0, Z\rangle \subset \mathsf{NS}(X)$ *generated by* C_0 *and* E_i*, with* $0 \leq i \leq N-1$*, is free as a* $\mathbb{Z}$*-module.*

(ii) *The subgroup* $\langle Z\rangle \subset \mathsf{NS}(X)$ *coincides with the root lattice of an affine Lie algebra of type ADE. Moreover, the intersection pairing is identified with the negative of the affine Cartan matrix and the multiplicities* m_i *are the dual Coxeter numbers of the associated affine Dynkin diagram for* $0 \leq i \leq N-1$.

Remark 8.6. As a consequence of Lemma 8.5, note the intersection numbers

$$f^2 = f \cdot E_i = 0 \ \text{ for } \ 0 \leq i \leq N-1 , \quad f \cdot C_0 = 1 ,$$
$$C_0 \cdot E_0 = 1 , \quad C_0 \cdot E_i = 0 \ \text{ for } \ 1 \leq i \leq N-1 .$$

Here $f := [E]$. △

8.2. **Cellular structure for moduli stacks of semistable sheaves.** Let (H, B) be a twisted stability condition on X with B generic as in Definition 7.1. The goal of this section is to prove that moduli stacks $\mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(\widehat{X}_Z; \gamma, n)$ of (H, B)-semistable sheaves on X with set theoretic support on Z have cellular structure as in Definition B.16. As a first step we will establish the existence of coarse moduli spaces by relating twisted stability to parabolic stability.

8.2.1. *Twisted stability and parabolic structure.*

Definition 8.7.

(1) We say that a coherent sheaf on X of dimension at most one is *vertical* if any each irreducible component of its reduced set-theoretic support is a closed subscheme of a fiber of π.

(2) We say that coherent sheaf on X of dimension at most one is *horizontal* if it does not contain any vertical subsheaves.

(3) We say that a closed subscheme $S \subset X$ is *vertical* (resp. *horizontal*) if its structure sheaf is vertical (resp. horizontal) respectively.

⊘

Remark 8.8. Note that any zero-dimensional coherent sheaf on X is vertical. △

The goal of this section is to show that for certain twisted stability parameters $B \in \mathsf{NS}(X)_{\mathbb{Q}}$ any moduli stack of vertical (H, B)-semistable sheaves on X with fixed topological invariants admits a coarse projective moduli scheme[12]. This will be done by identifying these moduli stacks with certain moduli stacks of *parabolic sheaves* on X.

We start with some preliminary results. Let $\Gamma \subset \mathsf{NS}(X)$ be the subgroup spanned by vertical divisor classes. Clearly, if $\mathcal{E}$ is a vertical sheaf as in Definition 8.7, then $\mathsf{ch}_1(\mathcal{E}) \in \Gamma$. The result below shows that the converse is also true.

Lemma 8.9. *Let $\mathcal{E}$ be a nonzero purely one dimensional sheaf on X with $\mathsf{ch}_1(\mathcal{E}) \in \Gamma$. Then $\mathcal{E}$ is a vertical sheaf.*

Proof. Let $\langle C_0, \Gamma\rangle \subset \mathsf{NS}(X)$ be the subgroup generated by C_0 and Γ. By [SS10, Theorem 6.3 and Proposition 6.6], the quotient $\mathsf{NS}(X)/\langle C_0, \Gamma\rangle$ is isomorphic to the Mordell-Weil group $\mathsf{MW}(X)$ generated by the sections of π, excluding C_0.

Let $\mathcal{E}_h \subset \mathcal{E}$ be the maximal horizontal subsheaf of $\mathcal{E}$ and let $\mathcal{E}_v := \mathcal{E}/\mathcal{E}_h$. Suppose that $\mathcal{E}_h$ is nonzero. Hence $\mathsf{ch}_1(\mathcal{E})$ is a nonzero horizontal effective divisor class on X, while $\mathsf{ch}_1(\mathcal{E}_v) \in \Gamma$. Moreover, the image of $\mathsf{ch}_1(\mathcal{E}_h)$ in $\mathsf{MW}(X)$ coincides with the image of $\mathsf{ch}_1(\mathcal{E})$, which is zero by assumption. Hence $\mathsf{ch}_1(\mathcal{E}_h) \in \langle C_0, \Gamma\rangle$.

By [SS10, Proposition 6.6], the subgroup $\langle C_0, \Gamma\rangle$ is freely generated by C_0, the fiber class f, and the equivalence classes of all irreducible components of singular fibers which do not intersect C_0. Since $\mathcal{E}_h$ is horizontal, this implies that $\mathsf{ch}_1(\mathcal{E}_h) \in \langle C_0\rangle$. At the same time, $\mathsf{ch}_1(\mathcal{E})$ and $\mathsf{ch}_1(\mathcal{E}_v)$ belong to the subgroup Γ, which implies that $\mathsf{ch}_1(\mathcal{E}_h)$ also belongs Γ. This leads to a contradiction since $\langle C_0\rangle \cap \Gamma = 0$ by [SS10, Proposition 6.6].

□

Next note the following transversality result.

Lemma 8.10. *Let D be an horizontal effective divisor on X. Let $\mathcal{E}$ be a vertical pure one-dimensional coherent sheaf on X with $\mathsf{ch}_1(\mathcal{E}) \neq 0$. Then*

$$\mathcal{T}or_k(\mathcal{E}, \mathcal{O}_D) = 0$$

for all $k \geq 1$. In particular, the canonical morphism $\mathcal{E} \otimes \mathcal{O}_X(-D) \to \mathcal{E}$ is injective. Moreover, $\mathcal{E} \otimes \mathcal{O}_D$ is a zero-dimensional sheaf of length $D \cdot \mathsf{ch}_1(\mathcal{E})$.

Proof. Since D is horizontal, the set-theoretic intersection of D with the support of $\mathcal{E}$ is zero-dimensional. Since $\mathcal{E}$ is purely one-dimensional, the claim follows from the long exact $\mathcal{T}or_X(\mathcal{E}, -)$-sequence associated to the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0\,.$$

□

Using Lemma 8.10, we next prove the following.

Lemma 8.11. *Let D be an horizontal effective divisor on X. Let $\mathcal{E}$ be a vertical purely one-dimensional sheaf with $\mathsf{ch}_1(\mathcal{E}) \neq 0$ and let $0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$ be a proper nontrivial saturated subsheaf of $\mathcal{E}$. Then $\mathcal{F} \cap (\mathcal{E} \otimes \mathcal{O}_X(-D)) = \mathcal{F} \otimes \mathcal{O}_X(-D)$ as subsheaves of $\mathcal{E}$.*

[12] Note that a similar result was proven for torsion free sheaves on surfaces in [MW97].

Proof. First, note that the quotient $\mathcal{E}/\mathcal{F}$ is a pure one-dimensional sheaf since $\mathcal{F}$ is saturated. Moreover, it is vertical. Lemma 8.10 shows that $\mathcal{T}or_1(\mathcal{E}/\mathcal{F}, \mathcal{O}_D) = 0$. Therefore, the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}/\mathcal{F} \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \mathcal{F} \otimes \mathcal{O}_D \longrightarrow \mathcal{E} \otimes \mathcal{O}_D \longrightarrow \mathcal{E}/\mathcal{F} \otimes \mathcal{O}_D \longrightarrow 0\,.$$

Then, one has a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} \otimes \mathcal{O}_X(-D) & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F} \otimes \mathcal{O}_D & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E} \otimes \mathcal{O}_X(-D) & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E} \otimes \mathcal{O}_D & \longrightarrow & 0 \end{array}$$

with exact rows, where all vertical arrows are injective. This yields the identifications

$$\begin{aligned} \mathcal{F} \cap (\mathcal{E} \otimes \mathcal{O}_X(-D)) &= \ker(\mathcal{F} \to \mathcal{E} \to \mathcal{E} \otimes \mathcal{O}_D) = \ker(\mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_D \to \mathcal{E} \otimes \mathcal{O}_D) \\ &= \ker(\mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_D) = \mathcal{F} \otimes \mathcal{O}_X(-D) \end{aligned}$$

as subsheaves of $\mathcal{E}$. □

In the following D will be a fixed horizontal effective divisor on X. Given any purely one dimensional vertical sheaf $\mathcal{E}$, the natural morphism $\mathcal{E}(-D) \to \mathcal{E}$ is injective by Lemma 8.10. Moreover, by Lemma 8.9, any purely one dimensional sheaf $\mathcal{E}$ with $0 \neq \mathsf{ch}_1(\mathcal{E}) \in \Gamma$ is vertical. This allows us to formulate the following.

Definition 8.12. A vertical *parabolic sheaf* on X with respect to D is defined by the datum $(\mathcal{E}_\bullet, \alpha_\bullet)$, where $\mathcal{E}$ is a purely one dimensional nonzero sheaf on X with $\mathsf{ch}_1(\mathcal{E}) \in \Gamma$, $\mathcal{E}_\bullet$ is a filtration

$$\mathcal{E} \otimes \mathcal{O}_X(-D) =: \mathcal{E}_s \subset \mathcal{E}_{s-1} \subset \cdots \subset \mathcal{E}_0 := \mathcal{E}\,,$$

and $\alpha_\bullet = (\alpha_0, \ldots, \alpha_{s-1}) \in \mathbb{Q}^s$ is a collection of rational weights so that

$$0 \leq \alpha_0 \leq \cdots \leq \alpha_{s-1} < 1\,.$$

⊘

Remark 8.13. In Definition 8.12, let $S \subset X$ be the reduced set-theoretic support of $\mathcal{E} \otimes \mathcal{O}_D$, which is a zero dimensional subscheme of X by Lemma 8.10. Then the monomorphism $\mathcal{E}_i \to \mathcal{E}_{i-1}$ restricts to an isomorphism on the complement $X \smallsetminus S$. This implies that each successive quotient $\mathcal{E}_{i-1}/\mathcal{E}_i$ is zero dimensional for all $1 \leq i \leq s$.

In conclusion, the topological invariants of a vertical parabolic sheaf as in Definition 8.12 consist of the Mukai vector $v(\mathcal{E}) \in \Gamma \times \mathbb{Z}$ and the numerical invariants

$$\chi_\bullet := (\chi(\mathcal{E}_i))_{1 \leq i \leq s-1}\,,$$

where

$$\chi(\mathcal{E}_{s-1}) \leq \cdots \leq \chi(\mathcal{E}_1)\,.$$

In particular, $\chi(\mathcal{E}_1) = \chi(\mathcal{E}(-D))$ if and only if all monomorphisms $\mathcal{E}(-D) \to \mathcal{E}_i$ are isomorphisms for $1 \leq i \leq s-1$. △

Definition 8.14. Let H be an ample divisor. Let $(\mathcal{E}_\bullet, \alpha_\bullet)$ be a vertical parabolic sheaf with $\mathsf{ch}_1(\mathcal{E}) \neq 0$. The *parabolic slope* of $(\mathcal{E}_\bullet, \alpha_\bullet)$ is defined as

$$\mu_H(\mathcal{E}_\bullet, \alpha_\bullet) := \frac{1}{H \cdot \mathsf{ch}_1(\mathcal{E})} \left(\chi(\mathcal{E}) + \sum_{i=0}^{s-1} \alpha_i\, \chi(\mathcal{E}_i/\mathcal{E}_{i+1}) \right)\,.$$

⊘

Let $(\mathcal{E}_\bullet, \alpha_\bullet)$ be a vertical parabolic sheaf, where $\mathcal{E}$ is pure one-dimensional and $\mathrm{ch}_1(\mathcal{E}) \neq 0$. Let $0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$ be a proper nontrivial saturated subsheaf of $\mathcal{E}$. Then, one has a canonical induced filtration

$$\mathcal{F}_\bullet\colon \quad \mathcal{F}_s \subset \mathcal{F}_{s-1} \subset \cdots \subset \mathcal{F}_0 := \mathcal{F}$$

where $\mathcal{F}_i := \mathcal{F} \cap \mathcal{E}_i$ for $1 \leq i \leq s$.

By Lemma 8.11, one has $\mathcal{F} \cap \mathcal{E}_s = \mathcal{F} \otimes \mathcal{O}_X(-D)$ as subsheaves of E, hence the pair $(\mathcal{F}_\bullet, \alpha_\bullet)$ defines a parabolic structure on $\mathcal{F}$. For convenience this parabolic structure will be denoted by $\mathcal{F} \cap (\mathcal{E}_\bullet, \alpha_\bullet)$.

Definition 8.15. We say that a vertical parabolic sheaf $(\mathcal{E}_\bullet, \alpha_\bullet)$ with $\mathrm{ch}_1(\mathcal{E}) \neq 0$ is *(semi)stable* if

$$\mu_H(\mathcal{F}_\bullet, \alpha_\bullet) \ (\leq) \ \mu_H(\mathcal{E}_\bullet, \alpha_\bullet)$$

for any proper nontrivial saturated subsheaf $0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$. ⊘

For any $v = (\gamma, n) \in (\Gamma \times \mathbb{Z})^{\times s}$, and any $\chi_\bullet \in \mathbb{Z}^{s-1}$, with $\gamma \neq 0$, let $\mathfrak{Coh}^{\mathsf{ss}}_H(X; v, \chi_\bullet, \alpha_\bullet)$ be the moduli stack of H-semistable parabolic sheaves with fixed invariants $(v, \chi_\bullet)$ and parabolic weights $\alpha_\bullet$. Let $\mathfrak{Coh}^{\mathsf{s}}_H(X; v, \chi_\bullet, \alpha_\bullet) \subset \mathfrak{Coh}^{\mathsf{ss}}_H(X; v, \chi_\bullet, \alpha_\bullet)$ be the stable locus. By [Tal17, Theorem 3.2.29], one has the following.[13]

Proposition 8.16. *For any $(v, \chi_\bullet, \alpha_\bullet)$ as above, $\mathfrak{Coh}^{\mathsf{ss}}_H(X; v, \chi_\bullet, \alpha_\bullet)$ is a global quotient stack which admits a coarse projective moduli scheme $\mathcal{M}^{\mathsf{ss}}_H(X; v, \chi_\bullet, \alpha_\bullet)$, parametrizing S-equivalence classes of semistable objects.*

Now, set

$$B := \frac{1}{p} D$$

with $p \in \mathbb{Z}$, $p \geq 2$.

Remark 8.17. Let $\mathcal{E}$ be a vertical pure one-dimensional sheaf with $\mathrm{ch}_1(\mathcal{E}) \neq 0$. Then, note that the data

$$\mathcal{E}_\bullet\colon \quad \mathcal{E} \otimes \mathcal{O}_X(-D) =: \mathcal{E}_p = \cdots = \mathcal{E}_1 \subset \mathcal{E} \tag{8.1}$$

$$\alpha_\bullet := \left(\frac{p-1}{p}, \ldots, \frac{1}{p} \right)$$

determines a parabolic structure on $\mathcal{E}$. The choice of weights is justified by the correspondence between parabolic sheaves and sheaves on the root stack $\mathcal{X} \to X$ determined by the pair (D, p). In the present form this was proven in [BV12]. Note also that

$$\mu_{(H,B)}(\mathcal{E}) = \mu_H(\mathcal{E}_\bullet, \alpha_\bullet) \ .$$

△

Lemma 8.18. *Under the conditions of Remark 8.17, $\mathcal{E}$ is (H, B)-(semi)stable if and only if $(\mathcal{E}_\bullet, \alpha_\bullet)$ is H-(semi)stable. Moreover, given any semistable parabolic sheaf $(\mathcal{E}_\bullet, \alpha_\bullet)$ with $\chi(\mathcal{E}_i) = \mathcal{E}(-D)$ for all $1 \leq i \leq s$, the monomorphism $\mathcal{E}(-D) \to \mathcal{E}_i$ is an isomorphism for all $1 \leq i \leq s$, as in equation (8.1).*

Proof. For the first part, it suffices to note that the construction in Remark 8.17 is compatible with induced parabolic structures by Lemma 8.11. Given a saturated subsheaf $0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$ of $\mathcal{E}$, the parabolic structures $\mathcal{F}_\bullet$ and $\mathcal{F} \cap \mathcal{E}_\bullet$ coincide.

The second part follows from Remark 8.13. □

Let $\mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(X; v)$ be the moduli stack of (H, B)-semistable pure one-dimensional vertical sheaves with Mukai vector $v = (\gamma, n)$, where $\gamma \neq 0$. Then Lemma 8.18 yields the following.

[13] Note that moduli spaces of torsion-free parabolic sheaves were constructed in [MY92].

Corollary 8.19. *Using the construction in Remark 8.17, the assignment*

$$\mathcal{E} \longmapsto (\mathcal{E}_\bullet, \alpha_\bullet)$$

determines an equivalence of stacks

$$\varphi\colon \mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(X;v) \longrightarrow \mathfrak{Coh}^{\mathsf{ss}}_{H}(X;v,\chi_\bullet,\alpha_\bullet)$$

with $\chi_\bullet := (n - D\cdot\gamma, \dots, n - D\cdot\gamma)$, *which preserves the stable loci. In particular,* $\mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(X;v)$ *is a global quotient stack which admits a coarse projective moduli scheme* $\mathcal{M}^{\mathsf{ss}}_{(H,B)}(X,v)$ *parametrizing S-equivalence classes of semistable objects.*

8.2.2. *Stable vertical sheaves.* For any effective divisor class $\gamma \in \mathsf{NS}(X)$, let Δ_γ be the set of all nonzero effective classes $\gamma' \in \mathsf{NS}(X) \smallsetminus \{0\}$ so that $\gamma - \gamma'$ is a nonzero effective class. Recall that $Z = E_{\mathsf{red}}$ for E a fixed reducible singular fiber of $\pi\colon X \to C$, and $f = [E] \in \mathsf{NS}(X)$.

Lemma 8.20. *Let* $\mathcal{E}$ *be an* (H,B)*-stable sheaf on* X *so that* $\mathsf{ch}_1(\mathcal{E}) \in \langle Z \rangle$*. Then, one of the following cases holds:*

- *(i)* $\mathsf{ch}_1(\mathcal{E}) = kf$, *with* $k \geq 1$, *and* $\dim \mathsf{Ext}^1_X(\mathcal{E},\mathcal{E}) = 2$.
- *(ii)* $\mathsf{ch}_1(\mathcal{E})^2 = -2$ *and* $\dim \mathsf{Ext}^1_X(\mathcal{E},\mathcal{E}) = 0$. *Moreover, the reduced set theoretic support of* $\mathcal{E}$ *is contained in* Z.

Proof. By Lemma 8.5, the subgroup $\langle Z \rangle \subset \mathsf{NS}(X)$ is freely generated by the reduced irreducible components of Z. Furthermore, it is naturally isomorphic to the root lattice of the associated affine ADE Lie algebra, the intersection form being identified with the negative Cartan matrix. This implies that $\mathsf{ch}_1(\mathcal{E})^2$ is even, $\mathsf{ch}_1(\mathcal{E})^2 \leq 0$, and $\mathsf{ch}_1(\mathcal{E})^2 = 0$ if and only if $\mathsf{ch}_1(\mathcal{E}) \in \langle f \rangle$. Furthermore, by Lemma 8.3, Z satisfies Assumption 0.9, i.e., there exists an open subscheme $U \subset X$, containing Z, so that $\omega_X|_U \simeq \mathcal{O}_U$. Then the claim follows from Lemma 7.4. □

For any $v = (\gamma, n) \in \Gamma \times \mathbb{Z}$, with $\gamma \neq 0$, let $\mathcal{S}^{\mathsf{ss}}_{(H,B)}(v)$ denote the set of isomorphism classes of (H,B)-semistable sheaves on X with Mukai vector v. Let $\mathcal{S}^{s}_{(H,B)}(v)$ be the subset of isomorphism classes of stable objects.

Below we determine the set $\mathcal{S}^{\mathsf{ss}}_{(H,B)}(v)$ for any Mukai vector $v = (\gamma, n) \in \langle Z \rangle \smallsetminus \langle f \rangle$ with $\gamma \neq 0$, assuming $B \in \mathsf{NS}_{\mathbb{R}}(X)$ to be generic.

