Ist. Mat. I - CIA

22/2/23

$$\frac{\deg \leq 1}{\deg = 2}$$

$\Delta > 0$ $$\frac{ax+b}{(cx+d)(ex+f)} = \underbrace{\frac{\alpha}{cx+d}}_{1} + \underbrace{\frac{\beta}{ex+f}}_{1}$$

$\Delta = 0$ $$\frac{ax+b}{(cx+d)^2} = \underbrace{\frac{\alpha}{cx+d}}_{1} + \underbrace{\frac{\beta}{(cx+d)^2}}_{2}$$

$\Delta < 0$ $$\frac{ax+b}{(cx+d)^2+e^2} = \underbrace{\frac{\alpha \cdot (cx+d)}{(cx+d)^2+e^2}}_{3} + \underbrace{\frac{\beta}{(cx+d)^2+e^2}}_{4}$$

1. $$\int \frac{1}{x+\gamma} = \log |x+\gamma|$$

2. $$\int \frac{1}{(x+\gamma)^2} = -\frac{1}{x+\gamma}$$

3. $$\int \frac{cx+d}{(cx+d)^2+e^2} = \frac{1}{2} \log ((cx+d)^2+e^2)$$

4. $$\int \frac{1}{(\gamma x+\delta)^2+1} = \frac{1}{\gamma} \arctan (\gamma x+\delta)$$

$$\frac{*}{\deg > 2} \rightsquigarrow \frac{f(x)}{g(x)} \qquad \deg(f(x)) < \deg(g(x))$$

$$g(x) = g_1(x) \cdots g_m(x) \qquad \deg(g_j(x)) \leq 2$$

$$\frac{f(x)}{g(x)} = \frac{f_1(x)}{g_1(x)} + \cdots + \frac{f_m(x)}{g_m(x)} \qquad \deg(f_j(x)) < \deg(g_j(x))$$

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Integrazione per parti

- $\int f(x) \cdot g'(x)\, dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\, dx$
- $\int_a^b f(x) \cdot g'(x)\, dx = f(x) \cdot g(x) \Big|_a^b - \int_a^b f'(x) \cdot g(x)\, dx$

Spiegazione $\quad (f \cdot g)' = f' \cdot g + f \cdot g'$

$$\Rightarrow f \cdot g = \int f' g + \int f \cdot g'$$

$$\Rightarrow \int f \cdot g' = f \cdot g - \int f' \cdot g$$

non conviene

Es: $\int x \cdot e^{2x}\, dx = \int \left(\frac{1}{2}x^2\right)' \cdot e^{2x} = \frac{1}{2}x^2 \cdot e^{2x} - \int \frac{1}{2}x^2 \cdot 2e^{2x}$

$$\int x \cdot \left(\frac{1}{2}e^{2x}\right)' = x \cdot \frac{1}{2}e^{2x} - \int 1 \cdot \frac{1}{2}e^{2x}$$

$$= x \cdot \frac{1}{2}e^{2x} - \frac{1}{4}e^{2x} + c$$

Es: $\int \sin(x)\cdot\cos(x)\,dx = \int \sin(x)\cdot(\sin(x))'\,dx$

$$= \sin^2(x) - \int \cos(x)\cdot\sin(x)\,dx$$

$$\Rightarrow \int \sin(x)\cdot\cos(x)\,dx = \boxed{\frac{1}{2}\sin^2(x) + c}$$

$$\left(\int \sin(x)\cos(x)\,dx = \frac{1}{2}\int \sin(2x)\,dx = \boxed{-\frac{1}{4}\cos(2x) + c} \right.$$

$$\left. \cos(2x) = 1 - 2\sin^2(x) \right)$$

Sostituzione

Prop: se esiste $f\circ g$ e $\int f = F$

allora $\int f(g(x))\cdot g'(x)\,dx = F(g(x)) + c$

Spiegazione: vale $* = * \iff *' = *'$

$$f(g(x))\cdot g'(x) = \underbrace{F'}_{f}(g(x))\cdot g'(x)$$

I uso: vedo f, g, g', calcolo $F = \int$ quindi uso la formula sostituendo I membro con II

Es: $\int \underbrace{\cos}_{f}(\underbrace{x^2}_{g(x)})\cdot\underbrace{2x}_{g'(x)}\,dx = \sin(x^2) + c$

II uso:

$$\int f(g(x))\cdot g'(x)\,dx = F(g(x)) + c$$

Al posto di x scrivo y e pongo $g(y) = x$ e riscrivo

$$\int f(g(y))\cdot g'(y)\,dy = F(x) + c$$

$$\int f(x)\,dx = \int f(g(y))\cdot g'(y)\,dy$$

dove $x = g(y)$

Uso pratico:

- $\int f(x)\,dx$
- si pone $x = g(y)$ con g sia bigettiva tra opportuni intervalli
- si sostituisce $\int f(x)\,dx = \int f(g(y))\cdot g'(y)\,dy$
- si integra (se ci si riesce)
- sostituire $y = g^{-1}(x)$

