Lezione 10

Modello basato sui tassi a breve

$$\begin{cases} dr(t) = \mu(t, r(t))\,dt + \sigma(t, r(t))\,dW_t \\ r(0) = r^*(0) \end{cases}$$

sotto prob. martingala

$$B(t,T) = E\left[e^{-\int_t^T r(s)\,ds} \,\middle|\, \mathcal{F}_t\right]$$

Vasicek $\quad dr = (b - ar)\,dt + \sigma\,dW_t$

C.I.R. $\quad dr = a(b - r)\,dt + \sigma\sqrt{r}\,dW_t$

Vasicek

$$r(t) = \frac{b}{a} + \left(r^*(0) - \frac{b}{a}\right)e^{-at} + \sigma\int_0^t e^{-a(t-s)}\,dW_s$$

$$E[r(t)] \longmapsto \frac{b}{a}$$

$$E\left[e^{-\int_t^T r(s)\,ds} \,\middle|\, \mathcal{F}_t\right] ? \qquad E\left[\frac{f(B(S,T))}{e^{\int_t^S r(u)\,du}} \,\middle|\, \mathcal{F}_t\right] ?$$

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equazione deriv. parziali – $u(t,x)$ $\begin{smallmatrix} C^1 \text{ in } t \\ C^2 \text{ in } x \end{smallmatrix}$

$$\boxed{e^{-\int_0^t r(s)\,ds}\,u(t, r(t))} - \int_0^t e^{-\int_0^s r(v)\,dv} \Bigg($$

$$\underbrace{\left(-ru + \frac{\partial u}{\partial t} + \mu\frac{\partial u}{\partial r} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial r^2}\right)}_{=0}(s, r(s))\,ds$$

— mart. locale

$$\begin{cases} -ru + \dfrac{\partial u}{\partial t} + \mu\dfrac{\partial u}{\partial r} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 u}{\partial r^2} = 0 & 0 < t < T \\ & r \in \mathbb{R} \\ u(T, r) = f(r) \end{cases}$$

↑ Term structure equation

$$E\left[e^{-\int_0^t r(s)\,ds} \cdot f(r(T)) \,\middle|\, \mathcal{F}_t\right] = u(t, r(t))$$

$$\boxed{B(t,T) = F^T(t, r(t))}$$

$$\begin{cases} -rF^T + \dots & 0 < t < T \\ & r \in \mathbb{R} \\ F^T(T, r) = 1 \end{cases}$$

$0 \leq t \leq S < T$

$$E\left[e^{-\int_t^S r(u)\,du}\,\underbrace{f(B(S,T))} \,\middle|\, \mathcal{F}_t\right] = G(t, r(t))$$

$$\begin{cases} -rG + \dots = 0 & 0 < t < S \\ & r \in \mathbb{R}^2 \\ G(S, r) = f(F^T(S, r)) \end{cases}$$

nuove E.D.P.

Struttura a termine affine (A.T.S.)

(Duffie-Kan 1996)

$$\boxed{B(t,T) = \exp\big(a(t,T) - b(t,T)\,r(t)\big)} \quad \text{A.T.S.}$$

Teorema D.K.

$$\mu(t, r) = \alpha(t)\,r + \beta(t)$$

$$\sigma(t, r) = \sqrt{\gamma(t)\,r + \delta(t)}$$

il modello ha A.T.S.

$$\begin{cases} b_t(t,T) + \alpha(t)\,b(t,T) - \frac{1}{2}\gamma(t)\,b^2(t,T) = -1 \\ b(T,T) = 0 \end{cases}$$

$$\begin{cases} a_t(t,T) = \beta(t)\,b(t,T) - \frac{1}{2}\delta(t)\,b^2(t,T) \\ a(T,T) = 0 \end{cases}$$

Principio dimostrazione

$B(t,T) = F^T(t, r(t))$ $\qquad$ F^T risolve term struct. equation

$$F^T(t, r) = \exp\big(a(t,T) - b(t,T)\,r\big)$$

$$\exp(\dots)\Big[\underset{\overset{\shortparallel}{0}}{(\dots)} + r\,\underset{\overset{\shortparallel}{0}}{(\dots)}\Big] = 0 \quad \forall t, \forall r$$

$$B(t,T) = \exp\big(a(t,T) - b(t,T)\,r(t)\big)$$

$$dB(t,T) = B(t,T)\Big(\underset{\overset{\shortparallel}{r(t)}}{(\dots)}\,dt - \underline{b(t,T)\,\sigma(t, r(t))\,dW_t}\Big)$$