Lemma 8.21. *Assume that* $B \in \mathsf{NS}_{\mathbb{R}}(X)$ *is generic. Let* $v = (\gamma, n) \in \langle Z \rangle \smallsetminus \langle f \rangle$ *with* $\gamma \neq 0$*. Then* $\mathcal{S}^{\mathsf{ss}}_{(H,B)}(v)$ *is either empty, or it consists of a unique polystable element* $\mathcal{E}_v^{\oplus \ell}$, *with* $\mathcal{E}_v$ *an* (H,B)*-stable sheaf with set theoretic support contained in* Z.

Proof. Suppose $\mathcal{S}^{\mathsf{ss}}_{(H,B)}(v)$ is not empty. Let $\mathcal{E}$ be a purely one dimensional (H,B)-semistable sheaf on X so that $v(\mathcal{E}) = v$. Let $\mathcal{E}_i$ be its Jordan-Hölder subquotients with $1 \leq i \leq j$. By Lemma 7.3, for each $1 \leq i \leq j$ there exists $\lambda_i \in \mathbb{Q}$, with $\lambda_i > 0$, so that

$$\mathsf{ch}_1(\mathcal{E}) = \lambda_i \mathsf{ch}_1(\mathcal{E}_i) \quad \text{and} \quad \chi(\mathcal{E}) = \lambda_i \chi(\mathcal{E}_i)\ .$$

In addition note that $\mathcal{E}$ is a vertical sheaf by Lemma 8.9. Therefore each $\mathcal{E}_i$ is a vertical sheaf as well for $1 \leq i \leq j$. Moreover, since $\mathcal{E}_i$ is (H,B)-stable, its reduced set theoretic support is connected for $1 \leq i \leq j$.

By [SS10, Proposition 6.6], the subgroup $\Gamma \subset \mathsf{NS}(X)$ consisting of vertical classes is freely generated by the fiber class f and all irreducible components of the singular fibers which do not intersect the section C_0. In particular Γ admits a direct sum decomposition

$$\Gamma \simeq \langle Z \rangle \oplus \Gamma'\ ,$$

where Γ' is freely generated by the irreducible components of the singular fibers contained in $X \smallsetminus Z$ which do not intersect the section C_0. Then, as $\mathsf{ch}_1(\mathcal{E}) \in \langle Z \rangle \subset \Gamma$, relations 8.2.2 imply that $\mathsf{ch}_1(\mathcal{E}_i) \in \langle Z \rangle$ as well, for all $1 \leq i \leq j$. Moreover, given the above direct sum decomposition, the reduced set theoretic support of $\mathcal{E}_i$ is a union of reduced fibers and irreducible components of Z. Since it is also connected, the reduced set theoretic support of $\mathcal{E}_i$ is therefore contained in Z. In conclusion, by Lemma 8.20, for each $1 \leq i \leq j$, one has two cases:

(1) $\mathsf{ch}_1(\mathcal{E}_i) \in \langle f \rangle$, or

(2) $\mathsf{ch}_1(\mathcal{E}_i)^2 = -2$ and $\mathcal{E}_i$ is rigid as in Lemma 8.20–(ii).

In both cases, the set theoretic support of $\mathcal{E}_i$ is contained in Z.

Suppose (1) holds for some $1 \leq i \leq \ell$. Since B is generic, Lemma 7.3 implies

$$\lambda \mathsf{ch}_1(\mathcal{E}) \in \langle f \rangle$$

for some $\lambda \in \mathbb{Q}$, with $\lambda \neq 0$. Then $\mathsf{ch}_1(\mathcal{E})^2 = 0$, leading to a contradiction. In conclusion all Jordan-Hölder subquotients of $\mathcal{E}$ are of type (2). Then, in completely analogy to the proof of Lemma 7.8, one concludes that $\mathcal{E} \simeq \mathcal{E}_v{}^{\oplus \ell}$ for an (H,B)-stable sheaf $\mathcal{E}_v$ with $\mathsf{ch}_1(\mathcal{E}_v)^2 = -2$. Moreover, $\mathcal{E}_v$ and ℓ are uniquely determined by v. □

Lemma 8.21 yields the following.

Theorem 8.22. *Assume that $B \in \mathsf{NS}_{\mathbb{R}}(X)$ is generic. Then, for any $v = (\gamma, n) \in \langle Z \rangle \times \mathbb{Z}$, with $\gamma \notin \angle f \rangle$, the natural map $\mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(\widehat{X}_Z; v) \to \mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(X; v)$ is an isomorphism. Moreover, $\mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(X; v)$ is either empty or it is equivalent to the classifying stack of a general linear group. In particular, Assumption 0.8 holds in this case, with $\ell = 0$.*

We next concentrate on moduli of stable vertical sheaves with first Chern class in $\langle f \rangle$, starting with primitive Mukai vectors as defined below.

Definition 8.23. We say that a Mukai vector $v \in \langle f \rangle \times \mathbb{Z}$ is *primitive* if $v = (kf, n)$ with $k \geq 1$ and (k, n) coprime. ⊘

Then note the following.

Lemma 8.24. *Let $B \in \mathsf{NS}(X)_{\mathbb{R}}$ be generic. Let $v := (kf, n)$, with $k, n \in \mathbb{Z}$, $k > 0$, be a primitive Mukai vector. Then, for any ample class $H \in \mathsf{NS}(X)$ one has $\mathcal{S}^{\mathsf{ss}}_{(H,B)}(v) = \mathcal{S}^{\mathsf{s}}_{(H,B)}(v)$.*

Proof. Assume that $\mathcal{E}$ is strictly (H,B)-semistable, with $v(E) = v$. Then, there exists an (H,B)-semistable subsheaf $0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$ of $\mathcal{E}$ with $\mu_{H,\beta}(\mathcal{F}) = \mu_{H,\beta}(\mathcal{E})$ and the quotient $\mathcal{E}/\mathcal{F}$ is a nonzero (H,B)-semistable sheaf of the same slope. Hence $\mathsf{ch}_1(\mathcal{F}) \in \Delta_{kf}$.

Since B is generic, by Lemma 7.3, the slope identity $\mu_{(H,B)}(\mathcal{F}) = \mu_{(H,B)}(\mathcal{E})$ implies that

$$kf = \lambda\, \mathsf{ch}_1(\mathcal{F}) \quad \text{and} \quad n = \lambda\, \chi(\mathcal{F}) \tag{8.2}$$

for some $\lambda \in \mathbb{Q}$, with $\lambda > 0$.

Since $k \geq 1$ and k, n are coprime, one has either $(k, n) = (1, 0)$ or $n \neq 0$. If $(k, n) = (1, 0)$, Lemma 8.5 implies that $\lambda = 1$ and $\mathsf{ch}_1(\mathcal{F}) = f$. This contradicts the condition $\mathsf{ch}_1(\mathcal{F}) \in \Delta_{kf}$, which requires $\mathsf{ch}_1(\mathcal{F}) = k'f$, with $0 < k' < k$.

Assume now that $n \neq 0$. Lemma 8.5 implies that $\mathsf{ch}_1(\mathcal{F}) = \ell f$ for some positive integer $\ell \in \mathbb{Z}$, with $\ell \geq 1$. Since $\mathsf{ch}_1(\mathcal{F}) \in \Delta_{kf}$, one has $\ell \leq k - 1$. Furthermore, the relations (8.2) yield $n\ell = k\chi(\mathcal{F})$. Since (k, n) are coprime, this implies that ℓ is a multiple of k which leads again to a contradiction. □

Next, let $\mathcal{K}_{\mathbb{R}}(X) \subset \mathsf{NS}_{\mathbb{R}}(X)$ be the real Kähler cone of X and let $\mathcal{K}_{\mathbb{Q}}(X)_{\mathbb{Q}} = \mathcal{K}_{\mathbb{R}}(X) \cap \mathsf{NS}_{\mathbb{Q}}(X)$.

Lemma 8.25. *Let $B \in \mathcal{K}_{\mathbb{R}}(X)$ be generic and let $v := (\gamma, n) \in \langle f \rangle \times \mathbb{Z}$ be a primitive Mukai vector. Then there exists $B' \in \mathcal{K}_{\mathbb{Q}}(X)$ (depending on v) so that $\mathcal{S}^{\mathsf{ss}}_{H,B'}(v) = \mathcal{S}^{\mathsf{s}}_{H,B'}(v)$ and $\mathcal{S}^{\mathsf{ss}}_{H,B'}(v) = \mathcal{S}^{\mathsf{ss}}_{H,B}(v)$.*

Proof. Recall that $\Delta_\gamma \in \mathsf{NS}(X)$ is the set of all nonzero effective classes $\gamma' \in \mathsf{NS}(X) \smallsetminus \{0\}$ so that $\gamma - \gamma'$ is a nonzero effective class. Note that this is a finite set since $\mathsf{NS}(X)$ is finitely generated. For any $(\gamma', n') \in \Delta_\gamma \times \mathbb{Z}$, let $\mathcal{W}_{(\gamma', n')} \subset \mathsf{NS}(X)_\mathbb{R}$ be the affine hyperplane defined by

$$\frac{n' - B \cdot \gamma'}{H \cdot \gamma'} = \frac{n - B \cdot \gamma}{H \cdot \gamma} .$$

Let

$$\mathcal{W}_\mathbb{R} := \bigcup_{(\gamma', n') \in \Delta_\gamma \times \mathbb{Z}} \mathcal{W}_{\gamma', n'} \quad \text{and} \quad \mathcal{W}_\mathbb{Q} := \mathcal{W}_\mathbb{R} \cap \mathsf{NS}_\mathbb{Q}(X) .$$

Note that $B \notin \mathcal{W}_\mathbb{R}$ by the genericity assumption. Let $\mathcal{C}_B \subset \mathcal{K}(X)_\mathbb{R}$ be the connected component of the complement of $\mathcal{K}(X)_\mathbb{R} \smallsetminus \mathcal{W}_\mathbb{R}$ containing B. We show below that $\mathcal{S}^{\mathsf{ss}}_{H,B'}(v) = \mathcal{S}^{\mathsf{s}}_{H,B'}(v)$ and $\mathcal{S}^{\mathsf{ss}}_{H,B'}(v) = \mathcal{S}^{\mathsf{ss}}_{H,B}(v)$ for any $B' \in \mathcal{C}_B$.

The first claim follows immediately from the fact that B' does not belong to $\mathcal{W}_\mathbb{R}$. This implies that there are no strictly (H, B')-semistable sheaves with Mukai vector v for any $B' \in \mathcal{C}_B$.

The second claim follows by a standard argument (see for example [Yos96, Lemma 2.2]). The details are provided below for completeness. We will first show that $\mathcal{S}^{\mathsf{ss}}_{H,B'}(v) \subset \mathcal{S}^{\mathsf{ss}}_{H,B}(v)$. Suppose $\mathcal{E}$ is an (H, B)-semistable sheaf which is not (H, B')-semistable. Let $\mathcal{E}_1 \subset \mathcal{E}$ be the first step in the Harder-Narasimhan filtration of $\mathcal{E}$ with respect to (H, B')-stability. Let $v(\mathcal{E}_1) = (\gamma_1, n_1)$. Under the current assumptions, $\mathcal{E}/\mathcal{E}_1$ is a non-zero purely one dimensional sheaf. Hence $\gamma_1 \in \Delta_\gamma$ and

$$\frac{\gamma_1 - B' \cdot \gamma_1}{H \cdot \gamma_1} > \frac{\gamma - B' \cdot \gamma}{H \cdot \gamma} .$$

Moreover, since $\mathcal{E}$ is (H, B)-stable, one has

$$\frac{\gamma_1 - B \cdot \gamma_1}{H \cdot \gamma_1} < \frac{\gamma - B \cdot \gamma}{H \cdot \gamma} .$$

Let

$$t \in [0,\, 1] \longmapsto \beta_t$$

be a linear path in $\mathsf{NS}_\mathbb{R}(X)$ so that $\beta(0) = B$ and $\beta(1) = B'$. Then the above inequalities imply that the linear function

$$t \longmapsto \frac{\gamma_1 - \beta(t) \cdot \gamma_1}{H \cdot \gamma_1} - \frac{\gamma - \beta(t) \cdot \gamma}{H \cdot \gamma}$$

changes sign in the interval $(0,\, 1)$. Therefore there exists $t_0 \in (0,\, 1)$ so that

$$\frac{\gamma_1 - \beta(t_0) \cdot \gamma_1}{H \cdot \gamma_1} - \frac{\gamma - \beta(t_0) \cdot \gamma}{H \cdot \gamma} = 0 .$$

In conclusion, B and B' are separated by the wall $\mathcal{W}_{H, \beta(t_0)}$, in contradiction with the assumption that $B' \in \mathcal{C}_B$. The inverse inclusion, $\mathcal{S}^{\mathsf{ss}}_{H,B}(v) \subset \mathcal{S}^{\mathsf{ss}}_{H,B'}(v)$ is proven by an identical argument. Hence, indeed, $\mathcal{S}^{\mathsf{ss}}_{H,B}(v) = \mathcal{S}^{\mathsf{ss}}_{H,B'}(v)$.

In conclusion, note that the intersection $\mathcal{K}_\mathbb{Q}(X) \cap \mathcal{C}_B$ is nonempty since Δ_γ is a finite set, hence $\Delta_\gamma \times \mathbb{Z} \simeq \mathbb{Z}^{|\Delta_\gamma|}$. Then any $B' \in \mathcal{K}_\mathbb{Q}(X) \cap \mathcal{C}_B$ satisfies the conditions stated in Lemma 8.25. □

We conclude this section with:

Proposition 8.26. *Assume that $B \in \mathsf{NS}_\mathbb{R}(X)$ is generic. For any primitive vector $v = (kf, n) \in \mathsf{NS}(X) \times \mathbb{Z}$, with $k \geq 1$, the moduli stack $\mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(X; v)$ is a smooth global quotient stack which admits a coarse projective moduli scheme $\mathcal{M}^{\mathsf{ss}}_{(H,B)}(X; v)$ parametrizing isomorphism classes of stable objects. Moreover, the latter is a two dimensional smooth projective scheme with symplectic structure.*

Proof. The first part follows from Corollary 8.19 and Lemma 8.24 using Lemma 8.25. In particular, since B' in Lemma 8.25 is a rational ample class, it can be written as

$$B' = \frac{[D]}{p}$$

where D is a smooth connected ample divisor on X and $p \in \mathbb{Z}$, with $p \geq 1$. The ampleness condition implies that $D \notin \Gamma$, hence Corollary 8.19 applies.

The second part is analogous to [Muk84], or, alternatively, follows from [PTVV13]. □

In the next surface it will be shown that the coarse moduli space in Proposition 8.26 is in fact isomorphic to X as an elliptic surface over C.

8.2.3. *Moduli via Fourier-Mukai functors.* In this section, H is a fixed ample class and B is a fixed generic twisted stability parameter. We shall show below that the moduli space $\mathcal{M}^{\mathsf{ss}}_{(H,B)}(X;v)$, with v primitive, admits a natural projection to C and it is isomorphic to X as a C-scheme. This is analogous to [Bri98], where the same result is proven for H-stable vertical sheaves. First recall the following well known result.

Lemma 8.27. *Let $\mathcal{E}$ be an (H,B)-stable vertical sheaf with set-theoretic support contained in a (scheme-theoretic) fiber X_p, with $p \in B$. Then the scheme-theoretic support of $\mathcal{E}$ is a closed subscheme of X_p. In particular, if X_p is irreducible, $\mathcal{E}$ is the pushforward of a torsion-free sheaf on X_p.*

Moreover, let X_p be an irreducible fiber of π. Under the current assumptions, X_p is reduced. Recall that a nonzero torsion-free sheaf $\mathcal{E}$ on X_p is (semi)stable if and only if

$$\frac{\chi(\mathcal{F})}{\mathsf{rk}(\mathcal{F})} \ (\leq) \ \frac{\chi(\mathcal{E})}{\mathsf{rk}(\mathcal{E})}$$

for any proper nontrivial subsheaf $0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$. Then, Lemma 8.27 yields the following.

Corollary 8.28. *Let $\mathcal{E}$ be an nonzero purely one dimensional vertical sheaf, set-theoretically supported on an irreducible fiber X_p, with $p \in B$. Then, $\mathcal{E}$ is (H,B)-stable if and only if $\mathcal{E}$ is isomorphic to the pushforward of a stable torsion-free sheaf on X_p.*

Proof. Under the stated assumptions, $\mathsf{ch}_1(\mathcal{F}) \in \mathbb{Z}\langle f \rangle$ for any nontrivial subsheaf $0 \subsetneq \mathcal{F} \subset \mathcal{E}$. Then, the claim follows from the slope identity

$$\mu_{(H,B)}(\mathcal{F}) = \frac{\chi(\mathcal{F})}{H \cdot \mathsf{ch}_1(\mathcal{F})} + \frac{B \cdot f}{H \cdot f} \,,$$

observing that the second term in the right-hand-side is independent of $\mathcal{F}$. □

Now let $\mathcal{H}(X;kf,0)$ denote the Hilbert scheme of pure one-dimensional closed subschemes $Z \subset X$ with

$$\mathsf{ch}_1(\mathcal{O}_Z) = kf \quad \text{and} \quad \chi(\mathcal{O}_Z) = 0 \,.$$

Recall the construction of the *determinant* of a pure one-dimensional sheaf $\mathcal{E}$ on X. Given any locally free resolution

$$\mathcal{F}_{-1} \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{E}$$

of $\mathcal{E}$, one obtains an injective morphism

$$\det(\mathcal{F}_{-1}) \otimes \det(\mathcal{F}_0)^{-1} \longrightarrow \mathcal{O}_X \,.$$

The image of this morphism is the 0-*th fitting ideal* $\mathsf{Fitt}_0(\mathcal{E})$, as defined in [Sta25, Tag 0C3C], which is independent of the chosen resolution. Moreover, the cokernel of the above morphism is the structure sheaf of a divisor $\Delta_{\mathcal{E}}$ so that

$$\mathsf{ch}_1(\mathcal{O}_{\Delta_{\mathcal{E}}}) = \mathsf{ch}_1(\mathcal{E}) \,.$$

This is called the *determinant* of $\mathcal{E}$. Then we have the following result.

Lemma 8.29. *For any primitive vector $v = (kf, n) \in \mathsf{NS}(X) \times \mathbb{Z}$, with $k \geq 1$, there is a canonical morphism*

$$\Delta\colon \mathcal{M}^{\mathsf{ss}}_{(H,B)}(X; v) \longrightarrow \mathcal{H}(X; kf, 0) \tag{8.3}$$

mapping the isomorphism class of a sheaf $\mathcal{E}$ to its determinant $\Delta_{\mathcal{E}}$.

Proof. Given a vertical (H, B)-semistable sheaf $\mathcal{E}$, with $\mathsf{ch}_1(\mathcal{E}) = kf$, using Lemma 8.3, the adjunction formula implies that

$$\chi(\mathcal{O}_{\Delta_{\mathcal{E}}}) = 0\ .$$

Therefore $\Delta_{\mathcal{E}}$ determines a point in the Hilbert scheme $\mathcal{H}(X; kf, 0)$. In order to prove the claim, we have to show that this construction works for flat families.

Let T be a parameter scheme of finite type over $\mathbb{C}$, and let $\mathcal{E}$ be a T-flat family of (H, B)-semistable sheaves on X with Mukai vector v. Since $\mathcal{E}$ is a flat family of pure one-dimensional sheaves, [HL10, Proposition 2.10] proves that $\mathcal{E}$ admits a two term locally free resolution

$$\mathcal{F}_{-1} \longrightarrow \mathcal{F}_0$$

as an $\mathcal{O}_{X\times T}$-module. Furthermore, this resolution restricts to a locally free resolution of $\mathcal{E}_t$ for all $t \in T$. The 0-th fitting ideal of $\mathcal{E}$ is again defined as the image of the morphism

$$\det(\mathcal{F}_{-1}) \otimes \det(\mathcal{F}_0)^{-1} \longrightarrow \mathcal{O}_{X\times T}\ .$$

Note that this morphism is injective since its restriction to any $t \in T$ coincides with the morphism

$$\det(\mathcal{F}_{-1,t}) \otimes \det(\mathcal{F}_{0,t})^{-1} \longrightarrow \mathcal{O}_{X_t}\ ,$$

which is injective. Then its cokernel is the structure sheaf of a T-flat divisor $\Delta_{\mathcal{E}}$ on $X \times T$. Moreover, $\Delta_{\mathcal{E}}$ restricts to $\Delta_{\mathcal{E}_t}$ on each fiber X_t. Therefore, one obtains a morphism $T \to \mathcal{H}(X; kf, 0)$. □

Recall that under the current assumptions the projection $\pi\colon X \to C$ has a section $\sigma\colon C \to X$ whose scheme-theoretic image is denoted by C_0. As a consequence of Lemma 8.29, we obtain:

Corollary 8.30. *For any primitive vector $v = (kf, n) \in \mathsf{NS}(X) \times \mathbb{Z}$, with $k \geq 1$, there is an additional morphism*

$$\delta\colon \mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(X; v) \longrightarrow \mathsf{Sym}^k(C) \tag{8.4}$$

mapping the isomorphism class of a sheaf $\mathcal{E}$ to the scheme-theoretic inverse image $\sigma^{-1}(\Delta_{\mathcal{E}})$.

