Cohomological Hall algebras, moduli spaces, and quantum groups

Plan

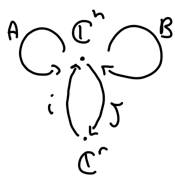
1. Motivation: moduli spaces vs. vertex algebras
2. Pletora of Cohomological Hall Algebras
3. COHA of the minimal resolution of a ADE singularity

1. moduli spaces vs. vertex algebras

Consider

$\mathcal{M}(r,n)$ = moduli space of framed sheaves ($\mathcal{E}$ torsion-free, $\phi : \mathcal{E}|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r}$ framing) on $\mathbb{P}^2 = \mathbb{C}^2 \cup \ell_\infty$ of $\mathrm{rk} = r \in \mathbb{Z}_{\geq 1}$, $c_1 = 0$, $c_2 = n \in \mathbb{Z}_{\geq 0}$

$$\simeq \left\{ (A,B,i,j) : [A,B] + ij = 0 \ ; \ \nexists\, 0 \subsetneq S \subsetneq \mathbb{C}^n \text{ s.t. } A(S) \subseteq S, B(S) \subseteq S, \mathrm{Im}(i) \subseteq S \right\} / GL_n(\mathbb{C})$$



This is the ADHM description of $\mathcal{M}(r,n)$

Via conjugation: $(gAg^{-1}, gBg^{-1}, gi, jg^{-1})$

= smooth quasi-projective variety of dimension $2rn$

Rmk

- $\mathcal{M}(1,n) \simeq \mathrm{Hilb}^n(\mathbb{C}^2)$ = Hilbert scheme of n pts in $\mathbb{C}^2$

- $\mathcal{M}(r,n)$ = Nakajima quiver variety associated to the 1-loop quiver

Fact: $\exists\, T := \underbrace{(\mathbb{C}^*)^2}_{\ni (t_1,t_2)} \times \underbrace{(\mathbb{C}^*)^r}_{\ni D} \curvearrowright \mathcal{M}(r,n)$:

torus of $\mathbb{P}^2$; D = diagonal matrix

- $(t_1,t_2,D)\cdot(\mathcal{E},\phi) = \left((F^{-1}_{t_1,t_2})^*\mathcal{E},\ \tilde{\phi} : (F^{-1}_{t_1,t_2})^*\mathcal{E} \longrightarrow (F^{-1}_{t_1,t_2})^*\mathcal{O}^{\oplus r}_{\ell_\infty} \longrightarrow \mathcal{O}^{\oplus r}_{\ell_\infty} \xrightarrow{\cdot D} \mathcal{O}^{\oplus r}_{\ell_\infty}\right)$

- $(t_1,t_2,D)\cdot(A,B,i,j) = (t_1A,\, t_2B,\, iD^{-1},\, t_1t_2Dj)$

The main player from the geometric side is the following vector space:

Def. $\mathbb{L}^{(r)}_n := H^*_T(\mathcal{M}(r,n))$ module over $H^*_T(pt) \simeq \mathbb{C}[\varepsilon_1,\varepsilon_2,a_1,\dots,a_r] =: R_r$

$\mathbb{L}^{(r)} := \bigoplus_{n\geq 0} \mathbb{L}^{(r)}_n$; $\mathbb{L}^{(r)}_K := \mathbb{L}^{(r)} \otimes_{R_r} K_r$ $\quad (K_r := \mathrm{Frac}(R_r)$ = field of fractions$)$

The main player from the algebraic side is:

Recall: $\mathcal{W}(sl(r))$ = $\mathbb{Z}$-graded vertex algebra generated by $\widetilde{W}_i(z) = \sum_{\ell\in\mathbb{Z}} \widetilde{W}_{i,\ell}\, z^{-\ell-i}$
for $i=2,\dots,r$

$$\mathcal{W}(gl(r)) := \mathcal{W}(sl(r)) \otimes \text{Heisenberg algebra}$$

$$\| $$

$$\langle b_\ell, c : \ell\in\mathbb{Z}\setminus\{0\}\rangle / [b_\ell, c]=0,\ [b_\ell, b_{-k}] = \ell\,\delta_{\ell,k}\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)c$$

