ES 1 $\quad$ 2 R , 2 B $\longrightarrow$ urna con due palline

(i) $\quad X = \# R$

$$P(X=k) = 0 \quad \text{se} \quad k \neq 0,1,2$$

$$P(X=0) = \frac{\binom{2}{0}\binom{2}{2}}{\binom{4}{2}} = \frac{1}{6}$$

$$P(X=2) = \frac{1}{6}$$

$$P(X=1) = 1 - 2\cdot\frac{1}{6} = \frac{2}{3} = \frac{\binom{2}{1}\binom{2}{1}}{\binom{4}{2}}$$

(ii) $\quad Y =$ estrazione di una pallina rossa

$$P(X=2 \mid Y) = \frac{P(X=2 \cap Y)}{P(Y)} =$$

$$= \frac{P(Y \mid X=2)\, P(X=2)}{P(Y)} =$$

$$= \frac{P(Y \mid X=2)\, P(X=2)}{\sum_{k=0}^{2} P(Y \mid X=k)\, P(X=k)}$$

$$P(Y \mid X=k) \begin{cases} = 0, & k=0 \\ = 1/2, & k=1 \\ = 1, & k=2 \end{cases}$$

$$P(X=2 \mid Y) = \frac{1\cdot\frac{1}{6}}{0\cdot\frac{1}{6} + \frac{1}{2}\cdot\frac{2}{3} + 1\cdot\frac{1}{6}} = \frac{1}{3}$$

ES. 2

(i) $F(x) = \begin{cases} \dfrac{1+e^x}{2}, & x < 0 \\ 1, & x \geq 0 \end{cases}$

- $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} F(x) = 1$

$\boxed{NO}$ Ok

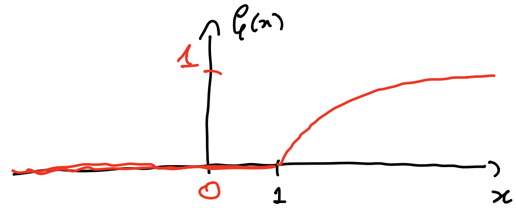
$\lim_{x \to -\infty} \dfrac{1+e^x}{2} = \dfrac{1}{2} \neq 0$

(ii) $G(x) = \begin{cases} 0, & x < 1 \\ 1 - \dfrac{1}{x^4}, & x \geq 1 \end{cases}$

- $\lim_{x \to -\infty} G(x) = 0$ Ok

$\lim_{x \to +\infty} G(x) = 1$, $\lim_{x \to +\infty} \left(1 - \dfrac{1}{x^4}\right) = 1$ Ok

- $G(x)$ crescente, non necessariamente strettamente



- $G(x)$ è continua.

$E[X^k] = \int_{-\infty}^{+\infty} x^k \, g(x) \, dx$ — densità

$g(x) = \begin{cases} 0, & x < 1 \\ \dfrac{4}{x^5}, & x \geq 1 \end{cases}$

$$E[X^k] = \int_1^{+\infty} x^k \frac{4}{x^5}\, dx = 4\int_1^{+\infty} x^{k-5}\, dx < +\infty$$

$$\iff k-5 < -1 \iff k < 4$$

X ha momenti finiti per $k = 1, 2, 3$

$$k<4, \quad E[X^k] = 4\int_1^{+\infty} x^{k-5}\, dx = \frac{4}{k-4} x^{k-4}\Big|_1^{+\infty} = \frac{4}{4-k}$$

$$E[X] = \frac{4}{3} \qquad E[X^2] = 2$$

$$Var(X) = E[X^2] - E[X]^2 = 2 - \frac{16}{9} = \frac{2}{9}$$

ES. 3 $\quad X_1, \dots, X_n$, $n = 10000$, var. aleat. di Bernoulli

$X=0 \leftrightarrow$ NO, $\quad X = 1 \leftrightarrow$ SÌ

$$\hat{p} = \frac{4900}{10000}$$

(i) Precisione della stima $< 10^{-3}$

Intervallo di fiducia $(1-\alpha)$ $= \left[\hat{p} - q_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + q_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$