Proof. Lemma 8.5 implies that any closed subscheme Z which belongs to $\mathcal{H}(X; kf, 0)$ is pure one-dimensional and set-theoretically supported on a union of fibers of π. Hence, its scheme-theoretic intersection with C_0 is a length k zero-dimensional subscheme of C_0. It is straightforward to check that this construction works for flat families, hence, it yields a morphism $\mathcal{H}(X; kf, 0) \to \mathsf{Sym}^k(C)$, mapping a closed subscheme $Z \subset X$ to $\sigma^{-1}(Z)$. The morphism (8.4) is obtained by composition with (8.3). □

On the other hand, by Lemmas 8.27 and 8.27, for any primitive Mukai vector v, one also has a set-theoretic map

$$\mathcal{S}^{\mathsf{ss}}_{H,B}(v) \longrightarrow C \tag{8.5}$$

mapping the isomorphism class of a sheaf $\mathcal{E}$ to the unique point $p \in \mathbb{C}$ so that the scheme theoretic support of $\mathcal{E}$ is contained in X_p.

Lemma 8.31. *For any primitive vector $v = (kf, n) \in \mathsf{NS}(X) \times \mathbb{Z}$, with $k \geq 1$, the morphism (8.4) determines a proper morphism $\mathcal{M}^{\mathsf{ss}}_{(H,B)}(X; v) \to C$ so that the induced map of closed points coincides with (8.5).*

Proof. We first note that the morphism (8.4) factors through the projection $\mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(X;v) \longrightarrow \mathcal{M}^{\mathsf{ss}}_{(H,B)}(X;v)$ since the coarse moduli space co-represents the functor of points [HL10, Theorem 4.3.4]. Hence one obtains a morphism $\mathcal{M}^{\mathsf{ss}}_{(H,B)}(X;v) \to C$.

Since both the domain and the target in (8.4) are projective, δ is proper. Since the moduli scheme $\mathcal{M}^{\mathsf{ss}}_{(H,B)}(X;v)$ is smooth, hence reduced, the scheme-theoretic image of δ is a reduced closed subscheme of $\mathsf{Sym}^k(C)$. Furthermore, by Lemma 8.27, the induced map of sets of closed points coincides with (8.5). This further implies that the scheme-theoretic image of δ is a closed subscheme of the reduced small diagonal in $\mathsf{Sym}^k(C)$. Hence, one obtains a proper morphism, as claimed. □

Next, let $U \subset C$ be the irreducible locus of π, i.e., the open subscheme of C so that X_p is irreducible for any $p \in U$. Let $X_U := X \times_C U$ and let $\pi_U \colon X_U \to U$ be the canonical projection. Note that π_U is projective, with irreducible reduced fibers and X_U is smooth. For any coprime pair (k,n), with $k \geq 1$, let $\mathfrak{Coh}^s(\pi_U;k,n) \to U$ denote the relative moduli stack of rank k stable torsion-free sheaves on the fibers of $\pi_U \colon X_U \to U$ with Euler characteristic n. Note that strictly semistable objects are absent for (k,n) coprime, hence $\mathfrak{Coh}^s(\pi_U;k,n)$ coincides with $\mathfrak{Coh}^{ss}(\pi_U;k,n)$. As shown in [Sim94, Theorem 1.21], this stack admits a quasi-projective relative coarse moduli scheme $\mathcal{M}(\pi_U;k,n) \to U$. Since (k,n) are assumed to be coprime, this scheme is in fact projective over U. Moreover, the following result follows from [HRLMSGTP09, Corollary 1.28]:

Theorem 8.32. *For any coprime pair (k,n), the relative moduli space $\mathcal{M}^s(\pi_U;k,n)$ is isomorphic to X_U as an U-scheme.*

Now, let $\mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(X_U;v)$ and $\mathcal{M}^{\mathsf{ss}}_{(H,B)}(X_U;v)$ be defined by the pullbacks

$$\begin{array}{ccc} \mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(X_U;v) & \longrightarrow & \mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(X;v) \\ \downarrow & & \downarrow \\ \mathcal{M}^{\mathsf{ss}}_{(H,B)}(X_U;v) & \longrightarrow & \mathcal{M}^{\mathsf{ss}}_{(H,B)}(X;v) \\ \downarrow & & \downarrow{\scriptstyle \delta} \\ U & \longrightarrow & C \end{array} .$$

By Corollary 8.28, taking pushforwards yields a morphism of stacks over U:

$$\mathfrak{Coh}^s(\pi_U;k,n) \longrightarrow \mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(X_U;v) .$$

By composition, one obtains a morphism

$$\mathfrak{Coh}^{\mathsf{s}}(\pi_U;k,n) \longrightarrow \mathcal{M}^{\mathsf{ss}}_{(H,B)}(X_U;v) \tag{8.6}$$

to the coarse moduli space. Since the coarse moduli space $\mathcal{M}^{\mathsf{s}}(\pi_U;k,n)$ co-represents the functor of points by [Sim94, Theorem 1.21], the morphism (8.6) factors through a morphism

$$\mathcal{M}^{\mathsf{s}}(\pi_U;k,n) \longrightarrow \mathcal{M}^{\mathsf{ss}}_{(H,B)}(X_U;v) . \tag{8.7}$$

of schemes over U.

Lemma 8.33. *The morphism (8.7) is an isomorphism of schemes over U.*

Proof. By Corollary 8.28, the morphism (8.7) induces an isomorphism on sets of points. Furthermore the domain is a smooth surface by Theorem 8.32, while the target is a also a smooth surface by Proposition 8.26. This implies that (8.7) is an isomorphism, as claimed. □

Let $Y_{k,n}$ be the connected component of $\mathcal{M}^{\mathsf{ss}}_{(H,B)}(X;v)$ containing the open subscheme $\mathcal{M}^{\mathsf{ss}}_{(H,B)}(X_U;v)$. Note that the latter is isomorphic to X_U by Theorem 8.32 and Lemma 8.33, hence it is in particular connected.

Let $\rho\colon Y_{k,n} \to C$ be the restriction of the morphism $\delta\colon \mathcal{M}^{\mathsf{ss}}_{(H,B)}(X;v) \to C$ in Lemma 8.31 to $Y_{k,n}$.

Proposition 8.34. *For any coprime pair (k,n), with $k \geq 1$, the component $Y_{k,n}$ is isomorphic to X as a C-scheme.*

Proof. First, $\rho\colon Y_{k,n} \to C$ is a proper morphism by Lemma 8.31. By Theorem 8.32 and Lemma 8.33, the inverse image $\rho^{-1}(U)$ is isomorphic to X_U as an U-scheme. Furthermore, by Proposition 8.26, $Y_{k,n}$ is a symplectic smooth projective surface. Therefore, $Y_{k,n}$ is a smooth symplectic elliptic surface over C, which is isomorphic to X_U over U. In particular, it is also relatively minimal over C. Since both X and $Y_{k,n}$ are relatively minimal over C, they are isomorphic as surfaces over C by [BPVdV84, Proposition III.8.4] or [SS10, Theorem 5.1]. □

The next result proves the existence of a universal family.

Lemma 8.35. *For any primitive vector $v = (kf, n) \in \mathsf{NS}(X) \times \mathbb{Z}$, with $k \geq 1$, there exists a universal family $\mathcal{P}$ over $\mathcal{M}^{\mathsf{ss}}_{(H,B)}(X;v) \times X$.*

Proof. This follows from [Muk87, Theorem A.6]. As proven in *loc. cit.*, it suffices to show that the greatest common divisor of all $\chi(\mathcal{E} \otimes \mathcal{V})$, with $\mathcal{V}$ a vector bundle on X, is one. Note that

$$\chi(\mathcal{E} \otimes \mathcal{O}_X(C_0)) = n + k$$

by the Grothendieck-Riemann-Roch theorem. Since (k,n) are coprime, it follows that $\chi(\mathcal{E} \otimes \mathcal{O}_X)$ and $\chi(\mathcal{E} \otimes \mathcal{O}_X(C_0))$ are coprime. □

We shall also denote by $\mathcal{P}$ the restriction of the universal sheaf to $Y_{k,n} \times X$. Now, let

$$\begin{array}{ccc} & Y_{k,n} \times X & \\ {}^{q}\swarrow & & \searrow^{p} \\ Y_{k,n} & & X \end{array}$$

denote the canonical projections. Set

$$\mathcal{P}^\vee := \mathbb{R}\mathcal{H}\mathsf{om}_{X \times Y_{k,n}}(\mathcal{P}, \mathcal{O}_{X \times Y_{k,n}}) \quad \text{and} \quad \mathcal{Q} := \mathcal{P}^\vee \otimes p^*\omega_X[1] .$$

By analogy to [Bri98, Lemma 5.1], we have the following.

Lemma 8.36. *$\mathcal{Q}$ is a perfect complex of amplitude $[0,0]$. Moreover both $\mathcal{P}$ and $\mathcal{Q}$ are flat over X and $Y_{k,n}$.*

As in [Bri98], let

$$\Phi\colon \mathsf{D}^{\mathsf{b}}(Y_{k,n}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X) \quad \text{and} \quad \Psi\colon \mathsf{D}^{\mathsf{b}}(X) \longrightarrow \mathsf{D}^{\mathsf{b}}(Y_{k,n})$$

be the Fourier-Mukai functors

$$\Phi(-) := \mathbb{R}p_*(\mathcal{P} \otimes^{\mathbb{L}} \mathbb{L}q^*(-)) \quad \text{and} \quad \Psi(-) := \mathbb{R}q_*(\mathcal{Q} \otimes^{\mathbb{L}} \mathbb{L}p^*(-)) .$$

Then, in complete analogy to [Bri98, Theorem 5.3], one has the following.

Theorem 8.37. *The functors Φ and Ψ are equivalences of derived categories so that*

$$\Psi \circ \Phi = \mathsf{id}_{\mathsf{D}^{\mathsf{b}}(Y_{k,n})}[-1] \quad \textit{and} \quad \Phi \circ \Psi = \mathsf{id}_{\mathsf{D}^{\mathsf{b}}(X)}[-1] .$$

The next result determines the action of the Fourier-Mukai functors in Theorem 8.37 of (H,B)-stable vertical sheaves.

Lemma 8.38. *For any closed point $y \in Y_{k,n}$, one has*

$$\Phi(\mathcal{O}_y) \simeq \mathcal{P}_y \quad \textit{and} \quad \Psi(\mathcal{P}_y) \simeq \mathcal{O}_y[-1] .$$

Proof. Consider the cartesian diagram

$$\begin{array}{ccc} X_y & \xrightarrow{\ j\ } & X \times Y_{k,n} \\ \downarrow{\scriptstyle q_y} & & \downarrow{\scriptstyle q} \\ y & \xrightarrow{\ i\ } & Y_{k,n} \end{array}$$

where i is the canonical morphism. By construction,

$$\Phi(i_* \mathcal{O}_y) = \mathbb{R}p_*(\mathcal{P} \otimes^{\mathbb{L}} \mathbb{L}q^* i_* \mathcal{O}_y) \ .$$

Since q is flat, we get

$$\mathbb{L}q^* i_* \mathcal{O}_y \simeq q^* i_* \mathcal{O}_y \simeq j_* q_y^* \mathcal{O}_y \simeq j_* \mathcal{O}_{X_y} \ .$$

The projection formula yields an isomorphism

$$\mathcal{P} \otimes^{\mathbb{L}} \mathbb{L}q^* i_* \mathcal{O}_y \simeq \mathbb{R}j_*(\mathbb{L}j^* \mathcal{P} \otimes^{\mathbb{L}} \mathcal{O}_{X_y}) \ .$$

Moreover, since $\mathcal{P}$ is flat over Y by Lemma 8.36, [Bri02, Lemma 3.1.1] shows that $\mathbb{L}j^*\mathcal{P} \simeq j^*\mathcal{P}$. Then, $\mathbb{L}j^*\mathcal{P} \otimes^{\mathbb{L}} \mathcal{O}_{X_y} \simeq j^*\mathcal{P}$. Hence, $\mathcal{P} \otimes^{\mathbb{L}} j_* \mathcal{O}_{X_y} \simeq j_* \mathcal{P}_y$. Therefore,

$$\Phi(i_* \mathcal{O}_y) \simeq \mathbb{R}p_*(j_* \mathcal{P}_y) \simeq \mathcal{P}_y \ ,$$

since the composition $p \circ j \colon X_y \to X$ is the identity.

The second equality follows from Theorem 8.37. □

We next prove a scheme-theoretic support result for $\mathcal{P}$. For simplicity, set $Y_{k,n} = Y$ in the following. We start by proving the following.

Lemma 8.39. *The canonical morphism $X \times_C Y \to X \times Y$ is a closed immersion.*

Proof. Let W be defined as the upper left corner of the pull-back square

$$\begin{array}{ccc} W & \longrightarrow & C \\ \downarrow & & \downarrow{\scriptstyle \delta} \\ X \times Y & \xrightarrow{\ \pi \times \rho\ } & C \times C \end{array} \ .$$

Since the diagonal morphism $\delta \colon C \to C \times C$ is a closed immersion, [Sta25, Tag 01JU] shows that the morphism $W \to X \times Y$ is a closed immersion. Therefore, it suffices to prove that the canonical morphism $X \times_C Y \to X \times Y$ factors through an isomorphism onto W.

Let T be the fiber product defined by the following pull-back square

$$\begin{array}{ccc} T & \xrightarrow{\ \tau_X\ } & X \times C \\ {\scriptstyle \tau_Y}\downarrow & & \downarrow{\scriptstyle \pi \times \mathrm{id}_C} \\ C \times Y & \xrightarrow[\ \mathrm{id}_C \times \rho\ ]{} & C \times C \end{array} \ . \tag{8.8}$$

Note the commutative diagram

$$\begin{array}{ccccc} X \times Y & & \xrightarrow{\qquad p_X \qquad} & & X \\ & \searrow{\scriptstyle \mathrm{id}_X \times \rho} & & & \uparrow{\scriptstyle q_X} \\ \downarrow{\scriptstyle p_Y} & & T & \xrightarrow{\ \tau_X\ } & X \times C \\ & \searrow{\scriptstyle \pi \times \mathrm{id}_Y} & \downarrow{\scriptstyle \tau_Y} & & \downarrow{\scriptstyle \pi \times \mathrm{id}_C} \\ Y & \xleftarrow[\ q_Y\ ]{} & C \times Y & \xrightarrow[\ \mathrm{id}_C \times \rho\ ]{} & C \times C \end{array} \ ,$$

where $\mathrm{id}_X \times \rho \colon X \times Y \to X \times C$ and $\pi \times \mathrm{id}_Y \colon X \times Y \to C \times Y$.

where p_X, p_Y and q_X, q_Y are the canonical projections. Using the defining properties of the fiber product, this determines two unique morphisms

$$f\colon X \times Y \to T \quad \text{and} \quad g\colon T \to X \times Y \tag{8.9}$$

satisfying the relations

$$\tau_X \circ f = \mathrm{id}_X \times \rho \quad \text{and} \quad \tau_Y \circ f = \pi \times \mathrm{id}_Y ,$$

respectively

$$p_X \circ g = q_X \circ \tau_X \quad \text{and} \quad p_Y \circ g = q_Y \circ \tau_Y .$$

The above relations yield:

$$p_X \circ g \circ f = q_X \circ \tau_X \circ f = q_X \circ (\mathrm{id}_X \times \rho) = p_X$$

and

$$p_Y \circ g \circ f = q_Y \circ \tau_Y \circ f = q_Y \circ (\pi \times \mathrm{id}_Y) = p_Y .$$

Then the defining properties of the fiber product uniquely identify $g \circ f = \mathrm{id}_{X\times Y}$. A similar argument shows that $f \circ g = \mathrm{id}_T$, as well. Thus, f and g are canonical isomorphisms.

Next let X' and Y' be defined through the pull-back diagrams

$$\begin{array}{ccc} X' & \longrightarrow & X \times C \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi\times\mathrm{id}_C} \\ C & \xrightarrow{\ \delta\ } & C \times C \end{array} \quad \text{and} \quad \begin{array}{ccc} Y' & \longrightarrow & C \times Y \\ {\scriptstyle \rho'}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_C\times\rho} \\ C & \xrightarrow{\ \delta\ } & C \times C \end{array} .$$

Then note that the canonical morphisms $X' \to X$ and $Y' \to Y$ defined by the compositions

$$X' \longrightarrow X \times C \xrightarrow{q_X} X \quad \text{and} \quad Y' \longrightarrow C \times Y \xrightarrow{q_Y} Y \tag{8.10}$$

are isomorphisms of schemes over C. This follows from the double commutative diagrams

$$\begin{array}{ccccc} X' & \longrightarrow & X \times C & \xrightarrow{q_X} & X \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi\times\mathrm{id}_C} & & \big\downarrow{\scriptstyle \pi} \\ C & \xrightarrow{\ \delta\ } & C \times C & \xrightarrow{p_2} & C \end{array} \quad \text{and} \quad \begin{array}{ccccc} Y' & \longrightarrow & C \times Y & \xrightarrow{q_Y} & Y \\ \big\downarrow{\scriptstyle \rho'} & & \big\downarrow{\scriptstyle \mathrm{id}_C\times\rho} & & \big\downarrow{\scriptstyle \rho} \\ C & \xrightarrow{\ \delta\ } & C \times C & \xrightarrow{p_1} & C \end{array} ,$$

observing that $p_1 \circ \delta = p_2 \circ \delta = \mathrm{id}_C$.

By base change with respect to the diagonal immersion $\delta\colon C \to C \times C$, the pull-back square (8.8) yields a cartesian diagram

$$\begin{array}{ccc} T \times_{C\times C} C & \longrightarrow & X' \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi'} \\ Y' & \xrightarrow{\ \rho'\ } & C \end{array} .$$

Since the morphisms (8.10) are isomorphisms over C, one obtains a commutative diagram

$$\begin{array}{cccc} T \times_{C\times C} C & & & \\ & \searrow \ \searrow \ \searrow & & \\ & & X \times_C Y & \longrightarrow & X \\ & & \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ & & Y & \xrightarrow{\ \rho\ } & C \end{array} ,$$

where the square is cartesian, and the map $T \times_{C\times C} C \to W$ is an isomorphism.

On the other hand, since the morphisms (8.9) are isomorphisms, one also has an isomorphism

$$T \times_{C\times C} C \simeq W$$

of schemes over $C \times C$. This concludes the proof. □

Given Lemma 8.39, we next prove the following.

Lemma 8.40. *The universal sheaf $\mathcal{P}$ is scheme-theoretically supported on the closed subscheme $X \times_C Y \subset X \times Y$. In particular, $\mathcal{P}_y$ is scheme-theoretically supported on the fiber $X_{\rho(y)}$ for any closed point $y \in Y$.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_{X\times Y}$ be the defining ideal sheaf of the closed subscheme $X \times_C Y \subset X \times Y$. Then, one has to show that the multiplication map $\mu\colon \mathcal{I} \otimes \mathcal{P} \to \mathcal{P}$ is identically zero.