<u>Es</u>: $\int \frac{x-1}{x+1}\,dx$ pongo $y = x+1$ cioè $x = y-1$

$$= \int \frac{y-2}{y}\cdot 1\cdot dy = \int\left(1-\frac{2}{y}\right)dy = y - 2\log|y|$$

$= x \cancel{+1} - 2\log|x+1| + c$

Es: $\int \sqrt{1-x}\,dx$

pongo $1-x = y^2$
cioè $x = 1-y^2$, $y = \sqrt{1-x}$
$dx = -2y\,dy$

$\parallel$

$$\int y\cdot(-2y\,dy) = -2\int y^2 dy = -\frac{2}{3}y^3 + c = -\frac{2}{3}(1-x)^{3/2} + c$$

Es: $\int \sqrt{1-x^2}\,dx$

$x = \sin(y)$ $\quad y = \arcsin(x)$
$dx = \cos(y)dy$

$\parallel$

$\int \cos^2(y)\,dy = $ ... per parti ... ... $\Big|_{y=\arcsin(x)}$
... $\cos(2y)$ ...

<u>Sostituzione per $\int_a^b$</u>

Se $f:[a,b]\to\mathbb{R}$ $\quad g:[c,d]\to\mathbb{R}$ t.c.
$f\circ g$ ha senso e $g(\{c,d\}) = \{a,b\}$ allora

$$\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))\cdot g'(t)\,dt$$

dove $\int_d^c = -\int_c^d$

Spiegazione: $\int f = F$ $\qquad \int_a^b f(x)\,dx = F(b) - F(a)$

inoltre $\int (f\circ g)\cdot g' = F\circ g$

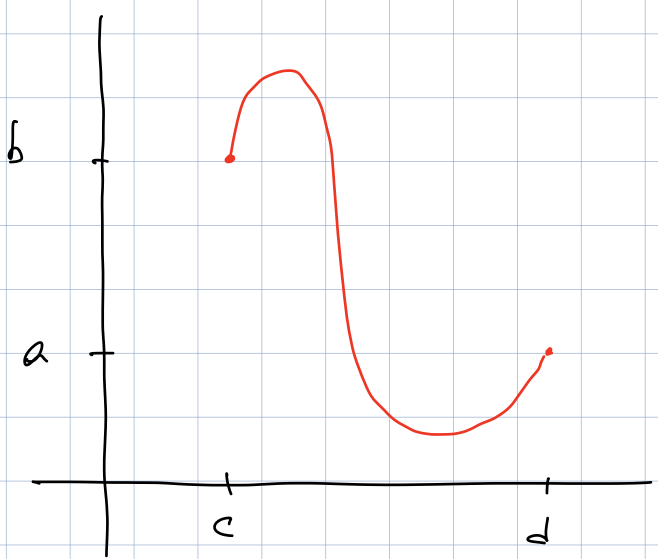
$g(c) = a$
$g(d) = b$

$$\int_c^d (f\circ g)\cdot g' = (F\circ g)\Big|_c^d = F(b) - F(a)$$

$g(c) = b$
$g(d) = a$

$$-\int_c^d (f\circ g)\cdot g' = -(F\circ g)\Big|_c^d = -\big(F(a) - F(b)\big) = F(b) - F(a)$$

Oss: le ipotesi consentono:



In pratica si usa solo con $g : [c,d] \to [a,b]$ bigettiva (crescente o decrescente)

I uso: riconosco $\int_c^d f(g(x))\cdot g'(x)\,dx$

Es: $\displaystyle\int_{-2}^{\sqrt{3}} \boxed{2x\cdot e^{x^2}}\,dx = \int_{-2}^{\sqrt{3}} \underset{f}{\underset{\shortparallel}{\exp}}\,(\underset{g(x)}{\underset{\shortparallel}{x^2}})\cdot \underset{g'(x)}{\underset{\shortparallel}{2x}}\,dx = \dots$

$g(-2)=4 \qquad g(\sqrt{3})=3 \qquad [a,b]=[3,4]$

ma applico con $-$

$\displaystyle \dots = -\int_3^4 e^y\,dy = -(e^4-e^3) = e^3-e^4$

$D\left(e^{x^2}\right) \qquad \displaystyle\int_{-2}^{\sqrt{3}} = e^{x^2}\Big|_{-2}^{\sqrt{3}}$

II modo: • Parto da $\displaystyle\int_a^b f(x)\,dx$

• Sostituisco $x=g(t)$ in modo che $\left(g:[c,d]\to[a,b]\right)$ bijetiva

$dx \longrightarrow g'(t)\,dt$

$a \longrightarrow g^{-1}(a)$

$b \longrightarrow g^{-1}(b)$

Es: $\displaystyle\int_{-1}^{0} \sin\left(\sqrt{x+1}\right)dx = \dots$ pongo $t=\sqrt{x+1}$ $\qquad x=t^2-1$

