Lezione 3-11

Brevi richiami nell'integrale di Itô

Modello a tempi continui

$\mathcal{T} = [0,T]$ $\quad S^0_t = e^{rt}$ $\quad S_t \leftarrow$ proc. stocastico

Filtrazione $(\mathcal{F}_t)_{0 \le t \le T}$ su $(\Omega, \mathcal{F}, P)$

Bachelier

$dS_t = \mu S_t \, dt$ + disturbo casuale

$S_t = S_0 e^{\mu t} + \sigma W_t$ — processo di Wiener (moto Browniano)

$W_0 = 0$

$s < t$, $(W_t - W_s)$ indip. da $\mathcal{F}_s$

$(W_t - W_s) \sim N(0, t-s)$.

le traiettorie sono continue

traiettoria (su ω) $\quad t \longmapsto X(t,\omega)$ $\quad (X_t)_{0 \le t \le T}$ proc. stocastico

W_t processo di Wiener rispetto a $(\mathcal{F}_t)$

W_t adattato, W_t è $\mathcal{F}_t$-mis.

$\mathcal{F}^W_t = \sigma(W_s, s \le t)$ $\quad \mathcal{F}_t \supseteq \mathcal{F}^W_t$

Samuelson (Black-Scholes)

$dS_t = S_t(\mu \, dt + \sigma \, dW_t)$ — σ volatilità

→ integrale stocastico di Itô

$(W_t)_{0 \le t \le T}$ $\quad (\mathcal{F}_t)_{0 \le t \le T}$

$$\int_0^t H_s \, dW_s = \boxed{\lim \sum H_i (W_{t_{i+1}} - W_{t_i})}$$

(H_s) progressivamente misurabile (adattato)

$$m^2 = m^2_{W,[0,T]} = \left\{ (H_s) \text{ prog. mis.} \;\middle|\; E\left[\int_0^T H_s^2 \, ds\right] < +\infty \right\}$$

$$\iint_{\Omega \times [0,T]} H^2(s,\omega) \, ds \, dP(\omega)$$

è definito $\left(\int_0^t H_s \, dW_s\right)_{0 \le t \le T}$ → martingala q. ovunque traiettorie continue

isometria

$$E\left[\left(\int_0^T H_s \, dW_s\right)^2\right] = E\left[\int_0^T H_s^2 \, ds\right]$$

$$\Lambda^2 = \left\{ H \;\middle|\; \int_0^T H^2(s,\omega) \, ds < +\infty \text{ q.c.} \right\}$$

è definito $\left(\int_0^t H_s \, dW_s\right)$ è una martingala locale

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$$\Lambda^p = \left\{ H \;\middle|\; \int_0^T |H(s,\omega)|^p \, ds < +\infty \text{ q.c.} \right\}$$

$\Lambda^1 \quad \Lambda^2$

$H_s \in \Lambda^2$, $K_s \in \Lambda^1$

$$\underbrace{\int_0^t H_s \, dW_s}_{\text{"Itô"}} + \underbrace{\int_0^t K_s \, ds}$$ ← traiett. continue

$$\int_0^t K(s,\omega) \, ds$$

$$X_t = X_0 + \int_0^t H_s \, dW_s + \int_0^t K_s \, ds$$

$H \in \Lambda^2$ $\quad K \in \Lambda^1$ $\quad$ processo di Itô

$dX_t = H_t \, dW_t + K_t \, dt$

$$\int_0^t L_s \, dX_s = \int_0^t L_s H_s \, dW_s + \int_0^t L_s K_s \, ds$$

Variazione quadratica

$$[X]_t = \lim \sum (X_{t_{i+1}} - X_{t_i})^2 = \int_0^t H_s^2 \, ds$$

$(dW_t)^2 = dt$ $\quad dW_t \cdot dt = 0$ $\quad dt \, dt = 0$

$$d[X]_t = (dX_t)^2 = (H_t \, dW_t + K_t \, dt)^2 = H_t^2 \, dt$$

Formula di Itô

F classe C^2 $\quad X$ processo di Itô

$$dF(X_t) = F'(X_t) \, dX_t + \frac{1}{2} F''(X_t) (dX_t)^2$$

$$F(X_t) = F(X_0) + \int_0^t F'(X_s) \, dX_s + \frac{1}{2} \int_0^t F''(X_s) \, d[X]_s$$

$$\int_0^t F''(X_s) H_s^2 \, ds$$

fissato ω

$t \mapsto F'(X_s(\omega))$

è unif. limitato su $[0,T]$

$$[X,Y]_t = \frac{[X+Y]_t - [X]_t - [Y]_t}{2}$$

$dX_t = H^1_t \, dW_t + K^1_t \, dt$

$dY_t = H^2_t \, dW_t + K^2_t \, dt$

$$d[X,Y]_t = (dX_t)(dY_t) = H^1_t H^2_t \, dt$$

$$dF(X^1_t, \dots, X^n_t) = \sum_{i=1}^n \frac{\partial F}{\partial x_i}(\dots) \, dX^i_t + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 F}{\partial x_i \partial x_j}(\dots)(dX^i_t)(dX^j_t)$$

$F(x,y) = xy$

$$d(X_t Y_t) = Y_t \, dX_t + X_t \, dY_t + d[X,Y]_t$$

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s \, dY_s + \int_0^t Y_s \, dX_s + [X,Y]_t$$