First note that it suffices to prove the claimed vanishing result for the restriction of μ to a certain open subscheme of Y. Recall that the open subscheme $U \subset C$ was defined as the irreducible locus of π. Then it is also the irreducible locus of $\rho\colon Y \to C$ since X and Y are isomorphic over C. We claim that it suffices to prove that the restriction of μ to $X \times Y_U$ is identically zero, where $Y_U \coloneqq Y \times_C U$. If this is the case, its image is a subsheaf of $\mathcal{P}$ with reduced set theoretic support $X \times S_{\mathsf{red}}$, where $S \subset Y$ is the scheme-theoretic union of the reducible fibers of $\rho\colon Y \to C$. Since $\mathcal{P}$ is flat over Y, and $S_{\mathsf{red}} \subset Y$ is a closed subscheme of codimension 1, this implies that the image of μ is identically zero.

Next note that $X \times_C Y$ is flat over Y since X is flat over C. This implies that the restriction $\mathcal{I}_y \coloneqq \mathcal{I}|_y$ coincides with the defining ideal sheaf of $(X \times_C Y)_y \subset X \times \{y\}$ for any $y \in Y$. Therefore, since Y_U is reduced, it suffices to prove that $\mathcal{P}_y$ is scheme theoretically supported on $(X \times_C Y)_y$ for any $y \in Y$.

For any $y \in Y$, one has a canonical isomorphism $(X \times_C Y)_y \to X_{\rho(y)}$ induced by the double cartesian diagram

$$\begin{array}{ccccc} (X \times_C Y)_y & \longrightarrow & X \times_C Y & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \{y\} & \longrightarrow & Y & \xrightarrow{\rho} & C \end{array} .$$

Then Lemma 8.33, shows that $\mathcal{P}_y$ is indeed scheme-theoretically supported on $(X \times_C Y)_y$ for any $y \in Y_U$. Since $X \times_C Y$ is a flat over Y and $\mathcal{P}$ is flat over Y, we get that $\mathcal{P}$ is scheme-theoretically supported on $X \times_C Y$. □

Using Theorem 8.37 and Lemmas 8.38 and 8.40, we next prove some structural results for the moduli stacks $\mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(X; v)$ of vertical (H,B)-semistable sheaves.

Lemma 8.41. *Let $\mathcal{E}$ and $\mathcal{F}$ be (H,B)-stable sheaves on X with Mukai vectors $v(\mathcal{E}) = (kf, n)$ and $v(\mathcal{F}) = m(kf, n)$, where $k, m, n \in \mathbb{Z}$, $k, m \geq 1$, and (k, n) are coprime. Assume that $\mathcal{E}$ and $\mathcal{F}$ are not isomorphic if $m = 1$. Then $\mathsf{Ext}^k_X(\mathcal{E}, \mathcal{F}) = 0$ for all $k \in \mathbb{Z}$.*

Proof. Under the stated conditions, $\mathcal{E}$ and $\mathcal{F}$ are non-isomorphic (H,B)-stable sheaves of equal slope. Then, one has

$$\mathsf{Ext}^0_X(\mathcal{E}, \mathcal{F}) = 0 \quad \text{and} \quad \mathsf{Ext}^0_X(\mathcal{F}, \mathcal{E}) = 0 .$$

Moreover, by Grothendieck-Riemann-Roch theorem, one has $\chi(\mathcal{E}, \mathcal{F}) = 0$ since $\mathsf{ch}_1(\mathcal{E}) \cdot \mathsf{ch}_1(\mathcal{F}) = 0$ and $K_X \cdot \mathsf{ch}_1(\mathcal{E}) = K_X \cdot \mathsf{ch}_1(\mathcal{F}) = 0$. Then the claim follows by Serre duality, keeping in mind that $\mathcal{E} \otimes \omega_X \simeq \mathcal{E}$ and $\mathcal{F} \otimes \omega_X \simeq \mathcal{F}$ by Lemma 8.3. □

Theorem 8.37 and Lemma 8.41 yield:

Corollary 8.42. *For any primitive vector $v = (kf, n) \in \mathsf{NS}(X) \times \mathbb{Z}$, with $k \geq 1$, the moduli space $\mathcal{M}_{(H,B)}(X; v)$ consists of only one connected component $Y_{k,n}$.*

Proof. Suppose $\mathcal{E}$ is an (H,B)-stable sheaf on X which does not belong to $Y_{k,n}$. In order to obtain a contradiction, it suffices to show that $\Psi(\mathcal{E}) = 0$. Note that

$$\Psi(\mathcal{E}) \simeq \mathbb{R}q_* \mathbb{R}\mathcal{H}\mathsf{om}_{X\times Y_{k,n}}(\mathcal{P}, \mathbb{L}p^*\mathcal{E})[1] .$$

since $\mathcal{E} \otimes \omega_X \simeq \mathcal{E}$. Since $p\colon X \times Y_{k,n} \to X$ is flat, one has an isomorphism $\mathbb{L}p^*\mathcal{E} \simeq \mathcal{E}$. Moreover, $p^*\mathcal{E}$ is flat over $Y_{k,n}$. Since $\mathcal{P}$ is also flat over $Y_{k,n}$, in complete analogy to [HT10, Lemma 4.2], we conclude that $\Psi(\mathcal{E})$ is isomorphic in $\mathsf{D}^\mathrm{b}(Y_{k,n})$ to a finite locally free complex $F^\bullet$ so that

$$\mathcal{H}^j(F^\bullet) \simeq \mathsf{Ext}^{j-1}_{X_y}(\mathcal{P}_y, \mathcal{E}_y)\ .$$

By Lemma 8.41, we have $\mathsf{Ext}^k_{X_y}(\mathcal{P}_y, \mathcal{E}_y) = 0$ for all $k \in \mathbb{Z}$ since $\mathcal{E}$ and $\mathcal{P}_y$ are non-isomorphic for all $y \in Y_{k,n}$. Therefore, the complex $F^\bullet$ is exact, which proves the claim. □

A similar argument using Theorem 8.37 and Lemma 8.41 also proves the following.

Corollary 8.43. *The set of isomorphism classes of (H, B)-stable sheaves with Mukai vector $v = (\ell f, m)$, with $\ell, m \in \mathbb{Z}$ and $\ell \geq 1$, is empty for (ℓ, m) not coprime.*

Finally, using Lemmas 8.38 and 8.40, and Corollary 8.42, we show below that the Fourier-Mukai transform preserves set theoretic support. More precisely, recall that $E = \pi^{-1}(o)$ is a reducible scheme theoretic fiber of $\pi\colon X \to C$, where $o \in C$ is a closed point. Since $Y_{k,n}$ and X are isomorphic as elliptic surfaces over C, the fiber $F := \rho^{-1}(o)$ is isomorphic to E. Note any (H, B)-stable sheaf on X with Mukai primitive Mukai vector $v = (kf, n)$ is isomorphic to $\mathcal{P}_y$ for some closed point $y \in Y_{k,n}$. Then, the following holds.

Lemma 8.44. *The point $y \in Y$ belongs to F if and only if $\mathcal{E}$ is scheme theoretically supported on E.*

Proof. The direct implication follows immediately from Lemmas 8.38 and 8.40.

In order to prove the inverse implication, suppose $\mathcal{E}$ is scheme-theoretically supported on E while $y \notin F$. In particular, $\rho(y) \neq o$. At the same time, by Lemma 8.40, $\mathcal{P}_y$ is scheme theoretically supported on the fiber $X_{\rho(y)}$, which is disjoint from E. Since $\mathcal{E} \simeq \mathcal{P}_y$, this leads to a contradiction. □

We next analyze the structure of semistable objects.

Lemma 8.45. *Let $\mathcal{E}$ be an (H, B)-semistable sheaf on X with Mukai vector $v = (\ell k f, \ell n)$ where $k, \ell \geq 1$ and (k, n) are coprime. Let $\mathcal{E}_1, \ldots, \mathcal{E}_j$ be its Jordan-Hölder subquotients with respect to (H, B)-stability. Then, $v(\mathcal{E}_i) = (kf, n)$ for any $1 \leq i \leq j$.*

Proof. Since B is generic, the identity

$$\mu_{(H,B)}(\mathcal{E}_i) = \mu_{(H,B)}(v)$$

implies that for any $1 \leq i \leq j$ one has

$$\mathsf{ch}_1(\mathcal{E}_i) = \lambda \ell k f \quad \text{and} \quad \chi(\mathcal{E}_i) = \lambda \ell n\ ,$$

where $\lambda \in \mathbb{Q}$, with $\lambda > 0$.

If $n = 0$, one has $k = 1$, and $\mathsf{ch}_1(\mathcal{E}_i) = \lambda \ell f$ for all $1 \leq i \leq j$. Hence $\lambda \ell \in \mathbb{Z}$ by Lemma 8.5. Since $\mathcal{E}_i$ is a (nonzero) subquotient of $\mathcal{E}$, it follows that $\lambda \ell = 1$. This proves the claim in this case.

Assume that $n \neq 0$. Let $\lambda \ell k = a$ and note that $a \in \mathbb{Z}$, with $a \geq 1$, by Lemma 8.5. Moreover,

$$an = k\chi(\mathcal{E}_i)$$

for all $1 \leq i \leq j$. Since (k, n) are coprime, this implies that

$$a = ck \quad \text{and} \quad \chi(E_i) = cn$$

for some $c \in \mathbb{Z}$. This yields

$$\mathsf{ch}_1(\mathcal{E}_i) = ck \quad \text{and} \quad \chi(\mathcal{E}_i) = cn\ .$$

Then, $c = 1$ by Corollary 8.43. □

Let $Z := E_{\mathsf{red}}$ and $W := F_{\mathsf{red}}$. For simplicity, set $Y := Y_{k,n}$. Then, Theorem 8.37, Lemma 8.38, Lemma 8.44, and Lemma 8.45 yield the following.

Theorem 8.46. *For any $v = (\ell k f, \ell n)$ with $k, \ell \geq 1$ and (k, n) coprime, the Fourier-Mukai functor $\Psi \colon \mathsf{D}^{\mathsf{b}}(X) \to \mathsf{D}^{\mathsf{b}}(Y)$ yields an equivalence of indgeometric derived stacks*

$$\psi_{k,n} \colon \mathbf{Coh}^{\mathsf{ss}}_{(H,B)}(\widehat{X}_Z; v) \xrightarrow{\sim} \mathbf{Coh}(\widehat{Y}_W; \ell)$$

where $\mathfrak{Coh}(\widehat{Y}_W; \ell)$ is the indgeometric derived stack of length ℓ zero-dimensional sheaves on Y with set-theoretic support on W.

Remark 8.47. Note that similar results were proven in [HRMnP02, HRLMSGTP09] for *relative* moduli spaces of semistable sheaves associated to elliptic fibrations. For $\ell > 1$, a generic (H, B)-semistable sheaf $\mathcal{E}$ with Mukai vector $(\ell k f, \ell n)$ as in Theorem 8.46 is not necessarily scheme-theoretically supported on the fiber E. Therefore, *loc. cit.* do not imply Theorem 8.46. △

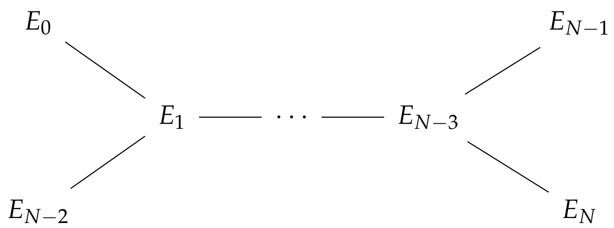
In conclusion, note that Theorems 8.22 and 8.46 provide a complete complete characterization of moduli stacks of (H, B)-semistable sheaves $\mathcal{E}$ with set-theoretic support contained in Z.

8.3. **Stratifications and amalgamation.** In this section we will construct a topologically surjective amalgamation map as in Theorem 4.21 for reducible elliptic fibers of type DE.

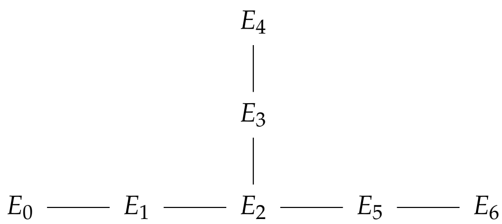
Using the same notation as in §8.1, let $Z := E_{\mathsf{red}}$, where E is a singular fiber of $\pi \colon X \to C$ of affine type D or E. Then note that Z satisfies Assumption 0.9 by Lemma 8.3.

Proceeding as in §7, we first construct a stratification of Z satisfying Assumptions 0.1 and 0.7. We first recall the dual intersection graphs associated to affine DE trees on a case by case basis. We shall denote by E_i the irreducible components of Z. Recall that $E_i \simeq \mathbb{P}^1$.

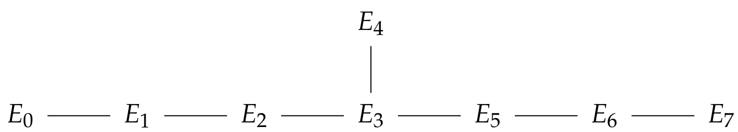
- Type affine D:



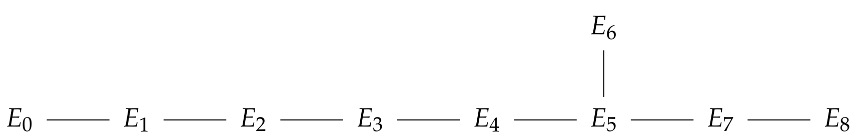
- Type affine E_6:



- Type affine E_7:



- Type affine E_8:



Then one constructs a stratification

$$\emptyset = Z^0 \subset Z^1 \subset \cdots \subset Z^{N+1} := Z\,. \tag{8.11}$$

by first setting $Z^{N+1} := Z$. The for each $1 \leq i \leq N+1$, let Z^{i-1} be the scheme-theoretic closure of the complement $Z^i \smallsetminus E_{i-1}$ for any $1 \leq i \leq N+1$. Note that $Z^0 = \emptyset$. Using Lemmas 8.2 and 8.5, one obtains:

Lemma 8.48. *The following hold.*

(i) For any $1 \leq i \leq N+1$ the stratification

$$\emptyset = Z^0 \subset Z^1 \subset \cdots \subset Z^{i-1} \subset Z^i$$

of Z^i satisfies Assumption 0.1 and Assumption 0.7.

(ii) Each stratum $Z_i^\circ := Z^i \smallsetminus Z^{i-1}$ is isomorphic to $\mathbb{A}^1$, while its scheme-theoretic closure $Z_i := \overline{Z^i \smallsetminus Z^{i-1}}$ coincides with E_{i-1}, for $2 \leq i \leq N+1$. Moreover, $Z_1^\circ = Z^1 = E_0$.

(iii) The scheme-theoretic intersection $Z_{i-1}^i := Z^{i-1} \cap Z_i$ is a length one zero-dimensional subscheme contained in the smooth loci of Z^{i-1} and Z_i for any $2 \leq i \leq N+1$, while $Z^0 = \emptyset$. In particular, the one-step stratification

$$Z_i^{i-1} \subset Z_i$$

satisfies Assumptions 0.1 and 0.7 for any $2 \leq i \leq N+1$.

Next, as in §4.4, let $W^i \subset Z$ be the scheme-theoretic closure of the complement $Z \smallsetminus Z^{i-1}$. Then W^i is a reduced closed subscheme of Z, containing Z_i, and the scheme-theoretic closure of the complement $W^i \smallsetminus Z_i$ coincides with W^{i+1}. Moreover, the scheme-theoretic intersection of W^i and Z_{i-1} coincides with Z_i^{i-1}. Therefore, Lemma 8.48 yields:

Corollary 8.49. *The one step stratification $Z_i \subset W^i$ satisfies Assumption 0.1 for all $1 \leq i \leq N+1$.*

Moreover, using Proposition 6.5, Theorems 8.22 and 8.46 yield the following.

Corollary 8.50. *Let (H, B) be a twisted stability condition on X with B generic in the sense of Definition 7.1. Given $\gamma \in \langle Z \rangle$ and $n \in \mathbb{Z}$ arbitrary, any nonempty moduli stack $\mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(\widehat{X}_Z; \gamma, n)$ is equivalent to either:*

(i) a classifying stack $B\mathsf{GL}(\ell, \mathbb{C})$, with $\ell \in \mathbb{N}$ and $\ell \geq 1$, if $\gamma \notin \mathbb{Z}\langle f \rangle$; or

(ii) a moduli stack $\mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(\widehat{X}_Z; \ell)$ of length ℓ zero-dimensional sheaves on X with set-theoretic support on Z, if $\gamma = \ell f$.

In particular, in the first case, $\mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(\widehat{X}_Z; \gamma, n)$ is T-equivariantly 0-cellular, while in the second case it is T-equivariantly 4-cellular. Furthermore, Assumption 0.8 holds.

For any $\alpha \in \mathbb{R}$ and $\gamma \in \langle Z \rangle \smallsetminus \{0\}$, let $\mathfrak{U}_\alpha(\gamma, n) \subset \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n)$ be the open substack parametrizing those coherent sheaves $\mathcal{E}$ such that $\mu_{(H,B)\text{-}\min}(\mathcal{E}) \geq \alpha$. Then, Proposition 6.6 yields:

Corollary 8.51. *Let (H, B) be a twisted stability condition with B generic. Then, $\mathfrak{U}_\alpha(\gamma, n)$ has a 5-cellular structure for all $\alpha \in \mathbb{R}$, $\gamma \in \langle Z \rangle \smallsetminus \{0\}$ and $n \in \mathbb{Z}$.*

In conclusion, as in §4.4, the stratification (8.11) yields an amalgamation map

$$\bar{\mathfrak{a}}_\gamma \colon \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_{N+1}}, \gamma_{N+1})) \widehat{\otimes}_{\mathbf{HA}^T_{X, Z^N_{N+1}}} \cdots \widehat{\otimes}_{\mathbf{HA}^T_{X, Z^1_2}} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}, \gamma_1)) \longrightarrow \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z, \gamma))$$

for any $\gamma \in \langle Z \rangle$, where $\gamma_i \in \langle Z_i \rangle$ are uniquely determined by the relation $\gamma = \sum_{i=1}^{N+1} \gamma_i$.

Set

$$\mathsf{H}^T_\bullet(\gamma_i, \ldots, \gamma_1) \coloneqq \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_i}, \gamma_i)) \widehat{\otimes}_{\mathbf{HA}^T_{X,Z_i^{i-1}}} \cdots \widehat{\otimes}_{\mathbf{HA}^T_{X,Z_2^1}} \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_1}, \gamma_1))$$

for all $1 \leq i \leq N+1$.

Given Lemma 8.48 and Corollaries 8.49 and 8.50, one checks that all assumptions of Theorem 4.21 are satisfied in complete analogy with Remark 7.10 and Corollary 7.15. Then, we obtain the following.

Theorem 8.52. *For any $\gamma \in \langle Z \rangle$ there is a commutative diagram*

$$\begin{array}{ccc} \mathsf{H}^T_\bullet(\gamma_{N+1}, \ldots, \gamma_1) & \xrightarrow{\quad \overline{\mathfrak{a}}_\gamma \quad} & \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma)) \\ \downarrow & & \downarrow \\ \mathsf{H}^T_\bullet(\mathfrak{U}_{N+1,k}(\gamma_{N+1})) \widehat{\otimes}_{\mathbf{HA}^T_{X,Z^N_{N+1}}} \mathsf{H}^T_\bullet(\gamma_N, \ldots \gamma_1)) & \xrightarrow{\overline{\tau}_{k,\gamma}} & \mathsf{H}^T_\bullet(\mathfrak{U}_k(\gamma)) \end{array}$$

where the vertical maps are determined by the restriction maps associated to the open immersions $\mathfrak{U}_{N+1,k}(\gamma_{N+1}) \to \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_{Z_{N+1}}; \gamma_{N+1})$ and $\mathfrak{U}_k(\gamma) \to \mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma)$, respectively. Moreover, both horizontal arrows as well as the bottom horizontal arrow are surjective. In particular the composition

$$\begin{array}{ccc} \mathsf{H}^T_\bullet(\gamma_{N+1}, \ldots, \gamma_1) & \xrightarrow{\overline{\mathfrak{a}}_\gamma} & \mathsf{H}^T_\bullet(\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma)) \\ & & \downarrow \\ & & \mathsf{H}^T_\bullet(\mathfrak{U}_k(\gamma)) \end{array}$$

is surjective for any $k \in \mathbb{N}$ and for all $\gamma \in \langle Z \rangle$.