Examples:

- $r=1$ $\quad \mathcal{W}(gl(1))$ = Heisenberg algebra
- $r=2$ $\quad \mathcal{W}(gl(2))$:= Virasoro algebra $\otimes$ Heisenberg algebra

Thm (Schiffmann-Vasserot, Maulik-Okounkov)

$\exists$ an action of $\mathcal{W}(gl(r))$ on $\mathbb{L}_K^{(r)}$ s.t. $\mathbb{L}_K^{(r)} \simeq$ Verma module with highest weight vector $[M(r,0)]$ (=fundamental class of $M(r,0)$), i.e., the module with basis

$$\left\{ \widetilde{W}_{i_1,-\ell_1} \cdots \widetilde{W}_{i_s,-\ell_s} [M(r,0)] : s\geq 0,\ \ell_i \geq 1 \right\}$$

Rmk

$r=1$: one recovers Nakajima and Grojnowski's result:
$\exists$ an action of the Heisenberg algebra on $\mathbb{L}_K^{(1)}$ s.t. $\mathbb{L}_K^{(1)} \simeq$ Fock space

Question: How did SV, MO prove the Theorem?

Attention ⚠: They are unable to construct the action directly, rather they use the action of another algebra:

Consider

- $F^c := \mathbb{C}(-\varepsilon_2/\varepsilon_1)[c_0, c_1, c_2, \ldots]$ — infinitely many variables
- $\exists\ F^c \longrightarrow K_r$, $c_k \longmapsto p_k(\frac{a_1}{\varepsilon_1}, \ldots, \frac{a_r}{\varepsilon_1})$ - kth power sum
- $\forall\ M$ = module over F^c, $M_{K_r} := M \otimes_{F^c} K_r$

Thm (SV, MO)

1. $\exists$ a faithful representation of $Y(\hat{gl}(1))_{K_r}$ (= affine Yangian of $\hat{gl}(1)$) on $\mathbb{L}^{(r)}_K$ s.t. $\mathbb{L}^{(r)}_K$ is generated by $[M(r,0)]$.

2. $\exists$ an embedding $Y(\hat{gl}(1))_{K_r} \hookrightarrow U(\mathcal{W}(gl(r)))$ (universal enveloping algebra)

as subalgebras of $\mathrm{End}(\mathbb{L}^{(r)}_K)$ s.t. $\exists$ an equivalence of categories:

$$\{\text{admissible } U(\mathcal{W}(gl(r)))\text{-modules}\} \xrightarrow{\sim} \{\text{admissible } Y(\hat{gl}(1))_{K_r}\text{-modules}\}$$

Rmk

- $Y(\widehat{\mathfrak{gl}}(1))$ = deformation of $U(\text{central extension of } \widehat{\mathfrak{gl}}(1)\otimes\mathbb{C}[z])$

  $\widehat{\mathfrak{gl}}(1)\otimes\mathbb{C}[z] = \mathbb{C}[t^{\pm 1}, z]$

  deformation $= \exists$ a $\mathbb{N}$-filtration $F^\bullet$ s.t. $gr(F^\bullet) \simeq U(\text{——})$

- The two representations are different:

depend on r

$$\overbrace{Y(\widehat{\mathfrak{gl}}(1))}^{\text{always the same}}{}_{K_r} \xrightarrow{\text{faithful}} \mathbb{L}^{(r)}_K \xleftarrow{\text{Verma module}} \mathcal{W}(\mathfrak{gl}(r))$$

$Y(\widehat{\mathfrak{gl}}(1))_{K_r}[M(r,0)] \simeq \mathbb{L}^{(r)}_K$

is obtained geometrically via:

$$\begin{cases} \text{MO: theory of stable envelopes + R-matrix realization} \\ \text{SV: definition via explicit operators} \\ \text{SV: COHA of 1-loop quiver} \end{cases}$$

Attention ⚠ : This is a 2d COHA and not 1d/3d Kontsevich-Soibelman COHA

Def.

