Lezione 6

Modelli a volatilità locale

$$dS_t = S_t\left(\mu(t,S_t)\,dt + \sigma(t,S_t)\,dW_t\right)$$

$\mathcal{F}^S_t = \mathcal{F}^W_t \rightarrow$ teorema rapp. martingale

completezza del mercato

prob. mart. equivalente

$$dS_t = S_t\left(r\,dt + \sigma(t,S_t)\,dW^*_t\right)$$

$$d\tilde S_t = \tilde S_t\,\sigma(t,S_t)\,dW^*_t$$

come determinare $\sigma(t,x)$?

Dupire (1994)

$C(t,K)$ $\qquad 0 \le t \le T$, $K > 0$

questi prezzi derivano da un modello con

$$\frac{K^2}{2}\sigma^2(t,K) = \frac{\frac{\partial C}{\partial t}(t,K) + rK\frac{\partial C}{\partial K}(t,K)}{\frac{\partial^2 C}{\partial K^2}(t,K)}$$

Problema: $\quad V_t = e^{-r(T-t)}E^*\left[f(S_T)\mid\mathcal{F}_t\right]$?

come calcolare H_t ?

$$dX_t = rX_t\,dt + \sigma(t,X_t)\,dW_t$$

$X_0 = x$

$$A_t = rx\frac{\partial}{\partial x} + \frac{x^2\sigma^2(t,x)}{2}\frac{\partial^2}{\partial x^2}$$

prese u C^1 in t $\quad C^2$ in x

$$e^{-rt}u(t,X_t) - \int_0^t e^{-rs}\left(\frac{\partial u}{\partial s} + A_s u - ru\right)(s,X_s)\,ds$$

è una mart. locale.

$$d\left(e^{-rt}u(t,X_t)\right) = e^{-rt}\left(\frac{\partial u}{\partial t} + A_t u - ru\right)(t,X_t)\,dt$$

$+ \ldots dW_t$

$$d\left(e^{-rt}u(t,X_t)\right) = -re^{-rt}u(t,X_t)\,dt + e^{-rt}\left(\frac{\partial u}{\partial t}(t,X_t) + \frac{\partial u}{\partial x}(t,X_t)\,dX_t\right) + e^{-rt}\underbrace{\frac{\partial^2 u}{\partial x^2}(t,X_t)}_{d[X]_t}$$

Corollario supponiamo che u risolve in senso classico l'equaz.

$$\frac{\partial u}{\partial t} + A_t u - ru = 0$$

allora $\left(e^{-rt}u(t,X_t)\right)$ è una mart. locale

Teorema: supponiamo che esista sol. classica di

$$\begin{cases} \frac{\partial F}{\partial t} + rx\frac{\partial F}{\partial x} + \frac{x^2\sigma^2(t,x)}{2}\frac{\partial^2 F}{\partial x^2} - rF = 0 & 0<t<T,\ x>0 \\ F(T,x) = f(x) \end{cases}$$

supponiamo di aver verificato che $e^{-rt}F(t,S_t)$ è una vera martingala

allora $V_t = F(t,S_t)$ $\quad H_t = \frac{\partial F}{\partial x}(t,S_t)$

$$e^{-rt}V_t = \tilde V_t = E^*\left[\tilde V_T\mid\mathcal{F}_t\right] = E^*\left[e^{-rT}f(S_T)\mid\mathcal{F}_t\right]$$

come in Black-Scholes

Modelli a volatilità stocastica

$$dS_t = S_t\left(\mu(t,S_t)\,dt + \sigma_t\,dW_t\right)$$

σ_t processo adattato $\mathcal{F}_t \supsetneq \mathcal{F}^W_t$ $\quad$ ($W_t - W_s$ indip. da $\mathcal{F}_s$ $s<t$)

$\mathcal{F}^S_t \supseteq \mathcal{F}^W_t$ $\quad$ non vale più teorema rappr. martingale

$$dS_t = S_t\left(\mu(t,S_t)\,dt + \sigma_t\,dW^1_t\right) \qquad r\,dt$$

$$d\sigma_t = \alpha(t,\sigma_t)\,dt + \beta(t,\sigma_t)\,dW^2_t$$

W^1, W^2 indip. $\qquad d[W^1,W^2] = 0$

par. correlato $\qquad d[W^1,W^2] = \rho\,dt \quad |\rho|<1$

W^1 e $\overline{W}$ indip. $\qquad W^2_t = \rho W^1_t + \sqrt{1-\rho^2}\,\overline{W}_t$

modello Stein-Stein

$$\begin{cases} dS_t = S_t\left(\mu\,dt + \sigma_t\,dW^1_t\right) \\ d\sigma_t = -\delta(\sigma_t - \theta)\,dt + k\,dW^2_t \\ d[W^1,W^2] = 0 \end{cases}$$

modello di Heston (C.I.R.)

$$\begin{cases} dS_t = S_t\left(\mu\,dt + \sqrt{v_t}\,dW^1_t\right) \\ dv_t = k(\theta - v_t)\,dt + \sigma\sqrt{v_t}\,dW^2_t \\ d[W^1,W^2] = 0 \end{cases}$$

$$d\sigma_t = \alpha(t,\sigma_t)\,dt + \beta(t,\sigma_t)\left[\rho W^1_t + \sqrt{1-\rho^2}\,W^2_t\right]$$

$$\frac{dP^*}{dP} = \exp\left(\int_0^T K^1_s\,dW^1_s + \int_0^T K^2_s\,dW^2_s - \frac{1}{2}\int_0^T \left((K^1_s)^2 + (K^2_s)^2\right)ds\right)$$

$W^1_t - \int_0^t K^1_s\,ds$, $\quad W^2_t - \int_0^t K^2_s\,ds$ $\quad$ sono Wiener indipendenti

$(K^1_t)_{0\le t\le T}$ vincolato

$(K^2_t)\ldots$ "libero"

non unicità prob. martingala.

$$d\tilde S_t = \sigma(t,S_t)\,\tilde S_t\,dW^1_t$$

Descrizione più realistica del mercato

come si possono fare i conti?

$$d\left[W^1_t, W^2_t\right] = 0$$

$\Omega = \Omega_1 \times \Omega_2 \qquad W^1_t(\omega_1) \qquad W^2_t(\omega_2)$

$P = P^1 \otimes P^2$

$$dS_t(\omega_1,\omega_2) = S_t(\omega_1,\omega_2)\left(r\,dt + \sigma_t(\omega_2)\,dW^1_t(\omega_1)\right)$$

$$E^*\left[e^{-rT}(S_T - K)^+\right] = \int_{\Omega_2} dP^*_2(\omega_2)\underbrace{\int_{\Omega_1}\left(S_T(\omega_1,\omega_2) - K\right)^+ e^{-rT}\,dP^*_1(\omega_1)}$$

fissato ω_2 — modello lognormale

$$C_0(S_0,K)(\omega_2) = S_0\,\Phi\left(d_1(\omega_2)\right) - Ke^{-rT}\Phi\left(d_2(\omega_2)\right)$$

$$d_{1,2}(\omega_2) = \frac{\log\left(\frac{S_0}{K}\right) + \int_0^T\left(r \pm \frac{\sigma^2_s(\omega_2)}{2}\right)ds}{\sqrt{\int_0^T \sigma^2_s(\omega_2)\,ds}}$$

$$C_0(S_0,K) = S_0\int_{\Omega_2}\Phi\left(d_1(\omega_2)\right)dP^*_2(\omega_2) - Ke^{-rT}\int_{\Omega_2}\Phi\left(d_2(\omega_2)\right)dP^*_2(\omega_2)$$