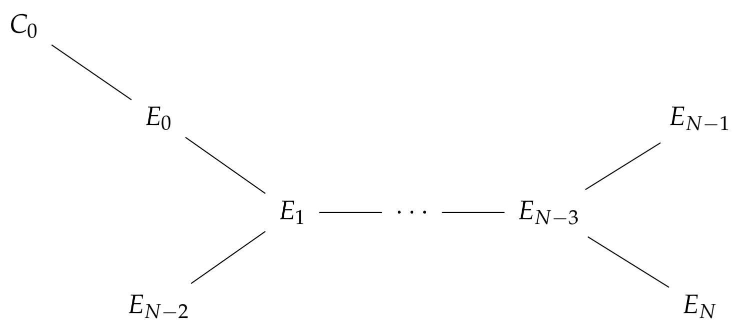
By analogy with Corollary 7.15, one also obtains the following.

Corollary 8.53. *The T-equivariant Borel-Moore homology of the stack $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma)$ is strongly generated by algebraic cycles for all values of the topological invariants.*

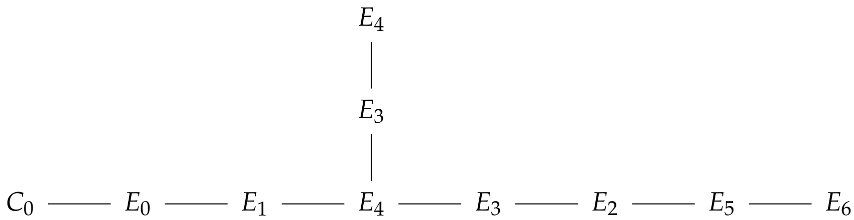
8.3.1. *Motivic variant.* We conclude this section with the observation that the assumptions of Theorem 4.22 hold for more general configurations of rational (-2)-curves on X. The main reason is that topological surjectivity of the motivic amalgation map in loc. cit. only requires the moduli stacks $\mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(\widehat{X}_{W_i}; \gamma, n)$ to have a cellular structure for $1 \leq i \leq N$. In the view of Proposition 4.13–(i), a cellular structure is not required for the moduli stacks $\mathfrak{Coh}^{\mathsf{ss}}_{(H,B)}(\widehat{X}_Z; \gamma, n)$ with $\gamma_i \neq 0$ for all $1 \leq i \leq N+1$.

For example, if X is a smooth elliptic $K3$-surface, one can take Z to be the scheme-theoretic union of the reduced fiber E_{red} and the section C_0. The resulting dual intersection graphs are listed below.

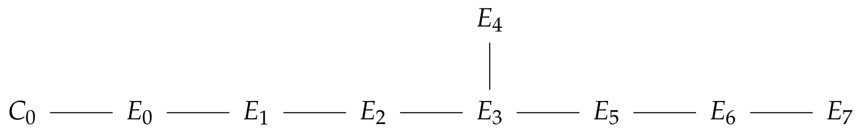
- Type affine D:



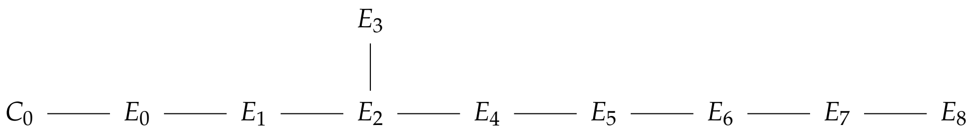
- Type affine E_6:



- Type affine E_7:



- Type affine E_8:



In all of the above examples we construct a stratification

$$\emptyset = Z^0 \subset Z^1 \subset \cdots \subset Z^{N+2} = Z \tag{8.12}$$

starting with $Z^{N+2} = Z$ and $Z^{N+1} = \overline{Z \smallsetminus C_0}$. In particular, note that Z^{N+1} coincides with the reduced fiber E_{red}.Then the remaining closed subschemes Z^i, with $1 \leq i \leq N$, are constructed as in Formula (8.11).

Set

$$\mathsf{H}_\bullet^{\mathsf{mot},T}(\gamma_i, \ldots, \gamma_1) \coloneqq \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_i}, \gamma_i)) \widehat{\otimes}_{\mathbf{HA}^T_{X,Z_i^{i-1}}} \cdots \widehat{\otimes}_{\mathbf{HA}^T_{X,Z_2^1}} \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_1}, \gamma_1))$$

for all $1 \leq i \leq N+2$.

Given the stratification (8.12), in complete analogy with Theorem 8.52, we obtain the following.

Theorem 8.54. *For any $\gamma \in \langle Z \rangle$ there is a commutative diagram*

$$\begin{array}{ccc}
\mathsf{H}_\bullet^{\mathsf{mot},T}(\gamma_{N+2}, \ldots, \gamma_1) & \xrightarrow{\overline{\mathfrak{a}}_\gamma} & \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma)) \\
\downarrow & & \downarrow \\
\mathsf{H}_\bullet^{\mathsf{mot},T}(\mathfrak{U}_{N+2,k}(\gamma_{N+2})) \widehat{\otimes}_{\mathbf{HA}^T_{X,Z^{N+1}_{N+2}}} \mathsf{H}_\bullet^{\mathsf{mot},T}(\gamma_{N+1}, \ldots \gamma_1)) & \xrightarrow{\overline{\tau}_{k,\gamma}} & \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathfrak{U}_k(\gamma))
\end{array}$$

where the vertical maps are determined by the restriction maps associated to the open immersions $\mathfrak{U}_{N+2,k}(\gamma_{N+2}) \to \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_{Z_{N+2}}; \gamma_{N+2})$ and $\mathfrak{U}_k(\gamma) \to \mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma)$, respectively. Moreover, both horizontal arrows as well as the bottom horizontal arrow are surjective. In particular the composition

$$\begin{array}{ccc}
\mathsf{H}_\bullet^{\mathsf{mot},T}(\gamma_{N+1}, \ldots, \gamma_1) & \xrightarrow{\overline{\mathfrak{a}}_\gamma} & \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathbf{Coh}_{\mathsf{ps}}^{\mathsf{nil}}(\widehat{X}_Z; \gamma); 0) \\
& & \downarrow \\
& & \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathfrak{U}_k(\gamma); 0)
\end{array}$$

is surjective for any $k \in \mathbb{N}$ and for all $\gamma \in \langle Z \rangle$.

Appendix A. Twisted stability for coherent sheaves on smooth surfaces

Let X be a smooth projective complex surface. In this section, we provide a finer characterization of the explicit admissible open exhaustion $\{\mathfrak{U}_\alpha(\widehat{X}_Z;\gamma,n)\}$ of $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z;\gamma,n)$ constructed in [DPS+25b, §4.6].

One can define a *twisted* version of the Hilbert polynomial depending on a fixed divisor $B \in N_1(X)$:

$$P_{(H,B)}(\mathcal{E},t) \coloneqq \chi(E\otimes \mathcal{O}_X(tH+B))\ .$$

Here, $N_1(X)$ denotes subgroup of numerical equivalence classes of divisors on X.

Thanks to the Grothendieck-Riemann-Roch theorem, this polynomial depends on the Chern classes of $\mathcal{E}, \mathcal{O}_X(H)$, and $\mathcal{O}_X(B)$, and the Todd classes of X. In particular, we can assume that $B \in N_1(X)_\mathbb{Q}$ or even $B \in N_1(X)_\mathbb{R}$ and extend the definition of $P_{(H,B)}(\mathcal{E},t)$ accordingly. Then, one can define a corresponding *twisted* semistability à la Yoshioka [Yos03], which is a generalization of the semistability introduced in [MW97].

When X has dimension two and Z has dimension one, we can replace Gieseker semistability with μ-semistability. In particular, we can consider the following notion of (H,B)*-slope* of a pure one-dimensional sheaf $\mathcal{E}$:

$$\mu_{(H,B)}(\mathcal{E}) \coloneqq \frac{\chi(\mathcal{E}) + B\cdot \mathsf{ch}_1(\mathcal{E})}{H\cdot \mathsf{ch}_1(\mathcal{E})}\ .$$

Lemma A.1. *Let X be a smooth projective surface, let $H \in N_1(X)$ be an ample divisor class and $B \in N_1(X)_\mathbb{R}$ be a real divisor class. Then, for any $\alpha \in \mathbb{R}$, and any $(\gamma,n) \in N_1(X)\times\mathbb{Z}$, with $\gamma \neq 0$, the set of isomorphism classes of torsion sheaves with fixed invariants (γ,n) satisfying the slope inequality*

$$\mu_{(H,B)\text{-}\min}(\mathcal{E}) \geq \alpha \tag{A.1}$$

is bounded.

Proof. We first reduce to the case of purely one-dimensional sheaves. Let $\mathcal{E}$ be a one-dimensional sheaf satisfying the inequality (A.1). Let $\mathcal{T} \subset E$ be the maximal zero-dimensional subsheaf of $\mathcal{E}$ and let $\mathcal{F} \coloneqq \mathcal{E}/\mathcal{T}$. Then condition (A.1) implies that $\mu_{(H,B)}(\mathcal{F}) \geq \alpha$, i.e.,

$$\chi(\mathcal{F}) \geq \alpha\, H\cdot\gamma - B\cdot\gamma\ ,$$

hence

$$\chi(\mathcal{T}) \leq n - \alpha\, H\cdot\gamma + B\cdot\gamma\ .$$

Since the number of possible values of $\chi(\mathcal{T})$ is finite and $\mathcal{T}$ is obviously semistable, the set of isomorphism classes $\{[\mathcal{T}]\}$ is bounded by [HL10, Theorem 3.3.7]. Then, by [Gro95, Proposition 1.2.ii], it suffices to show that the set of isomorphism classes $\{[\mathcal{F}]\}$ is bounded. Since

$$\alpha\, H\cdot\gamma - B\cdot\gamma \leq \chi(\mathcal{F}) \leq n\ ,$$

it follows that $\chi(\mathcal{F})$ takes finitely many values. Since boundedness is closed under finite unions, this further implies that it suffices to prove the Lemma under the assumption that $\mathcal{E}$ is pure one-dimensional.

Let $\mathcal{H} \subset \mathcal{E}$ be the maximal destabilizing subsheaf of $\mathcal{E}$ with respect to H-stability. By [HL10, Theorem 3.3.7], it suffices to prove that the set $\{\mu_H(\mathcal{H})\}$ is bounded above. Note that

$$\mu_{(H,B)}(\mathcal{H}) \leq \mu_{(H,B)\text{-}\max}(\mathcal{E})\ ,$$

which yields

$$\mu_H(\mathcal{H}) \leq \mu_{(H,B)\text{-}\max}(\mathcal{E}) - \frac{B\cdot \mathsf{ch}_1(\mathcal{H})}{H\cdot \mathsf{ch}_1(\mathcal{H})}\ .$$

Let $\Delta_\gamma \subset N_1(X)$ be the set of nonzero effective classes $\gamma' \in N_1(X)$ so that $\gamma-\gamma'$ is also effective. Note that this is a finite set since the Mori cone of X is finitely generated. Therefore it suffices to prove that the set $\{\mu_{(H,B)\text{-}\max}(\mathcal{E})\}$ is bounded above.

Let $\mathcal{E}' \subset \mathcal{E}$ be the maximal destabilizing subsheaf of $\mathcal{E}$ with respect to (H, B)-stability. Let $\mathcal{E}'' := \mathcal{E}/\mathcal{E}'$ and note that $\mathcal{E}''$ is pure one-dimensional.

If $\mathcal{E}'' = 0$, it follows that $\mathcal{E}$ is (H, B)-semistable, hence $\mu_{(H,B)\text{-}\max}(\mathcal{E}) = \mu_{(H,B)}(\mathcal{E})$. Hence, the claim follows.

Assume that $\mathcal{E}''$ is nonzero. Then condition (A.1) implies that $\mu_{(H,B)}(\mathcal{E}'') \geq \alpha$. By a straightforward computation, this yields

$$\mu_{(H,B)}(\mathcal{E}') \leq (\mu_{(H,B)}(\mathcal{E}) - \alpha)\frac{H \cdot \mathsf{ch}_1(\mathcal{E})}{H \cdot \mathsf{ch}_1(\mathcal{E}')} .$$

Since the set Δ_γ is finite, this implies that the set $\{\mu_{(H,B)\text{-}\max}(\mathcal{E})\}$ is indeed bounded above. □

Proposition A.2. *Fix $(\gamma, n) \in N_1(X) \times Z$ and $\alpha \in \mathbb{R}$. Then, there exist an integer N and a reduced quasi-projective G-scheme Q, where G is a general linear group, so that ${}^{\mathsf{red}}\mathfrak{U}_\alpha(\widehat{X}_Z; \gamma, n)$ is equivalent to the global quotient $[\mathsf{Q}/G]$.*

Proof. Given any $\alpha \in \mathbb{R}$, the set of isomorphism classes of properly supported coherent sheaves $\mathcal{E}$ on X with invariants (γ, n) satisfying the slope inequality

$$\mu_{(H,B)\text{-}\min}(\mathcal{E}) > \alpha$$

is bounded by Lemma A.1. Using the standard construction, this implies that there exists a parameter scheme P for all such sheaves, which is an open subscheme of a Quot scheme consisting of quotients $V \otimes \mathcal{O}_X(-NH) \to \mathcal{E}$, with V a finite dimensional vector space and N a positive integer. Recall that parameter scheme P is defined as the open subscheme of the Quot scheme where the induced global section map $V \to H^0(\mathcal{E}(NH))$ is an isomorphism. The stack $\mathfrak{U}_\alpha(X; \gamma, n)$ is then naturally equivalent to the global quotient $[\mathsf{P}/\mathsf{GL}(V)]$ with respect to the natural action of $\mathsf{GL}(V)$ on P. Let $\mathsf{Q} \subset \mathsf{P}$ be the reduced closed subscheme parametrizing quotients $V \otimes \mathcal{O}_X(-NH) \to \mathcal{E}$ so that $\mathcal{E}$ is set theoretically supported on Z. Clearly, Q is $\mathsf{GL}(V)$-invariant, and the stack ${}^{\mathsf{red}}\mathfrak{U}_\alpha(\widehat{X}_Z; \gamma, n)$ is equivalent to the global quotient $[\mathsf{Q}/GL(V)]$. □

Remark A.3. Under the Assumptions of [DPS+25b, Theorem 4.65], let T be an algebraic torus acting on X such that the action lifts to a T-action on Y and Z is T-invariant. One can show that the equivalence in Proposition A.2 is T-equivariant. Thus, $\mathfrak{U}_k(\widehat{X}_Z; \gamma, n)$ is T-equivariantly equivalent to a quotient stack. Therefore, $\mathbf{Coh}^{\mathsf{nil}}_{\mathsf{ps}}(\widehat{X}_Z; \gamma, n)$ is admissibly ind-stratifiable derived T-stack in the sense of Definition B.4[14] for any $(\gamma, n) \in N_1(X) \times \mathbb{Z}$ with $\gamma \neq 0$. △

Appendix B. Surjectivity results in Borel-Moore homology and Künneth isomorphisms

This section focuses on surjectivity results for (motivic) Borel-Moore homology and Künneth isomorphisms for specific classes of admissible ind-geometric derived stacks.

In this section, we shall use the notation $\mathsf{H}^T_\bullet(-)$ and $\mathsf{H}^{\mathsf{mot},T}_\bullet(-; n)$, with $n \in \mathbb{Z}$, following those introduced in [DPS+25b, Example 5.10]. Moreover, we set $\mathsf{H}_T := \mathsf{H}^T_\bullet(\mathsf{pt})$ and $\mathsf{H}^{\mathsf{mot}}_T := \mathsf{H}^{\mathsf{mot},T}_\bullet(\mathsf{pt}; 0)$. Note that the cycle map $\mathsf{H}^{\mathsf{mot}}_T \to \mathsf{H}_T$ is an isomorphism since both rings are canonically isomorphic to the character ring of T. Therefore we will not distinguish between $\mathsf{H}^{\mathsf{mot}}_T$ and H_T in the following, and we will implicitly use the identification with the character ring.

[14]See Notation B.5.

B.1. **Generation by algebraic cycles and (motivic) Künneth property.**

Definition B.1. Let $\mathcal{X}$ be a quasi-compact and quasi-separated geometric derived stack and let $T \times \mathcal{X} \to \mathcal{X}$ be an algebraic torus[15] action on $\mathcal{X}$. We say that $\mathcal{X}$ is T-equivariantly *stratifiable by quotient stacks* if ${}^{\mathsf{red}}\mathcal{X}$ admits a T-invariant stratification $\{\mathcal{X}_i\}$ into reduced locally closed substacks such that each $\mathcal{X}_i$ is T-equivariantly isomorphic to a quotient stack $[U_i/G_i]$, for a scheme U_i equipped with a $T \times G_i$-action, where G_i is a linear algebraic group. ⊘

Remark B.2. The above condition is quite mild. If instead of requiring the strata of the stratification to be quotient stacks we limit ourselves to require them to be gerbes and T to be trivial, then any quasi-compact geometric stack of finite presentation admits such a presentation, see [Sta25, Tag 06RB] or [MT10, Corollary II.3.3.4]. △

Theorem B.3. *Let $j\colon \mathcal{X} \to \mathcal{Y}$ be an open immersion of quasi-compact and quasi-separated geometric derived stacks. Let $\mathcal{Z} \subset \mathcal{Y}$ be the canonical closed complement of ${}^{\mathsf{red}}\mathcal{X}$ inside ${}^{\mathsf{red}}\mathcal{Y}$. Let $T \times \mathcal{Y} \to \mathcal{Y}$ be an algebraic torus action on $\mathcal{Y}$ preserving $\mathcal{X}$ and $\mathcal{Z}$. If the negative weight T-equivariant motivic Borel-Moore homology groups $\mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{Z};n)$ vanish for all $n < 0$, then the restriction map*

$$j^* \colon \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{Y};0) \longrightarrow \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X};0)$$

in weight zero T-equivariant motivic Borel-Moore homology is surjective.

In particular, this is the case if $\mathcal{Y}$ is locally of finite type and T-equivariantly stratifiable by global quotient stacks.

Proof. The first statement follows immediately from the localization sequence.

In order to prove the second statement, notice that $\mathcal{Z}$ is again T-equivariantly stratifiable by global quotients, so it is enough to prove that negative weights of the motivic homology groups vanish for this class of stacks. Proceeding by noetherian induction, and using once more the localization sequence, we immediately reduce to the case of a single quotient stack $[U/G]$ as in Definition B.1. In this case, $\mathsf{H}_\bullet^{\mathsf{mot},T}([U/G];0) \simeq \mathsf{H}_\bullet^{\mathsf{mot},T\times G}(U;0)$ coincides with equivariant Edidin-Graham cycles thanks to $\mathbb{A}^1$-homotopy invariance (cf. [Kha19, Example 2.10]). The conclusion in this case then follows from the vanishing of Bloch's cycle complex for negative codimension.[16] □

Now, we see the implication of the previous theorem in the theory of Borel-Moore homology of admissible indgeometric derived stacks.

Definition B.4. Let $\mathcal{X}$ be an admissible indgeometric derived stack and let $T \times \mathcal{X} \to \mathcal{X}$ be an algebraic torus action. We say that $\mathcal{X}$ is *T-equivariantly admissibly ind-stratifiable by global quotients* (or simply *T-equivariantly admissibly ind-stratifiable*) if it admits an admissible open exhaustion $\{\mathcal{V}_k\}$ preserved by the torus action such that each ${}^{\mathsf{red}}\mathcal{V}_k$ is T-equivariantly stratifiable by global quotients as in Definition B.1. ⊘

Notation B.5. For ease of exposition, a (admissible ind)geometric derived stack $\mathcal{X}$ equipped with an algebraic torus action $T \times \mathcal{X} \to \mathcal{X}$ will be called a *(admissible ind)geometric derived T-stack*. Furthermore, if $\mathcal{X}$ is T-equivariantly admissibly ind-stratifiable as in Definition B.4, $\mathcal{X}$ will be called an *admissibly ind-stratifiable derived T-stack*. ⊘

Lemma B.6. *Let $j\colon \mathcal{X} \to \mathcal{Y}$ be a T-equivariant open immersion of admissible indgeometric derived T-stacks. In addition, assume that $\mathcal{Y}$ is T-equivariantly admissibly ind-stratifiable as in Definition B.4. Then the induced restriction map $j^* \colon \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{Y};0) \to \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X};0)$ is surjective.*

[15] The case $T = \{1\}$ is allowed, in which case one recovers the usual definition of a stratification by global quotients.