- $$\mathrm{COHA}_{1\text{-loop}} = \bigoplus_{k\geq 0} H_*^{(\mathbb{C}^*)^2\times GL_k(\mathbb{C})}(C_k)$$ as a vector space

where

$$\underbrace{(\mathbb{C}^*)^2}_{\text{scaling}} \times \underbrace{GL_k(\mathbb{C})}_{\text{conjugation}} \curvearrowright C_k = \{(A,B)\in Mat(k,\mathbb{C}) : [A,B]=0\}$$ – commuting variety

- multiplication : $m = \bigoplus_{k_1,k_2} m_{k_1,k_2}$, where

$$m_{k_1,k_2} : H_*^{(\mathbb{C}^*)^2\times GL_{k_1}(\mathbb{C})}(C_{k_1}) \times H_*^{(\mathbb{C}^*)^2\times GL_{k_2}(\mathbb{C})}(C_{k_2}) \longrightarrow H_*^{(\mathbb{C}^*)^2\times GL_{k_1+k_2}(\mathbb{C})}(C_{k_1+k_2})$$

refined Gysin pullback

$m_{k_1,k_2} := q_* \circ p^*$ induced by the Hall convolution diagram:

$$\begin{tikzcd}
& C_{k_1+k_2} \cap \text{parabolic of } gl(k_1+k_2,\mathbb{C}) \arrow[dl, "p"'] \arrow[dr, "q"] & \\
C_{k_2} \times C_{k_1} & & C_{k_1+k_2}
\end{tikzcd}$$

corresponding to fixing $\mathbb{C}^{k_1} \subset \mathbb{C}^{k_1+k_2}$

The Key result is:

Thm (SV)

1. $(\mathrm{COHA}_{1\text{-loop}})_{K_r} \simeq Y^+(\hat{\mathfrak{gl}}(1))_{K_r} =: Y^+(\mathfrak{g}_{1\text{-loop}})_{K_r}$

2. $(\mathrm{COHA}_{1\text{-loop}})_{K_r} \curvearrowright \mathbb{L}^{(r)}_{K}$

Question: Where is the 1-loop quiver in the definition of $\mathrm{COHA}_{1\text{-loop}}$?

Answer:

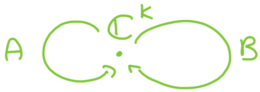
- $$\mathrm{COHA}_{1\text{-loop}} = \bigoplus_{k \geq 0} H_*^{(\mathbb{C}^*)^2 \times GL_k(\mathbb{C})}(C_k)$$

stack quotient

stack of repr.s of $\Pi_{1\text{-loop}}$ of dimension k

$$\simeq \bigoplus_{k\geq 0} H_*^{(\mathbb{C}^*)^2}\left(\left[C_k / GL_k(\mathbb{C})\right]\right) \simeq \bigoplus_{k \geq 0} H_*^{(\mathbb{C}^*)^2}\left(\underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}; k)\right)$$

$\cup$



$\longmapsto$ $\Pi_{1\text{-loop}}$ - module (= repr.)

$+$

$[A, B] = 0$

$\Pi_{1\text{-loop}} = $ preprojective algebra of the 1-loop quiver $= \mathbb{C}\langle x, y\rangle / [x,y]$

Set $\underline{\mathrm{Rep}}(\Pi_{1\text{-Loop}}) = \bigsqcup_{k\geq 0} \underline{\mathrm{Rep}}(\Pi_{1\text{-Loop}}, k)$