Prec. della stima $\quad q_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < 10^{-3}$

$$q_{1-\frac{\alpha}{2}}\sqrt{\frac{\frac{4900}{10^4}\cdot\frac{5100}{10^4}}{10^4}} = q_{1-\frac{\alpha}{2}}\frac{70}{10^6}\sqrt{5100} < 10^{-3}$$

$$q_{1-\alpha/2} < \frac{10^3}{70} \frac{1}{\sqrt{5100}} \sim \frac{10^2}{7 \cdot 71,414} \sim \frac{10^2}{499,9} \sim 0,2$$

$$\Phi(q_{1-\alpha/2}) = 1-\alpha/2 \qquad \Phi(0,2) \sim 0,57926 \sim 1-\alpha/2$$

$$\alpha/2 \sim 1-0,579 \qquad \alpha \sim 2 \cdot 0,421 \sim 0,842$$

$$1-\alpha \sim 0,158$$

(ii) $\mathcal{H}_0)$ $p \geq 0.5 = p_0$ $\qquad$ $\mathcal{H}_1)$ $p < 0.5$

$$C = \left\{ \frac{\sqrt{n}}{\sqrt{p_0(1-p_0)}} (\hat{p} - p_0) < q_\alpha \right\} \quad \text{livello } \alpha$$

$$\bar{\alpha} = \Phi\left(\frac{\sqrt{n}}{\sqrt{p_0(1-p_0)}} (\hat{p} - p_0) \right) = \Phi\left(\frac{\sqrt{10^4}}{\sqrt{\frac{1}{4}}} (0,49 - 0,5) \right) =$$

$$= \Phi(-2) = 1 - \Phi(2) \sim 1 - 0.97725 \sim 0.02275$$

Poco plausibile

ES 3 (vecchio programma)

$$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & * \\ 0 & * & * \end{pmatrix}$$

(i) Completare P affinché $\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right)$ sia invariante.

$$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & x & y \end{pmatrix}$$

$$\left(\tfrac{1}{6}\ \ \tfrac{1}{3}\ \ \tfrac{1}{2}\right) P = \left(\tfrac{1}{6}\ \ \tfrac{1}{3}\ \ \tfrac{1}{2}\right)$$

$$\begin{cases} \frac{1}{6} = \frac{1}{6} \\ \frac{1}{6} + \frac{1}{2}x = \frac{1}{3} & \qquad x = \frac{1}{3} \\ \frac{1}{6} + \frac{1}{2}y = \frac{1}{2} & \qquad y = \frac{2}{3} \end{cases}$$

(ii) $X_0 = (1\ \ 0\ \ 0)$

$X_1 = X_0 P = (0\ \ 1\ \ 0)$ $\qquad P(\text{stato } 2 \text{ al tempo } 1) = 1$

$X_2 = X_1 P = \left(\tfrac{1}{2}\ \ 0\ \ \tfrac{1}{2}\right)$ $\qquad P(\ \ "\ \ \ \ "\ \ 2) = 0$

$X_3 = X_2 P = \left(0\ \ \tfrac{2}{3}\ \ \tfrac{1}{3}\right)$ $\qquad P(\ \ "\ \ \ \ "\ \ 3) = \tfrac{2}{3}$

(iii) $X_0 = (1\ \ 0\ \ 0) \qquad (0\ \ 1\ \ 0) \qquad (0\ \ 0\ \ 1)$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$X_2 = \left(\tfrac{1}{2}\ \ \underline{\underline{0}}\ \ \tfrac{1}{2}\right) \qquad \left(0\ \ \underline{\underline{\tfrac{2}{3}}}\ \ \tfrac{1}{3}\right) \qquad \left(\tfrac{1}{6}\ \ \underline{\underline{\tfrac{2}{9}}}\ \ \tfrac{11}{18}\right)$

Maggiore prob.