[16] This argument was suggested to us by A. A. Khan and Y. Bae.

Proof. Let $\{\mathcal{V}_k\}_{k\in J}$ be a T-equivariant admissible open exhaustion of $\mathcal{Y}$ such that ${}^{\mathsf{red}}\mathcal{V}_k$ is T-equivariantly stratifiable by global quotients. For any $k \in J$, let $\{\mathcal{V}_{k,n}\}_{n\in I}$ be a T-equivariant presentation for $\mathcal{V}_k$. For each pair n, k, set

$$\mathcal{U}_{k,n} \coloneqq \mathcal{V}_{k,n} \times_{\mathcal{Y}_n} \mathcal{X}_n \ . \tag{B.1}$$

For any $k \in J$, let $\mathcal{U}_k$ be the quasi-compact quasi-separated indgeometric derived stack having as presentation $\{\mathcal{U}_{k,n}\}_{n\in I}$. Then $\{\mathcal{U}_k\}_{k\in\mathbb{N}}$ is a T-equivariant admissible open exhaustion of $\mathcal{X}$.

Moreover, note that ${}^{\mathsf{red}}\mathcal{U}_k$ and ${}^{\mathsf{red}}\mathcal{V}_k$ are quasi-compact geometric derived T-stacks for any $k \in J$. By construction, $\{{}^{\mathsf{red}}\mathcal{U}_k\}_{k\in\mathbb{N}}$ and $\{{}^{\mathsf{red}}\mathcal{V}_k\}_{k\in\mathbb{N}}$ form direct systems of stacks where the transition functions are canonical induced T-equivariant open immersions. Furthermore, one also has the canonical T-equivariant open immersions ${}^{\mathsf{red}}j_k \colon {}^{\mathsf{red}}\mathcal{U}_k \to {}^{\mathsf{red}}\mathcal{V}_k$ induced by the fiber product (B.1). In particular, since ${}^{\mathsf{red}}\mathcal{V}_k$ is T-equivariantly stratifiable for each $k \in J$, so is ${}^{\mathsf{red}}\mathcal{U}_k$.

Since ${}^{\mathsf{red}}\mathcal{V}_k$ and ${}^{\mathsf{red}}\mathcal{U}_k$ are reduced stacks, for each $k \in J$ there is a canonical closed geometric T-stack $\mathcal{Z}_k \subset {}^{\mathsf{red}}\mathcal{V}_k$ so that $(\mathcal{Z}_k, {}^{\mathsf{red}}\mathcal{U}_k)$ form a complementary pair in ${}^{\mathsf{red}}\mathcal{V}_k$. Note also that

$$\mathcal{Z}_k' \simeq \mathcal{Z}_k \times_{{}^{\mathsf{red}}\mathcal{V}_{k'}} {}^{\mathsf{red}}\mathcal{V}_k$$

for any pair $k' \leq k$, hence the stacks $\mathcal{Z}_k$ also form a direct system, where the transition functions are T-equivariant open immersions. In addition, $\mathcal{Z}_k$ is also T-equivariantly stratifiable for all $k \in J$.

Since we have

$$\mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X};0) \simeq \lim_k \mathsf{H}_\bullet^{\mathsf{mot},T}({}^{\mathsf{red}}\mathcal{U}_k;0) \quad \text{and} \quad \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{Y};0) \simeq \lim_k \mathsf{H}_\bullet^{\mathsf{mot},T}({}^{\mathsf{red}}\mathcal{V}_k;0) \ ,$$

the restriction map j^* is identified with the map of inverse systems determined by ${}^{\mathsf{red}}j_k^*$ for $k \in J$. Theorem B.3 shows that each map ${}^{\mathsf{red}}j_k^*$ is surjective, and one has an exact sequence

$$\mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{Z}_k;0) \longrightarrow \mathsf{H}_\bullet^{\mathsf{mot},T}({}^{\mathsf{red}}\mathcal{V}_k;0) \xrightarrow{{}^{\mathsf{red}}j_k^*} \mathsf{H}_\bullet^{\mathsf{mot},T}({}^{\mathsf{red}}\mathcal{U}_k;0) \longrightarrow 0$$

for each $k \in J$. Moreover, again thanks to Theorem B.3, for any $k' \to k$ there is a commutative diagram

$$\begin{array}{ccccccc}
\mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{Z}_k;0) & \longrightarrow & \mathsf{H}_\bullet^{\mathsf{mot},T}({}^{\mathsf{red}}\mathcal{V}_k;0) & \xrightarrow{{}^{\mathsf{red}}j_k^*} & \mathsf{H}_\bullet^{\mathsf{mot},T}({}^{\mathsf{red}}\mathcal{U}_k;0) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{Z}_{k'};0) & \longrightarrow & \mathsf{H}_\bullet^{\mathsf{mot},T}({}^{\mathsf{red}}\mathcal{V}_{k'};0) & \xrightarrow{{}^{\mathsf{red}}j_{k'}^*} & \mathsf{H}_\bullet^{\mathsf{mot},T}({}^{\mathsf{red}}\mathcal{U}_{k'};0) & \longrightarrow & 0
\end{array} \ ,$$

where the vertical maps are the restrictions maps associated to the canonical T-equivariant open immersions. The latter are surjective since ${}^{\mathsf{red}}\mathcal{V}_{k'}$, ${}^{\mathsf{red}}\mathcal{U}_{k'}$, and $\mathcal{Z}_{k'}$ are T-equivariantly stratifiable. In conclusion, the above diagram shows that the kernels $\ker({}^{\mathsf{red}}j_k^*) \subset \mathsf{H}_\bullet^{\mathsf{mot},T}({}^{\mathsf{red}}\mathcal{V}_k;0)$ form a surjective inverse system. In particular, it satisfies the Mittag-Leffler property, hence the map j^* is indeed surjective. □

Definition B.7. Let $\mathcal{X}$ be an admissible indgeometric derived T-stack. We say that the T-equivariant Borel-Moore homology of $\mathcal{X}$ is *generated by algebraic cycles* if

(1) the odd degree T-equivariant Borel-Moore homology of $\mathcal{X}$ is trivial, and

(2) the cycle map $\mathsf{cl} \colon \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X};0) \to \mathsf{H}_\bullet^T(\mathcal{X})$ is surjective.

We say that the T-equivariant Borel-Moore homology of $\mathcal{X}$ is *strongly generated by algebraic cycles* if (1) holds and

(2′) the cycle map $\mathsf{cl} \colon \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X};0) \to \mathsf{H}_\bullet^T(\mathcal{X})$ is an isomorphism.

We say that the T-equivariant (motivic) Borel-Moore homology of $\mathcal{X}$ is *flat* (or, equivalently, *locally free*) if it is flat as an H_T-module. ⊘

The following is a direct consequence of the above definition and Lemma B.6.

Proposition B.8. *Let $j\colon \mathcal{X} \to \mathcal{Y}$ be a T-equivariant open immersion of admissible indgeometric derived T-stacks and let the reduced closed substack $\mathcal{Z} \subset {}^{\mathsf{red}}\mathcal{Y}$ be the canonical complement of the open immersion ${}^{\mathsf{red}}j\colon {}^{\mathsf{red}}\mathcal{X} \to {}^{\mathsf{red}}\mathcal{Y}$. Assume that $\mathcal{Y}$ is T-equivariantly admissibly ind-stratifiable, and the T-equivariant Borel-Moore homology of $\mathcal{X}$ is generated by algebraic cycles. Then, the map*

$$j^*\colon \mathsf{H}_\bullet^T(\mathcal{Y}) \longrightarrow \mathsf{H}_\bullet^T(\mathcal{X})$$

is surjective and the long exact sequence associated to the complementary pair $(\mathcal{Z}, {}^{\mathsf{red}}\mathcal{X})$ in ${}^{\mathsf{red}}\mathcal{Y}$ splits into a short exact sequence

$$0 \longrightarrow \mathsf{H}_\bullet^T(\mathcal{Z}) \longrightarrow \mathsf{H}_\bullet^T(\mathcal{Y}) \longrightarrow \mathsf{H}_\bullet^T(\mathcal{X}) \longrightarrow 0\,.$$

Corollary B.9. *Let $\mathcal{X}$ be an admissible indgeometric derived T-stack and let $\{\mathfrak{U}_k\}_{k\in\mathbb{N}}$ be a T-equivariant admissible open exhaustion of it. Assume that $\mathcal{X}$ is T-equivariantly admissibly ind-stratifiable and ${}^{\mathsf{red}}\mathfrak{U}_{k\in J}$ is T-equivariantly stratifiable by global quotients for all $k \in J$.*

If the T-equivariant Borel-Moore homology of ${}^{\mathsf{red}}\mathfrak{U}_k$ is generated by algebraic cycles for any $k \in J$, then the odd degree T-equivariant Borel-Moore homology of $\mathcal{X}$ is trivial, and the restriction map $\mathsf{H}_\bullet^T(\mathcal{X}) \to \mathsf{H}_\bullet^T(\mathfrak{U}_k)$ is surjective for all $k \in J$.

If in addition, the T-equivariant Borel-Moore homology of ${}^{\mathsf{red}}\mathfrak{U}_k$ is strongly generated by algebraic cycles for any $k \in J$, then the T-equivariant Borel-Moore homology of $\mathcal{X}$ is strongly generated by algebraic cycles.

Proof. First, the odd degree T-equivariant Borel-Moore homology of $\mathcal{X}$ is trivial since this is the case for $\mathfrak{U}_k$ for any $k \in J$. Moreover, Proposition B.8 implies that the restriction map $\mathsf{H}_\bullet^T(\mathcal{X}) \to \mathsf{H}_\bullet^T(\mathfrak{U}_k)$ is surjective for all $k \in J$.

The second claim is evident. □

Now, we give the definition of (motivic) Künneth property.

Definition B.10. Let $(\mathcal{X}_1, \ldots, \mathcal{X}_\ell)$ be a sequence of admissible indgeometric derived T-stacks. We say that $(\mathcal{X}_1, \ldots, \mathcal{X}_\ell)$ has

(1) the *Künneth property* if the Künneth map

$$\mathsf{H}_\bullet^T(\mathcal{X}_1)\widehat{\otimes}_{\mathsf{H}_T} \cdots \widehat{\otimes}_{\mathsf{H}_T}\mathsf{H}_\bullet^T(\mathcal{X}_\ell) \longrightarrow \mathsf{H}_\bullet^T(\mathcal{X}_1 \times \cdots \times \mathcal{X}_\ell)$$

is an isomorphism.

(2) the *(weight zero) motivic Künneth property* if the Künneth map

$$\mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X}_1;0)\widehat{\otimes}_{\mathsf{H}_T} \cdots \widehat{\otimes}_{\mathsf{H}_T}\mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X}_\ell;0) \longrightarrow \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X}_1 \times \cdots \times \mathcal{X}_\ell;0)$$

is an isomorphism.

⊘

Remark B.11. Note that in the above definition, $\widehat{\otimes}$ denotes the *completed* tensor product discussed in [DPS22, Remark II.1.45] and [DPS+25b, Remark 3.1]. When the tensor product is between (T-equivariant) Borel-Moore homologies of quasi-compact quasi-separated geometric derived T-stacks, we shall simply use $\otimes$ since there is no completion involved in this case.
△

Proposition B.12. *Let $\mathcal{X}_i$ be an iterated vector bundle stack*[17] *over an admissible indgeometric derived T-stack $\mathcal{Y}_i$ for $1 \leq i \leq \ell$. Assume that the T-action on $\mathcal{Y}_i$ lifts to a T-action on $\mathcal{X}_i$ so that the canonical projection $\pi_i\colon \mathcal{X}_i \to \mathcal{Y}_i$ is equivariant for all $1 \leq i \leq \ell$.*

If the sequence of T-stacks $(\mathcal{Y}_1, \cdots, \mathcal{Y}_\ell)$ has the (motivic) Künneth property, then the same holds for $(\mathcal{X}_1, \cdots, \mathcal{X}_\ell)$.

[17] i.e., an iterated stack of cosections.

Proof. By functoriality of Künneth maps, one has a commutative diagram

$$\begin{array}{ccc} \mathsf{H}^T_\bullet(\mathcal{Y}_1)\widehat{\otimes}_{\mathsf{H}_T}\cdots\widehat{\otimes}_{\mathsf{H}_T}\mathsf{H}^T_\bullet(\mathcal{Y}_\ell) & \longrightarrow & \mathsf{H}^T_\bullet(\mathcal{Y}_1\times\cdots\times\mathcal{Y}_\ell) \\ \big\downarrow{\scriptstyle \pi_1^!\widehat{\otimes}\cdots\widehat{\otimes}\pi_\ell^!} & & \big\downarrow{\scriptstyle (\pi_1\times\cdots\times\pi_\ell)^!} \\ \mathsf{H}^T_\bullet(\mathcal{X}_1)\widehat{\otimes}_{\mathsf{H}_T}\cdots\widehat{\otimes}_{\mathsf{H}_T}\mathsf{H}^T_\bullet(\mathcal{X}_\ell) & \longrightarrow & \mathsf{H}^T_\bullet(\mathcal{X}_1\times\cdots\times\mathcal{X}_\ell) \end{array},$$

where the horizontal arrows are Künneth maps. By the iterated vector bundle stack property (cf. [DPS22, Remark II.1.32]) the vertical arrows are isomorphisms. Since the top horizontal arrow is an isomorphism by assumption, the bottom horizontal arrow is also an isomorphism. □

B.2. **ℓ-cellular stacks.** In this section, we shall introduce the notion of ℓ-cellularity for quasi-compact quasi-separated geometric classical T-stacks. As we shall see, this leads to a Künneth formula for T-equivariant Borel-Moore homology.

Definition B.13. A quasi-compact quasi-separated geometric classical T-stack $\mathcal{X}$ is called a *linear quotient T-stack* if it is T-equivariantly equivalent to a quotient stack $[V/G]$, where V is a finite-dimensional vector space equipped with a linear $T\times G$-action, where G is a finite product of general linear groups. ⊘

Proposition B.14. *Let $\mathcal{X}$ be a linear quotient T-stack. Then the negative weight T-equivariant motivic Borel-Moore homology of $\mathcal{X}$ is trivial, and its T-equivariant Borel-Moore homology of $\mathcal{X}$ is flat and strongly generated by algebraic cycles.*

Proof. Assume that $\mathcal{X}\simeq[V/G]$ as in Definition B.13. By [Kha19, Example 2.10], the motivic Borel-Moore homology $\mathsf{H}^{\mathsf{mot}}_\bullet([V/T\times G];n)$ is isomorphic to the equivariant Chow group $\mathsf{CH}^{T\times G}_\bullet(V;n)$ for any $n\in\mathbb{Z}$. Hence it vanishes for $n<0$.

Moreover, one has a commutative diagram

$$\begin{array}{ccc} \mathsf{H}^{\mathsf{mot},T}_\bullet(\mathcal{X};0) & \xrightarrow{\mathsf{cl}} & \mathsf{H}^T_\bullet(\mathcal{X}) \\ \big\downarrow & & \big\downarrow \\ \mathsf{CH}^{T\times G}_\bullet(V;0) & \xrightarrow{\mathsf{cl}} & \mathsf{H}^{T\times G}_\bullet(V) \end{array},$$

with $k\in\mathbb{Z}$, where the vertical arrows are canonical isomorphisms. There is also a second commutative diagram

$$\begin{array}{ccc} \mathsf{CH}^{T\times G}_\bullet(V;0) & \xrightarrow{\mathsf{cl}} & \mathsf{H}^{T\times G}_\bullet(V) \\ \big\downarrow & & \big\downarrow \\ \mathsf{CH}^\bullet_{T\times G}(V;0) & \xrightarrow{\mathsf{cl}} & \mathsf{H}^\bullet_{T\times G}(V) \end{array}$$

where the vertical arrows are again canonical isomorphisms. The right vertical arrow is just Poincaré duality, while the left one is provided by the bivariate formalism for equivariant Chow groups (cf. [EG98, §2.6]). Then the claim follows from the observation that both $\mathsf{CH}^\bullet_{T\times G}(V;0)$ and $\mathsf{H}^\bullet_{T\times G}(V)$ are canonically isomorphic to the character ring of $T\times G$. □

Since any direct product of linear quotient T-stacks is a linear quotient T-stack, Propositions B.12 and B.14 yield the following.

Corollary B.15. *The following hold for any sequence $(\mathcal{X}_1,\dots,\mathcal{X}_n)$ of linear quotient T-stacks.*

(i) The sequence $(\mathcal{X}_1,\dots,\mathcal{X}_n)$ has the (motivic) Künneth property.

(ii) The T-equivariant Borel-Moore homology of the product $\mathcal{X}_1\times\cdots\times\mathcal{X}_n$ is flat and strongly generated by algebraic cycles.

(iii) The negative weight T-equivariant motivic Borel-Moore homology of $\mathcal{X}_1\times\cdots\times\mathcal{X}_\ell$ is trivial.

Now, we are ready to give the definition of ℓ-cellular (classical) stacks.

Definition B.16.

(i) A quasi-compact quasi-separated geometric classical T-stack $\mathcal{X}$ is 0-*cellular* if $\mathcal{X}$ is equivalent to a T-equivariant iterated vector bundle stack over a linear quotient T-stack.

(ii) A quasi-compact quasi-separated geometric classical T-stack $\mathcal{X}$ is ℓ-*cellular*, with $\ell \geq 1$, if $\mathcal{X}$ admits a T-invariant stratification by closed substacks

$$\emptyset =: \mathcal{X}_0 \subset \mathcal{X}_1 \subset \cdots \subset \mathcal{X}_s := \mathcal{X} \tag{B.2}$$

with $s \geq 1$, such that each stratum

$$\mathcal{X}_i^\circ := \mathcal{X}_i \smallsetminus \mathcal{X}_{i-1}$$

is equivalent to a T-equivariant iterated vector bundle stack over a finite product of $(\ell - 1)$-cellular T-stacks, for $s \geq 1$.

We call the stratification (B.2) a *T-equivariant ℓ-cellular decomposition of* $\mathcal{X}$.

⊘

Remark B.17. Let $\mathcal{X}$ be a 0-cellular T-stack, i.e., $\mathcal{X}$ is isomorphic to an iterated vector bundle stack over a linear quotient stack. Then the following hold by [DPS22, Remark II.1.32], Proposition B.12, and Proposition B.14:

(1) The negative weight Borel-Moore homology of $\mathcal{X}$ vanishes.

(2) The Borel-Moore homology of $\mathcal{X}$ is flat and strongly generated by algebraic cycles.

Furthermore, the same statements hold for any finite product $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ of 0-cellular T-stacks by [DPS22, Remark II.1.32]), Corollary B.15, and Proposition B.12. △

The goal of this section is to show by induction that properties stated in Remark B.17, as well as both Künneth properties hold for ℓ-cellular T-stacks with $\ell \geq 1$. The proof of the inductive step consists of several lemmas, as follows.

Lemma B.18. *Fix $\ell \in \mathbb{N}$, with $\ell \geq 1$, and assume that for any finite product $\mathcal{Y}$ of T-equivariant $(\ell - 1)$-cellular stacks, the following properties hold:*

(i) The negative weight Borel-Moore homology of $\mathcal{Y}$ vanishes.

(ii) The Borel-Moore homology of $\mathcal{Y}$ is flat and strongly generated by algebraic cycles.

Let $\mathcal{X}$ be a ℓ-cellular T-stack and let

$$\emptyset =: \mathcal{X}_0 \subset \mathcal{X}_1 \subset \cdots \subset \mathcal{X}_s := \mathcal{X}$$

be a T-equivariant ℓ-cellular decomposition of $\mathcal{X}$, with $s \geq 1$. Then the following hold:

(1) The negative weight T-equivariant motivic Borel-Moore homology of $\mathcal{X}$ vanishes.