Vasicek

$$\begin{cases} b_t(t,T) - a\,b(t,T) = -1 \\ b(T,T) = 0 \end{cases} \qquad b(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right)$$

formule per a...

$$a(t,T) = \frac{\sigma^2}{2}\int_t^T b^2(s,T)\,ds - b\int_t^T b(s,T)\,ds$$

C.I.R.

$$\begin{cases} b_t(t,T) - a\,b(t,T) - \frac{\sigma^2}{2}b^2(t,T) = -1 \\ b(T,T) = 0 \end{cases}$$

equaz. di Riccati

$$\gamma = \frac{\sqrt{a^2 + \sigma^2}}{2} \qquad b(t,T) = \frac{\sinh(\gamma(T-t))}{\gamma\cosh(\gamma(T-t)) + \frac{a}{2}\sinh(\gamma(T-t))}$$

$$\begin{cases} a_t(t,T) = b\;b(t,T) \\ a(T,T) = 0 \end{cases}$$

$$a(t,T) = \frac{2b}{\sigma^2}\log\left\{\frac{\gamma e^{-a(T-t)^2}}{\gamma\cosh\dots}\right\}$$

$$E\left[e^{-\int_t^S r(u)\,du}\,f(B(S,T)) \,\middle|\, \mathcal{F}_t\right] ? \quad t < S < T$$

↳ equaz. derivate parziali

Completezza del modello

$$d\tilde{B}(t,T) = \tilde{B}(t,T)\big(-b(t,T)\,\sigma(t, r(t))\,dW_t\big)$$

$$\mathcal{F}_t^W = \mathcal{F}_t^{\tilde{B}}$$

Vasicek

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opzione call al tempo S su un bond $B(S,T)$

$B(S,T) = F^T(S, r(S))$ ....

~~coprire~~ replicare con un portafoglio basato su → B_t ← idealizzazione

→ un bond di qualsiasi scadenza posteriore a S.

Inversione della curva dei rendimenti

adattare (calibrare) il modello ai prezzi osservati.

$B^*(0,T)$ $\qquad 0 \leq T \leq T^*$

$f^*(0,T)$ $\qquad r^*(0) = f^*(0,0)$

modello $\qquad B(0,T) = F^T(0, r^*(0))$

↳ dipende da altri ~~parametri~~ parametri

$$B(0,T) \cong B^*(0,T) \quad ?$$

↗ modello $\qquad$ ↑ mercato

Vasicek $\quad$ C.I.R.

Modello Ho-Lee

$$dr(t) = \theta(t)\,dt + \sigma\,dW_t$$

$$r(0) = r^*(0)$$

$$\begin{cases} b_t(t,T) = -1 \\ b(T,T) = 0 \end{cases} \qquad b(t,T) = T - t$$

$$a(t,T) = \int_0^t \theta(s)(s - T)\,ds + \frac{\sigma^2}{2}\frac{(T-t)^3}{3}$$

$$a(0,T) - b(0,T)\,r^*(0) = \log B^*(0,T)$$

$$a_T(0,T) - b_T(0,T)\,r^*(0) = -f^*(0,T)$$

trovare $\theta(s)$ tale che

$$\int_0^T \theta(s)\,ds = f^*(0,T) + \frac{\sigma^2 T^2}{2} - f^*(0,0)$$

Hull-White

$$dr(t) = (\theta(t) - ar)\,dt + \sigma\,dW_t$$

$$\begin{cases} b_t(t,T) = a\,b(t,T) - 1 \\ b(T,T) = 0 \end{cases}$$

$$\begin{cases} a_t(t,T) = \theta(t)\,b(t,T) - \frac{\sigma^2}{2}b^2(t,T) \\ a(T,T) = 0 \end{cases}$$

Si può fare inversione curva rendimenti

$f^*_T(0,T)$ cioè $T \mapsto B^*(0,T)$ deriv. 2 volte