Hall convolution diagram becomes:

stack of extensions of finite-dim. reprs

$$\begin{tikzcd}
 & \underline{\mathrm{Rep}}^{ext}(\Pi_{1\text{-loop}}) \arrow[dl, bend right, "p"'] \arrow[dr, bend left, "q"] & \\
 & 0 \to E_1 \to E \to E_2 \to 0 \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
\underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}) \times \underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}) \ni (E_2, E_1) & & E \in \underline{\mathrm{Rep}}(\Pi_{1\text{-loop}})
\end{tikzcd}$$

$$m := q_* \circ p^* : H_*^{(\mathbb{C}^*)^2}(\underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}))^{\times 2} \longrightarrow H_*^{(\mathbb{C}^*)^2}(\underline{\mathrm{Rep}}(\Pi_{1\text{-loop}}))$$

This way of interpreting $\mathrm{COHA}_{1\text{-loop}}$ shows that the same construction can be applied to other cases:

2. Plethora of COHAs

Q = quiver = $(I = \{\text{vertices}\}, \Omega = \{\text{arrows}\})$

Schiffmann-Vasserot, Yang-Zhao: 2d (equivariant) COHA of Q:

$$\mathrm{COHA}_Q^{(A)} := H_*^{(A)}(\underline{\mathrm{Rep}}(\Pi_Q)) \quad + \text{ Hall multiplication}$$

certain torus $\subset \mathbb{C}^* \times (\mathbb{C}^*)^{\Omega}$

Characterization of the "quantum nature":

Thm (SV): $\mathrm{COHA}^A_Q \hookrightarrow Y^+(\mathfrak{g}^{MO}_Q)$ = Yangian of the Maulik-Okounkov Lie algebra of Q

Conjecture: —"— $\simeq$ ——"—— (True for Q = 1-loop, ADE, affine ADE)

X = smooth projective curve$/\mathbb{C}$

We define 3 COHAs associated to X:

- Schiffmann, Minets for $rk = 0$: Dolbeault COHA of X = $\mathrm{COHA}_{Dol}(X)$ =
  COHA of Higgs sheaves ($\mathcal{E}$ coherent sheaf, $\phi : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_X$) on X

Porta –: de Rham COHA of X = $\mathrm{COHA}_{dR}(X)$ =
  COHA of flat bundles ($\mathcal{F}$ vector bundle, $\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega^1_X$ conn. $\nabla^2 = 0$) on X

Porta –, Davison: Betti COHA of X = $\mathrm{COHA}_B(X)$ =
  COHA of $\pi_1(X) \longrightarrow GL_n(\mathbb{C})$ for some n

Rmk

$\exists$ COHA versions of

- Riemann-Hilbert correspondence: $COHA_{dR}(X) \simeq COHA_B(X)$
- non-abelian Hodge correspondence: $\exists$ $F^\bullet \subset COHA_{dR}(X)$ s.t. $gr_{F_\bullet} \simeq COHA_{Dol}(X;0)$

S smooth quasi-projective surface$/\mathbb{C}$

Schiffmann-Vasserot: COHA of zero-dimensional sheaves on $\mathbb{C}^2$ = $COHA_{1\text{-loop}}$

Kapranov-Vasserot, Yu Zhao for $rk=0$: COHA of properly supported coherent sheaves on S

Rmk

1. $H_*^{BM}(-) \rightsquigarrow K_0(-)$ = Grothendieck group of coherent sheaves

COHAs $\rightsquigarrow$ K-theoretical Hall algebras

Yangians $\rightsquigarrow$ quantum loop algebras

2. Porta- : $\exists$ categorified version: monoidal structure on $D^b(Coh(\text{———}))$

<u>Question</u>: What is the "quantum nature" of the COHAs of curves or the COHA of a surface?