(2) The T-equivariant Borel-Moore homology of $\mathcal{X}$ is flat and strongly generated by algebraic cycles.

(3) For each $1 \leq i \leq s$, the long exact sequence associated to the pair $(\mathcal{X}_{i-1}, \mathcal{X}_i^\circ)$ in $\mathcal{X}_i$ splits into short exact sequences

$$0 \longrightarrow \mathsf{H}_n^T(\mathcal{X}_{i-1}) \longrightarrow \mathsf{H}_n^T(\mathcal{X}_i) \longrightarrow \mathsf{H}_n^T(\mathcal{X}_i^\circ) \longrightarrow 0\,.$$

(4) Statement (3) also holds for weight zero motivic homology groups.

Proof. By definition, for $1 \leq i \leq s$ each stratum $\mathcal{X}_i^\circ$ is equivalent to a T-equivariant iterated vector bundle stack over a finite product of $(\ell - 1)$-cellular T-stacks. By the iterated vector bundle stack property (cf. [DPS22, Remark II.1.32]) and using the current assumptions, it follows that Claims (1) and (2) holds for $\mathcal{X}_i^\circ$ by Corollary B.15–(ii) and –(iii). In particular, the odd degree T-equivariant Borel-Moore homology $\mathsf{H}^T_{\mathsf{odd}}(\mathcal{X}_i^\circ)$ is trivial for all $1 \leq i \leq s$. This implies Claim (3) for the closed substacks $\mathcal{X}_i$ with $1 \leq i \leq s$.

Note that the negative weight Borel-Moore homology of all strata $\mathcal{X}_i^\circ$, with $1 \leq i \leq s$, is also trivial for any $n < 0$ and for all $1 \leq i \leq s$. By induction on $1 \leq i \leq s$, using the the exact sequences

$$\cdots \longrightarrow \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X}_{i-1}; n) \longrightarrow \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X}_i; n) \longrightarrow \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X}_i^\circ; n) \longrightarrow \cdots ,$$

we obtain that Claim (1) holds for all closed substacks $\mathcal{X}_i$ for $1 \leq i \leq s$. In particular, Claim (1) holds for $\mathcal{X}$.

Furthermore, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X}_{i-1}; 0) & \longrightarrow & \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X}_i; 0) & \longrightarrow & \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X}_i^\circ; 0) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathsf{cl}} & & \downarrow{\scriptstyle \mathsf{cl}} & & \downarrow{\scriptstyle \mathsf{cl}} & & \\ 0 & \longrightarrow & \mathsf{H}_\bullet^T(\mathcal{X}_{i-1}) & \longrightarrow & \mathsf{H}_\bullet^T(\mathcal{X}_i) & \longrightarrow & \mathsf{H}_\bullet^T(\mathcal{X}_i^\circ) & \longrightarrow & 0 \end{array}$$

for each $1 \leq i \leq s$. As a consequence of the snake lemma, note that if the outer vertical arrows are isomorphisms, then the same holds for the middle one, and the top row is exact to the left. Under the current assumptions, the right vertical arrow in the above diagram is an isomorphism for all $1 \leq i \leq s$. Therefore, by induction on $1 \leq i \leq s$, this implies Claim (2) for all closed substacks $\mathcal{X}_i$, with $1 \leq i \leq s$. In particular, Claim (2) holds for $\mathcal{X}$. In addition, it also implies that the top row in the above diagram is exact to the left for all $1 \leq i \leq s$, i.e., Claim (4) holds. □

Lemma B.19. *Fix $\ell \in \mathbb{N}$, with $\ell \geq 1$, and assume that any finite product $\mathcal{Y}$ of T-equivariant $(\ell - 1)$-cellular stacks satisfies properties (i) and (ii) in Lemma B.18, and*

(iii) $\mathcal{Y}$ has both the Künneth property (Definition B.10–(1)) as well as the weight zero motivic Künneth property (Definition B.10–(2)).

Let $\mathcal{X}$ be an ℓ-cellular T-stack and let $\mathcal{Y}$ be a (T-equivariant iterated vector bundle stack over) a finite product of $(\ell - 1)$-cellular T-stacks. Then, the negative weight T-equivariant motivic Borel-Moore homology of the product $\mathcal{X} \times \mathcal{Y}$ is trivial, and its T-equivariant Borel-Moore homology is flat and strongly generated by algebraic cycles. Moreover, the pair $(\mathcal{X}, \mathcal{Y})$ has the (motivic) Künneth property as well.

Proof. Let

$$\emptyset =: \mathcal{X}_0 \subset \mathcal{X}_1 \subset \cdots \subset \mathcal{X}_s := \mathcal{X}$$

be a T-equivariant ℓ-cellular decomposition of $\mathcal{X}$. Then

$$\emptyset = \mathcal{X}_0 \times \mathcal{Y} \subset \mathcal{X}_1 \times \mathcal{Y} \subset \cdots \subset \mathcal{X}_s \times \mathcal{Y} = \mathcal{X} \times \mathcal{Y}$$

is a T-equivariant ℓ-cellular decomposition of $\mathcal{X} \times \mathcal{Y}$. Therefore, under the current assumptions, Lemma B.18 shows that $\mathcal{X} \times \mathcal{Y}$ has indeed the first two properties stated in Lemma B.19.

In order to prove the (motivic) Künneth property, note that the long exact sequence associated to each pair $(\mathcal{X}_{i-1}, \mathcal{X}_i^\circ)$ in $\mathcal{X}_i$ splits into short exact sequences, as shown in Lemma B.18–(3). Moreover, the same holds for each pair $(\mathcal{X}_{i-1} \times \mathcal{Y},\ \mathcal{X}_i^\circ \times \mathcal{Y})$ in $\mathcal{X}_i \times \mathcal{Y}$. Thus, one then obtains a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{H}_\bullet^T(\mathcal{X}_{i-1}) \otimes_{\mathsf{H}_T} \mathsf{H}_\bullet^T(\mathcal{Y}) & \longrightarrow & \mathsf{H}_\bullet^T(\mathcal{X}_i) \otimes_{\mathsf{H}_T} \mathsf{H}_\bullet^T(\mathcal{Y}) & \longrightarrow & \mathsf{H}_\bullet^T(\mathcal{X}_i^\circ) \otimes_{\mathsf{H}_T} \mathsf{H}_\bullet^T(\mathcal{Y}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathsf{H}_\bullet^T(\mathcal{X}_{i-1} \times \mathcal{Y}) & \longrightarrow & \mathsf{H}_\bullet^T(\mathcal{X}_i \times \mathcal{Y}) & \longrightarrow & \mathsf{H}_\bullet^T(\mathcal{X}_i^\circ \times \mathcal{Y}) & \longrightarrow & 0 \end{array},$$

where the vertical arrows are Künneth maps. By assumption, the right vertical arrows are isomorphisms for all $1 \leq i \leq s$. Thus, by induction on $1 \leq i \leq s$, we get that the pair $(\mathcal{X}, \mathcal{Y})$ has the Künneth property.

The weight zero motivic Künneth property for the pair follows from its Borel-Moore counterpart using the strong generation condition. □

Lemma B.20. *Fix $\ell \in \mathbb{N}$, with $\ell \geq 1$, and assume that any finite product of T-equivariant $(\ell-1)$-cellular stacks satisfies properties (i) and (ii) in Lemma B.18, and property (iii) in Lemma B.19.*

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be ℓ-cellular T-stacks. Then, the negative weight T-equivariant motivic Borel-Moore homology of the product $\mathcal{X}_1 \times \mathcal{X}_2$ is trivial, and its T-equivariant Borel-Moore homology is flat and strongly generated by algebraic cycles. Moreover, the sequence $(\mathcal{X}_1, \mathcal{X}_2)$ has the (motivic) Künneth property.

Proof. Let

$$\emptyset =: \mathcal{X}_{n,0} \subset \cdots \subset \mathcal{X}_{n,s_n} := \mathcal{X}_n$$

be an ℓ-cellular decomposition of $\mathcal{X}_n$ for $n = 1, 2$. Lemma B.19 implies that the negative weight T-equivariant motivic Borel-Moore homology of $\mathcal{X}_1 \times \mathcal{X}_{2,i}^\circ$ is trivial and its T-equivariant Borel-Moore homology is flat and generated by algebraic cycles. In particular, since the odd degree T-equivariant Borel-Moore homology of each product $\mathcal{X}_1 \times \mathcal{X}_{2,i}^\circ$ is trivial, one obtains a short exact sequence

$$0 \longrightarrow \mathsf{H}_\bullet^T(\mathcal{X}_1 \times \mathcal{X}_{2,i-1}) \longrightarrow \mathsf{H}_\bullet^T(\mathcal{X}_1 \times \mathcal{X}_{2,i}) \longrightarrow \mathsf{H}_\bullet^T(\mathcal{X}_1 \times \mathcal{X}_{2,i}^\circ) \longrightarrow 0$$

for each $1 \leq i \leq s_2$. Since the odd T-equivariant Borel-Moore homology of each stratum $\mathcal{X}_{2,i}^\circ$ is also trivial, by Lemma B.18–(3), we also obtain an exact sequence

$$0 \longrightarrow \mathsf{H}_\bullet^T(\mathcal{X}_1) \otimes_{\mathsf{H}_T} \mathsf{H}_\bullet^T(\mathcal{X}_{2,i-1}) \longrightarrow \mathsf{H}_\bullet^T(\mathcal{X}_1) \otimes_{\mathsf{H}_T} \mathsf{H}_\bullet^T(\mathcal{X}_{2,i}) \longrightarrow \mathsf{H}_\bullet^T(\mathcal{X}_1) \otimes_{\mathsf{H}_T} \mathsf{H}_\bullet^T(\mathcal{X}_{2,i}^\circ) \longrightarrow 0$$

for each $1 \leq i \leq s_2$. Moreover, by Lemma B.18–(3), we also have an exact sequence

$$0 \longrightarrow \mathsf{H}_\bullet^T(\mathcal{X}_{2,i-1}) \longrightarrow \mathsf{H}_\bullet^T(\mathcal{X}_{2,i}) \longrightarrow \mathsf{H}_\bullet^T(\mathcal{X}_{2,i}^\circ) \longrightarrow 0$$

for each $1 \leq i \leq s_2$. Since $\mathsf{H}_\bullet^T(\mathcal{X}_1)$ is flat as an H_T-module by the same lemma, this yields an exact sequence

$$0 \longrightarrow \mathsf{H}_\bullet^T(\mathcal{X}_1) \otimes_{\mathsf{H}_T} \mathsf{H}_\bullet^T(\mathcal{X}_{2,i-1}) \longrightarrow \mathsf{H}_\bullet^T(\mathcal{X}_1) \otimes_{\mathsf{H}_T} \mathsf{H}_\bullet^T(\mathcal{X}_{2,i}) \longrightarrow \mathsf{H}_\bullet^T(\mathcal{X}_1) \otimes_{\mathsf{H}_T} \mathsf{H}_\bullet^T(\mathcal{X}_{2,i}^\circ) \longrightarrow 0$$

for each $1 \leq i \leq s_2$.

In addition, the above sequences fit in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{H}_\bullet^T(\mathcal{X}_1) \otimes_{\mathsf{H}_T} \mathsf{H}_\bullet^T(\mathcal{X}_{2,i-1}) & \longrightarrow & \mathsf{H}_\bullet^T(\mathcal{X}_1) \otimes_{\mathsf{H}_T} \mathsf{H}_\bullet^T(\mathcal{X}_{2,i}) & \longrightarrow & \mathsf{H}_\bullet^T(\mathcal{X}_1) \otimes_{\mathsf{H}_T} \mathsf{H}_\bullet^T(\mathcal{X}_{2,i}^\circ) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathsf{H}_\bullet^T(\mathcal{X}_1 \times \mathcal{X}_{2,i-1}) & \longrightarrow & \mathsf{H}_\bullet^T(\mathcal{X}_1 \times \mathcal{X}_{2,i}) & \longrightarrow & \mathsf{H}_\bullet^T(\mathcal{X}_1 \times \mathcal{X}_{2,i}^\circ) & \longrightarrow & 0 \end{array},$$

where the vertical arrows are Künneth maps. By Lemma B.19, the right vertical map in the above diagram is an isomorphism for each $1 \leq i \leq s_2$, and each term $\mathsf{H}_\bullet^T(\mathcal{X}_1 \times \mathcal{X}_{2,i}^\circ)$ is flat.

Finally, note that the right vertical map is an isomorphism for each $1 \leq i \leq s_2$, and each term $\mathsf{H}_\bullet^T(\mathcal{X}_1 \times \mathcal{X}_{2,i}^\circ)$ is flat by Lemma B.19. By induction on $1 \leq i \leq s_2$, we obtain that the pair $(\mathcal{X}_1, \mathcal{X}_2)$ has the Künneth property. In addition, $\mathsf{H}_\bullet^T(\mathcal{X}_1 \times \mathcal{X}_2)$ is also flat.

Similarly, since the negative weight motivic Borel-Moore homology of each product $\mathcal{X}_1 \times \mathcal{X}_{2,i}^\circ$ vanishes by Lemma B.19, by induction on $1 \leq i \leq s_2$, we obtain the exact sequences

$$\mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X}_1 \times \mathcal{X}_{2,i-1}; 0) \longrightarrow \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X}_1 \times \mathcal{X}_{2,i}; 0) \longrightarrow \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X}_1 \times \mathcal{X}_{2,i}^\circ) \longrightarrow 0 \quad .$$

Hence, in conclusion, we have a commutative diagram

$$\begin{array}{ccccccccc} & & \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X}_1 \times \mathcal{X}_{2,i-1};0) & \longrightarrow & \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X}_1 \times \mathcal{X}_{2,i};0) & \longrightarrow & \mathsf{H}_\bullet^{\mathsf{mot},T}(\mathcal{X}_1 \times \mathcal{X}_{2,i}^\circ;0) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathsf{cl}} & & \downarrow{\scriptstyle \mathsf{cl}} & & \downarrow{\scriptstyle \mathsf{cl}} & & \\ 0 & \longrightarrow & \mathsf{H}_\bullet^T(\mathcal{X}_1 \times \mathcal{X}_{2,i-1}) & \longrightarrow & \mathsf{H}_\bullet^T(\mathcal{X}_1 \times \mathcal{X}_{2,i}) & \longrightarrow & \mathsf{H}_\bullet^T(\mathcal{X}_1 \times \mathcal{X}_{2,i}^\circ) & \longrightarrow & 0 \end{array}$$

for each $1 \leq i \leq s_2$. By Lemma B.19, the right vertical map is an isomorphism for all $1 \leq i \leq s_2$. Then, by induction on $1 \leq i \leq s_2$, using the snake lemma, one obtains that the T-equivariant Borel-Moore homology the middle vertical arrow is an isomorphism as well. Moreover, the top row is also exact to the left.

Finally, note that the strong generation condition implies the Künneth property for weight zero T-equivariant motivic Borel-Moore as well. □

Proposition B.21. *Any sequence $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ of ℓ-cellular T-stacks has the (motivic) Künneth property for any $\ell \geq 0$ and $n \geq 2$.*

Moreover, the negative weight T-equivariant motivic Borel-Moore homology of the product $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ is trivial, and its Borel-Moore homology is flat and strongly generated by algebraic cycles.

Proof. We shall prove the proposition by induction on $\ell \geq 0$. When $\ell = 0$, the claims hold as already discussed in Remark B.17.

Now, let us assume that the claim is true for $(\ell-1)$-cellular stacks and for all values $n \geq 2$ and let us prove it for ℓ-cellular stacks and for all values $n \geq 2$. This inductive step will be proved by induction on $n \geq 2$. Note that case $n = 2$ follows from Lemmas B.18 and B.20. Therefore, it remains to prove the inductive step on n, keeping ℓ fixed.

As an intermediate step, we shall first show that the properties stated in the proposition hold for any sequence of ℓ-cellular T-stacks $\mathcal{X}_1, \ldots, \mathcal{X}_{n-1}$ and $\mathcal{Y}$, with $\mathcal{Y}$ a T-equivariant iterated vector bundle stack over a finite product of $(\ell-1)$-cellular T-stacks. Let

$$\emptyset =: \mathcal{X}_{n-1,0} \subset \cdots \subset \mathcal{X}_{n-1,s} =: \mathcal{X}_{n-1}$$

be an ℓ-cellular decomposition of $\mathcal{X}_{n-1}$. Then note that

$$\emptyset = \mathcal{X}_{n-1,0} \times \mathcal{Y} \subset \cdots \subset \mathcal{X}_{n-1,s} \times \mathcal{Y} = \mathcal{X}_{n-1} \times \mathcal{Y}$$

is an ℓ-cellular decomposition of $\mathcal{X}_{n-1} \times \mathcal{Y}$. Applying the inductive hypothesis with respect to n, it follows that the properties stated in the proposition hold for the sequence $(\mathcal{X}_1, \ldots, \mathcal{X}_{n-2}, \mathcal{X}_{n-1} \times \mathcal{Y})$. In particular, the negative weight T-equivariant motivic Borel-Moore homology of the product $\mathcal{X}_1 \times \cdots \times \mathcal{X}_{n-2} \times (\mathcal{X}_{n-1} \times \mathcal{Y})$ is trivial, and its Borel-Moore homology is flat and strongly generated by algebraic cycles. Moreover, the Künneth map

$$\mathsf{H}_\bullet^T(\mathcal{X}_1) \otimes_{\mathsf{H}_T} \cdots \otimes \mathsf{H}_\bullet^T(\mathcal{X}_{n-2}) \otimes_{\mathsf{H}_T} \mathsf{H}_\bullet^T(\mathcal{X}_{n-1} \times \mathcal{Y}) \longrightarrow \mathsf{H}_\bullet^T(\mathcal{X}_1 \times \cdots \times \mathcal{X}_{n-2} \times \mathcal{X}_{n-1} \times \mathcal{Y})$$

is an isomorphism. In addition, Lemma B.19 shows that the Künneth map

$$\mathsf{H}_\bullet^T(\mathcal{X}_{n-1}) \otimes \mathsf{H}_\bullet^T(\mathcal{Y}) \longrightarrow \mathsf{H}_\bullet^T(\mathcal{X}_{n-1} \times \mathcal{Y})$$

is an isomorphism. By composition, it follows that the Künneth map

$$\mathsf{H}_\bullet^T(\mathcal{X}_1) \otimes \cdots \otimes \mathsf{H}_\bullet^T(\mathcal{X}_{n-2}) \otimes \mathsf{H}_\bullet^T(\mathcal{X}_{n-1}) \otimes \mathsf{H}_\bullet^T(\mathcal{Y}) \longrightarrow \mathsf{H}_\bullet^T(\mathcal{X}_1 \times \cdots \times \mathcal{X}_{n-2} \times \mathcal{X}_{n-1} \times \mathcal{Y})$$

is also isomorphism. This completes the proof of the intermediate step.

In order to complete the proof of the inductive step on n, (hence also of the inductive step on ℓ), let

$$\emptyset =: \mathcal{X}_{n,0} \subset \cdots \subset \mathcal{X}_{n,s} := \mathcal{X}_n$$

be an ℓ-cellular decomposition of $\mathcal{X}_n$ and let

$$\mathcal{Z} := \mathcal{X}_1 \times \cdots \times \mathcal{X}_{n-1} .$$

One has the induced stratification

$$\emptyset =: \mathcal{Z}\times\mathcal{X}_{n,0} \subset \cdots \subset \mathcal{Z}\times\mathcal{X}_{n,s} \coloneqq \mathcal{Z}\times\mathcal{X}_n\ ,$$

with strata $\mathcal{Z}\times\mathcal{X}^\circ_{n,i}$, for $1\leq i\leq s$. Since each stratum $\mathcal{X}^\circ_{n,i}$ is an iterated vector bundle stack over a finite product of $(\ell-1)$-cellular stacks, the above intermediate step applies to all collections $(\mathcal{X}_1,\ldots,\mathcal{X}_{n-1},\mathcal{X}^\circ_{n,i})$, for $1\leq i\leq s$. In particular, the odd degree T-equivariant Borel-Moore homology $\mathsf{H}^T_{\mathsf{odd}}(\mathcal{Z}\times\mathcal{X}^\circ_{n,i})$ is trivial for all $1\leq i\leq s$. Hence the long exact sequence associated to each complementary pair $(\mathcal{Z}\times\mathcal{X}_{n,i-1},\ \mathcal{Z}\times\mathcal{X}^\circ_{n,i})$ in $\mathcal{Z}\times\mathcal{X}_{n,i}$ splits into short exact sequences

$$0\longrightarrow \mathsf{H}^T_\bullet(\mathcal{Z}\times\mathcal{X}_{n,i-1})\longrightarrow \mathsf{H}^T_\bullet(\mathcal{Z}\times\mathcal{X}_{n,i})\longrightarrow \mathsf{H}^T_\bullet(\mathcal{Z}\times\mathcal{X}^\circ_{n,i})\longrightarrow 0\ . \tag{B.3}$$

Moreover, the negative weight motivic T-equivariant Borel-Moore homology of $\mathcal{Z}\times\mathcal{X}^\circ_{n,i}$ is also trivial for all $1\leq i\leq s$. By induction over $1\leq i\leq s$, this yields an exact sequence

$$\mathsf{H}^{\mathsf{mot},T}_\bullet(\mathcal{Z}\times\mathcal{X}_{n,i-1};0)\longrightarrow \mathsf{H}^{\mathsf{mot},T}_\bullet(\mathcal{Z}\times\mathcal{X}_{n,i};0)\longrightarrow \mathsf{H}^{\mathsf{mot}}_\bullet(\mathcal{Z}\times\mathcal{X}^\circ_{n,i};0)\longrightarrow 0 \tag{B.4}$$

for each $1\leq i\leq s$. In addition, for each $1\leq i\leq s$, and each $k<0$, one has an exact sequence

$$\mathsf{H}^{\mathsf{mot},T}_\bullet(\mathcal{Z}\times\mathcal{X}_{n,i-1};k)\longrightarrow \mathsf{H}^{\mathsf{mot},T}_\bullet(\mathcal{Z}\times\mathcal{X}_{n,i};k)\longrightarrow \mathsf{H}^{\mathsf{mot},T}_\bullet(\mathcal{Z}\times\mathcal{X}^\circ_{n,i};k)\ .$$

By induction over $1\leq i\leq s$, this implies that the negative weight motivic T-equivariant Borel-Moore homology of $\mathcal{Z}\times\mathcal{X}_{n,i}$ is trivial for all $1\leq i\leq s$.

Next note that the exact sequences (B.3) and (B.4) fit in the commutative diagram

$$\begin{array}{ccccccccc} & & \mathsf{H}^{\mathsf{mot},T}_\bullet(\mathcal{Z}\times\mathcal{X}_{n,i-1};0) & \to & \mathsf{H}^{\mathsf{mot},T}_\bullet(\mathcal{Z}\times\mathcal{X}_{n,i};0) & \to & \mathsf{H}^{\mathsf{mot},T}_\bullet(\mathcal{Z}\times\mathcal{X}^\circ_{n,i};0) & \to & 0 \\ & & \downarrow{\scriptstyle \mathsf{cl}} & & \downarrow{\scriptstyle \mathsf{cl}} & & \downarrow{\scriptstyle \mathsf{cl}} & & \\ 0 & \longrightarrow & \mathsf{H}^T_\bullet(\mathcal{Z}\times\mathcal{X}_{n,i-1}) & \longrightarrow & \mathsf{H}^T_\bullet(\mathcal{Z}\times\mathcal{X}_{n,i}) & \longrightarrow & \mathsf{H}^T_\bullet(\mathcal{Z}_1\times\mathcal{X}^\circ_{n,i}) & \longrightarrow & 0 \end{array}\ ,$$

where the right vertical arrow in the above sequence is an isomorphism for all $1\leq i\leq s$. Then, using the snake lemma, an inductive argument over $1\leq i\leq s$, proves that the T-equivariant Borel-Moore homology of $\mathcal{Z}\times\mathcal{X}_n$ is flat and strongly generated by algebraic cycles. Moreover, the top row is exact to the left.

Finally, in order to prove the Künneth property, note the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathsf{H}^T_\bullet(\mathcal{Z})\otimes_{\mathsf{H}_T}\mathsf{H}^T_\bullet(\mathcal{X}_{n,i-1}) & \to & \mathsf{H}^T_\bullet(\mathcal{Z})\otimes_{\mathsf{H}_T}\mathsf{H}^T_\bullet(\mathcal{X}_{n,i}) & \to & \mathsf{H}^T_\bullet(\mathcal{Z})\otimes_{\mathsf{H}_T}\mathsf{H}^T_\bullet(\mathcal{X}^\circ_{n,i}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathsf{H}^T_\bullet(\mathcal{Z}\times\mathcal{X}_{n,i-1}) & \longrightarrow & \mathsf{H}^T_\bullet(\mathcal{Z}\times\mathcal{X}_{n,i}) & \longrightarrow & \mathsf{H}^T_\bullet(\mathcal{Z}\times\mathcal{X}^\circ_{n,i}) & \longrightarrow & 0 \end{array}\ ,$$

where the vertical arrows are Künneth maps. The right vertical arrows are isomorphisms for all $1\leq i\leq s$ by the intermediate step proven above. Then the claim follows by induction on $1\leq i\leq s$. This completes the proof of the inductive step on n, as well as the proof of the inductive step on ℓ.

Given the strong generation property, we conclude that the weight zero motivic Künneth property also holds for an arbitrary finite sequence of ℓ-cellular stacks. □

Definition B.22. We say that a T-invariant admissible open exhaustion $\{\mathcal{U}_k\}_{k\in J}$ of an admissible indgeometric derived T-stack $\mathcal{X}$ is T-equivariantly *ℓ-cellular* for some $\ell\geq 0$ if, for each $k\in J$, the reduced stack ${}^{\mathsf{red}}\mathcal{U}_k$ admits a T-equivariant ℓ-cellular

$$\emptyset =: {}^{\mathsf{red}}\mathcal{U}_{k,0}\subset {}^{\mathsf{red}}\mathcal{U}_{k,1}\subset\cdots\subset {}^{\mathsf{red}}\mathcal{U}_{k,\ell}\coloneqq {}^{\mathsf{red}}\mathcal{U}_k \tag{B.5}$$

satisfying the following conditions:

(1) The stratifications (B.5) are naturally compatible with the open immersions ${}^{\mathsf{red}}\mathcal{U}_k\to {}^{\mathsf{red}}\mathcal{U}_{k'}$ for any $0\leq k<k'$.

(2) For any pair $0 \leq k < k'$ the induced stratification
$$\mathcal{Z}_{k,k',i} \coloneqq \mathcal{Z}_{k,k'} \times_{{}^{\mathsf{red}}\mathcal{U}_{k'}} {}^{\mathsf{red}}\mathcal{U}_{k',i}$$
for $0 \leq i \leq s$, of the canonical reduced complement $\mathcal{Z}_{k,k'} \coloneqq {}^{\mathsf{red}}\mathcal{U}_{k'} \smallsetminus {}^{\mathsf{red}}\mathcal{U}_k$ is a T-equivariant ℓ-cellular decomposition.

We shall call $\{\mathcal{U}_k\}_{k\in J}$ a T-invariant *ℓ-cellular* open exhaustion of $\mathcal{X}$. ⊘

An immediate consequence of Theorem B.3 and Proposition B.21 is the following.

Corollary B.23. *Let $\mathcal{X}$ be an admissible indgeometric derived T-stack which admits a T-invariant ℓ-cellular open exhaustion $\{\mathcal{U}_k\}_{k\in J}$. Then the transition functions of the inverse systems*
$$\{\mathsf{H}^T_\bullet(\mathcal{U}_k)\}_{k\in J} \quad \textit{and} \quad \{\mathsf{H}^{\mathsf{mot},T}_\bullet(\mathcal{U}_k)\}_{k\in J}$$
are surjective. Moreover, the restriction maps
$$\mathsf{H}^T_\bullet(\mathcal{X}) \longrightarrow \mathsf{H}^T_\bullet(\mathcal{U}_k) \quad \textit{and} \quad \mathsf{H}^{\mathsf{mot},T}_\bullet(\mathcal{X}) \longrightarrow \mathsf{H}^{\mathsf{mot},T}_\bullet(\mathcal{U}_k)$$
are surjective for all $k \in J$.

We next show that Definition B.22 leads to a well-behaved Künneth map in the presence of T-invariant ℓ-cellular open exhaustions.

Proposition B.24. *Let $\mathcal{X}$ and $\mathcal{Y}$ be an admissible indgeometric derived T-stacks which admits a T-invariant admissible open exhaustions $\{\mathcal{U}_k\}_{k\in J}$ and $\{\mathcal{V}_k\}_{k\in J}$, respectively. Then, the following hold:*

(i) For any $k \in J$, one has a commutative diagram
$$\begin{array}{ccc} \mathsf{H}^T_\bullet(\mathcal{X})\widehat{\otimes}_{\mathsf{H}_T}\mathsf{H}^T_\bullet(\mathcal{Y}) & \longrightarrow & \mathsf{H}^T_\bullet(\mathcal{X}\times\mathcal{Y}) \\ \downarrow & & \downarrow \\ \mathsf{H}^T_\bullet(\mathcal{X})\widehat{\otimes}_{\mathsf{H}_T}\mathsf{H}^T_\bullet(\mathcal{V}_k) & \longrightarrow & \mathsf{H}^T_\bullet(\mathcal{X}\times\mathcal{V}_k) \end{array} , \tag{B.6}$$
where the vertical arrows are the canonical projections, and the horizontal arrows are Künneth maps.

(ii) If $\{\mathcal{U}_k\}_{k\in J}$ and $\{\mathcal{V}_k\}_{k\in J}$ are T-invariant ℓ-cellular open exhaustions for some $\ell \geq 0$, then the vertical maps in diagram (B.6) are surjective, and the bottom horizontal arrow is an isomorphism.

Proof. Recall that the Borel-Moore homology of $\mathcal{X}$ (resp. $\mathcal{Y}$) is isomorphic to the limit of Borel-Moore homologies of the $\mathcal{U}_k$'s (resp. $\mathcal{V}_k$'s) by [DPS22, Remark II.1.44] and [DPS+25b, Remark 3.1]. Hence, the top horizontal arrow in diagram (B.6) coincides with the natural composition map
$$\Big(\lim_h \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h)\Big)\widehat{\otimes}_{\mathsf{H}_T}\mathsf{H}^T_\bullet(\mathcal{Y}) \longrightarrow \lim_h \Big(\mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h)\widehat{\otimes}_{\mathsf{H}_T}\mathsf{H}^T_\bullet(\mathcal{Y})\Big) \longrightarrow \lim_h \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h \times \mathcal{Y}) ,$$
where the first map from the right-hand-side is induced by the Künneth maps
$$\mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h)\widehat{\otimes}_{\mathsf{H}_T}\mathsf{H}^T_\bullet(\mathcal{Y}) \longrightarrow \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h \times \mathcal{Y})$$
for $h \geq 0$. This yields the diagram (B.6), where the bottom horizontal map is a similar composition,
$$\Big(\lim_h \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h)\Big)\widehat{\otimes}_{\mathsf{H}_T}\mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{V}_k) \longrightarrow \lim_h \Big(\mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h) \otimes_{\mathsf{H}_T} \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{V}_k)\Big)$$
$$\longrightarrow \lim_h \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h \times {}^{\mathsf{red}}\mathcal{V}_k) , \tag{B.7}$$
for any fixed $k \in J$. Thus, claim (i) follows.

Now, we prove claim (ii). Surjectivity of the left vertical map in diagram (B.6) follows from Corollary B.23. Surjectivity of the right vertical map will follow once we prove that the bottom horizontal map is an isomorphism, as we shall show now.

First note that any pair $({}^{\mathsf{red}}\mathcal{U}_h, {}^{\mathsf{red}}\mathcal{V}_k)$ has the Künneth property under the stated conditions. This implies that the first map from the right-hand-side in (B.7) is an isomorphism of H_T-modules. Hence it suffices to prove that the first map from the left-hand-side, i.e.,

$$\Big(\lim_h \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h)\Big)\widehat{\otimes}_{\mathsf{H}_T}\mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{V}_k) \longrightarrow \lim_h \Big(\mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h)\otimes_{\mathsf{H}_T}\mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{V}_k)\Big), \tag{B.8}$$

is also an isomorphism of H_T-modules. This will be proven by induction on $\ell \geq 0$.

For each $k \geq 0$, let

$$\emptyset =: {}^{\mathsf{red}}\mathcal{V}_{k,0} \subset {}^{\mathsf{red}}\mathcal{V}_{k,1} \subset \cdots \subset {}^{\mathsf{red}}\mathcal{V}_{k,s} := {}^{\mathsf{red}}\mathcal{V}_k$$

for $s \geq 1$, be a T-invariant ℓ-cellular structure of ${}^{\mathsf{red}}\mathcal{V}_k$ satisfying the conditions of Definition B.22.

For $\ell = 0$, the reduced stack ${}^{\mathsf{red}}\mathcal{V}_k$ is a linear quotient T stack in the sense of Definition B.13, in particular it is T-equivariantly equivalent to quotient stack $[V/G]$, where V is a finite dimensional vector space equipped with a linear $T \times G$-action, where G is a finite product of general linear groups. Note also the canonical isomorphism

$$\mathsf{H}^{T\times G}_\bullet(V) \simeq \mathsf{H}^T_\bullet(\mathsf{pt})\otimes_{\mathbb{C}}\mathsf{H}^G_\bullet(V) \simeq (\mathsf{H}_T^{-\bullet})^\vee \otimes_{\mathbb{C}} \mathsf{H}^G_\bullet(V)\,.$$

Then one obtains an isomorphism

$$\mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h)\otimes_{\mathsf{H}_T}\mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{V}_k) \simeq \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h)\otimes_{\mathbb{C}}\mathsf{H}^G_\bullet(V)$$

for each $h \geq 0$. Moreover the above isomorphisms are clearly compatible with the restriction maps associated to the open immersions ${}^{\mathsf{red}}\mathcal{U}_h \to {}^{\mathsf{red}}\mathcal{U}_{h'}$, for $0 \leq h < h'$. Similarly, one also has an isomorphism

$$\Big(\lim_h \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h)\Big)\widehat{\otimes}_{\mathsf{H}_T}\mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{V}_k) \simeq \Big(\lim_h \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h)\Big)\widehat{\otimes}_{\mathbb{C}}\mathsf{H}^G_\bullet(V)\,.$$

Therefore the case $\ell = 0$ reduces to the claim that the natural map

$$\Big(\lim_h \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h)\Big)\widehat{\otimes}_{\mathbb{C}}\mathsf{H}^G_\bullet(V) \longrightarrow \lim_h \Big(\mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h)\otimes_{\mathbb{C}}\mathsf{H}^G_\bullet(V)\Big)$$

is an isomorphism. Making the homological degrees explicit, the map in question reads

$$\bigoplus_{i+j=m}\Big(\lim_h \mathsf{H}^T_i({}^{\mathsf{red}}\mathcal{U}_h)\Big)\widehat{\otimes}_{\mathbb{C}}\mathsf{H}^G_j(V) \longrightarrow \bigoplus_{i+j=m}\lim_h \Big(\mathsf{H}^T_i({}^{\mathsf{red}}\mathcal{U}_h)\otimes_{\mathbb{C}}\mathsf{H}^G_j(V)\Big)$$

for each fixed degree $m \in \mathbb{Z}$. This follows from the fact that $\mathsf{H}^G_j(V) = 0$ for all $j > 2\dim_{\mathbb{C}}(V)$, and that $\mathsf{H}^G_j(V)$ is a finite dimensional vector space for $j \leq 2\dim_{\mathbb{C}}(V)$. In particular, the direct sum over (i,j) in the above equation is finite. This proves the case $\ell = 0$.

In order to prove the inductive step, note that for each $1 \leq i \leq s$ we have an exact sequence

$$0 \longrightarrow \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{V}_{k,i-1}) \longrightarrow \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{V}_{k,i}) \longrightarrow \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{V}^\circ_{k,i}) \longrightarrow 0 \tag{B.9}$$

of flat H_T-modules. Here, ${}^{\mathsf{red}}\mathcal{V}^\circ_{k,i} := {}^{\mathsf{red}}\mathcal{V}_{k,i} \smallsetminus {}^{\mathsf{red}}\mathcal{V}_{k,i-1}$ is equivalent to a T-equivariant iterated vector bundle stack over a finite product of $(\ell-1)$-cellular T-stacks, for $1 \leq i \leq s$.

The sequence (B.9) yields an exact sequence

$$\begin{aligned}0 \longrightarrow \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h)\otimes_{\mathsf{H}_T}\mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{V}_{k,i-1}) &\longrightarrow \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h)\otimes_{\mathsf{H}_T}\mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{V}_{k,i}) \\ &\longrightarrow \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h)\otimes_{\mathsf{H}_T}\mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{V}^\circ_{k,i}) \longrightarrow 0\end{aligned}$$

for each $h \geq 0$. For ease of exposition, we will denote the above exact sequence by

$$0 \longrightarrow H_{h,i-1} \longrightarrow H_{h,i} \longrightarrow H^\circ_{h,i} \longrightarrow 0\,.$$

Then, for any $0 \leq h < h'$ we have a commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_{h',i-1} & \longrightarrow & H_{h',i} & \longrightarrow & H^\circ_{h',i} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_{h,i-1} & \longrightarrow & H_{h,i} & \longrightarrow & H^\circ_{h,i} & \longrightarrow & 0 \end{array},$$

where the vertical arrows are the natural restriction maps. In particular, the data $\{H_{h,i}\}_{h\in\mathbb{Z}}$ and $\{H^\circ_{h,i}\}_{h\in\mathbb{Z}}$ define inverse systems of H_T-modules for any $1 \leq i \leq i-1$. Moreover, given the above commutative diagrams one obtains an exact sequence

$$0 \longrightarrow \lim_h H_{h,i-1} \longrightarrow \lim_h H_{h,i} \longrightarrow \lim_h H^\circ_{h,i} \tag{B.10}$$

for each $1 \leq i \leq s$. Moreover, the inverse system $\{\mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h)\}_{h\in J}$ is surjective by Corollary B.23. Hence, the inverse system $\{H_{h,i}\}_{h\in J}$ is also surjective for any $1 \leq i \leq s$. This further implies that the sequence (B.10) is exact to the right. In conclusion, we are left with an exact sequence

$$0 \longrightarrow \lim_h H_{h,i-1} \longrightarrow \lim_h H_{h,i} \longrightarrow \lim_h H^\circ_{h,i} \longrightarrow 0$$

for each $1 \leq i \leq s$.

Next note that the exact sequence (B.9) also yields the exact sequence

$$0 \longrightarrow \Big(\lim_h \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h)\Big)\widehat{\otimes}_{\mathsf{H}_T}\mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{V}_{k,i-1}) \longrightarrow \Big(\lim_h \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h)\Big)\widehat{\otimes}_{\mathsf{H}_T}\mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{V}_{k,i})$$
$$\longrightarrow \Big(\lim_h \mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{U}_h)\Big)\widehat{\otimes}_{\mathsf{H}_T}\mathsf{H}^T_\bullet({}^{\mathsf{red}}\mathcal{V}^\circ_{k,i}) \longrightarrow 0$$

for any $1 \leq i \leq s$. Again, for ease of exposition, this sequence will be denoted by

$$0 \longrightarrow \widehat{H}_{i-1} \longrightarrow \widehat{H}_i \longrightarrow \widehat{H}^\circ_i \longrightarrow 0\,.$$

In order to conclude the proof, note that for each $1 \leq i \leq s$ one has a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{H}_{i-1} & \longrightarrow & \widehat{H}_i & \longrightarrow & \widehat{H}^\circ_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \lim_h H_{h,i-1} & \longrightarrow & \lim_h H_{h,i} & \longrightarrow & \lim_h H^\circ_{h,i} & \longrightarrow & 0 \end{array},$$

where the vertical arrows are natural maps as in Formula (B.8). Then the inductive step on ℓ is proven by straightforward induction on $1 \leq i \leq s$. □

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