3. COHA of ADE sing.

based on joint project with Diaconescu, Porta, Schiffmann, and Vasserot

Consider

- $G \subset SL(2,\mathbb{C})$ finite group $\longleftrightarrow$ ADE quiver Q_{fin}
- $\pi : Y \longrightarrow X := \mathbb{C}^2/G$ resolution of singularities
- $C := \pi^{-1}(0) = C_1 \cup \cdots \cup C_e$, $C_i \simeq \mathbb{P}^1$, $(C_i \cdot C_j) = -$ Cartan matrix of Q_{fin}
- torus $A \subset GL(2,\mathbb{C})$ centralizing G $(A = \{1\}, \mathbb{C}^*, \text{ or } \mathbb{C}^* \times \mathbb{C}^*)$

Set

$\underline{Coh}_C(Y) :=$ moduli stack of coherent sheaves on Y <u>set-theoretically</u> supported on C

$$\mathrm{COHA}^A_{Y,C} := H^A_*(\underline{Coh}_C(Y))$$

Thm (DP–SV)

1. $\mathrm{COHA}^A_{Y,C}$ depends only on the formal neighborhood $\hat{Y}_C$ of Y along C.

2.

- $\exists$ a surjective map, induced by the Hall multiplication:

$$\phi : H^A_*\left(\underline{\mathrm{Coh}}_{C_1}(Y) \times \cdots \times \underline{\mathrm{Coh}}_{C_r}(Y)\right) \longrightarrow \mathrm{COHA}^A_{Y,C}$$

- $\mathrm{Ker}(\phi)$ only depends on the classes supported at the intersections $\underline{\mathrm{Coh}}_{C_i \cap C_{i+1}}(Y)$

Rmk: (2) provides a PBW-type Theorem.

Attention ⚠: We also provide a description in terms of the Yangian of the affine ADE quiver

Notation:

- Q = affine ADE quiver $= (I = \{\text{vertices}\}, \Omega = \{\text{arrows}\})$
- $\mathfrak{g}_Q$ = (derived) Kac-Moody algebra of Q
- $L\mathfrak{g}_Q$ = (central extension of) $\mathfrak{g}_Q[z] \oplus K$, $K := \bigoplus_{\substack{n\neq 0 \\ \ell>0}} \mathbb{C} c_{n,\ell}$
- $\underline{d} \in \mathbb{Z}I \rightsquigarrow \underline{d} = \underline{d}_{fin} + n\delta \in \mathbb{Z}I_{fin} + \mathbb{Z}\delta$, δ = imaginary root

For any $\ell \in \mathbb{Z}$:

$$L\mathfrak{g}_\ell := \bigoplus_{\underline{d}} L\mathfrak{g}_{\underline{d}},$$

where $\underline{d} \in \mathbb{Z}I$ such that $\begin{cases} \underline{d}_{fin} \in \Delta^-_{fin}, \\ \ell(\check{\rho}_0, \underline{d}_{fin}) \leq n < (\ell+1)(\check{\rho}_0, \underline{d}_{fin}). \end{cases}$

sum of the fundamental coweights of Q_{fin}

Thm (DP-SV)

1. $\mathrm{COHA}_{Y,C} \simeq \text{"}\lim_{\ell\leq 0}\text{"}\, U(Lm)/(L\mathfrak{g}_{<\ell} \cdot U(Lm))$ as algebras

where

$$Lm := \bigoplus_{\underline{d}_{fin} \in \Delta^-_{fin} \cup \{0\}} L\mathfrak{g}_{\underline{d}}, \quad L\mathfrak{g}_{<\ell} := \bigoplus_{k<\ell} L\mathfrak{g}_k$$

2. $\mathrm{COHA}^{A}_{Y,C} \simeq \hat{Y}_Q := \text{"}\lim_{\ell \le 0}\text{"}\, \hat{Y}_{Q,\ell}$ as algebras

where

$\hat{Y}_{Q,\ell}$ is a certain quotient of the Yangian $Y^+(\mathfrak{g}_Q)$

Attention ⚠ :

1. This is the first example of an explicit characterization of a COHA of sheaves of dimension ≥ 0

2. $\mathrm{COHA}^{A}_{Y,C} \simeq$ new positive part of $\hat{Y}(\mathfrak{g}_Q)$ completion