$dx = 2t\,dt$

$x=-1 \longrightarrow t=0$

$x=0 \longrightarrow t=1$

$$\dots = \int_0^1 \sin(t)\cdot 2t\,dt = -\cos(t)\cdot 2t\Big|_0^1 + \int_0^1 \cos(t)\cdot 2\,dt = \dots$$

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Spesso quando l'integranda coinvolge sin/cos si risolve ponendo $t = \tan(x/2)$ cioè

$x = 2\arctan(t) \Rightarrow dx = \frac{2}{t^2+1}\,dt$ e

$$\cos(t) = \frac{1-t^2}{1+t^2} \qquad \sin(t) = \frac{2t}{1+t^2}$$

Formule parametriche razionali:

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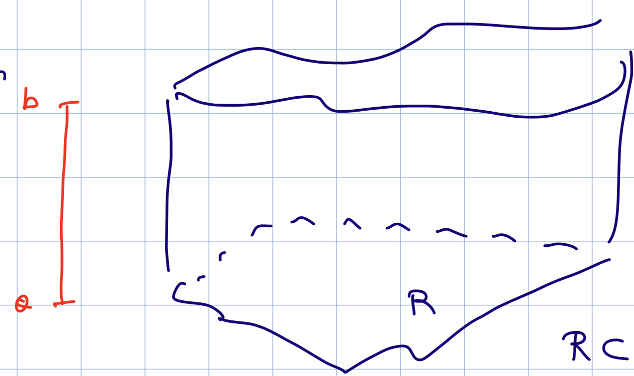
"Principio di Cavalieri"

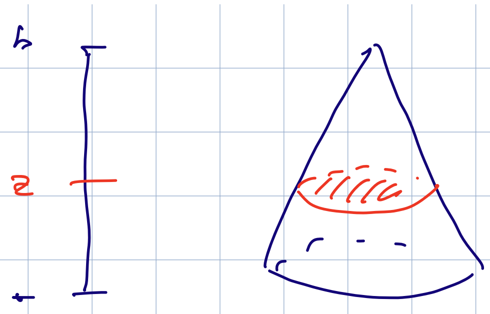


$R \subset \mathbb{R}^2$

$R \times [a,b]$

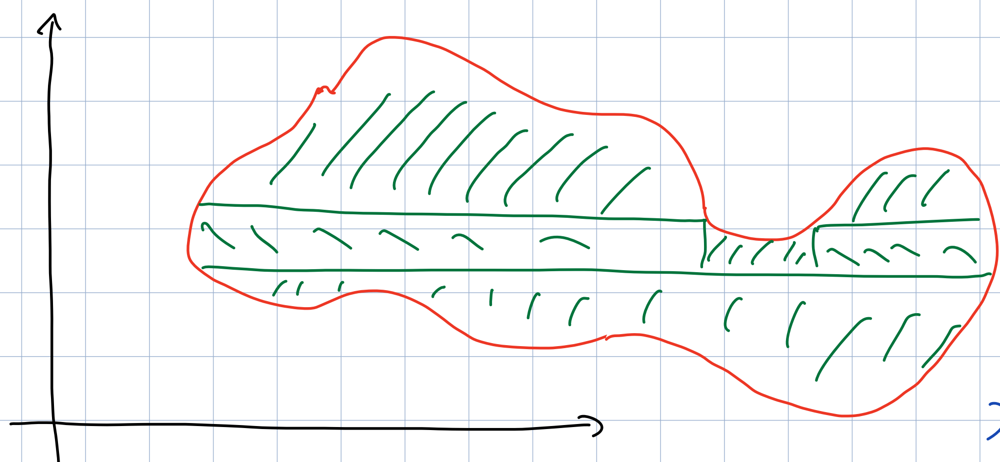
$$\Rightarrow \mathrm{vol}(S) = \mathrm{Area}(R)\cdot(b-a)$$

<u>Fatto</u>: se $S = \{(x,y,z) : z \in [a,b],\ (x,y) \in R(z)\}$



$$\Rightarrow \mathrm{vol}(S) = \int_a^b A(R(z))\,dz \qquad \text{(in ipotesi ragionevoli)}$$

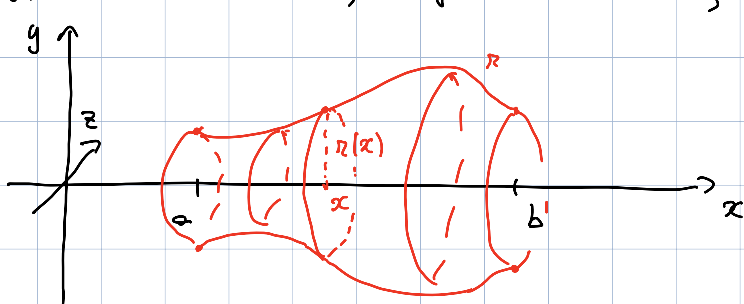
($A(R)$ = ottenute scomponendo in sottografici :



)

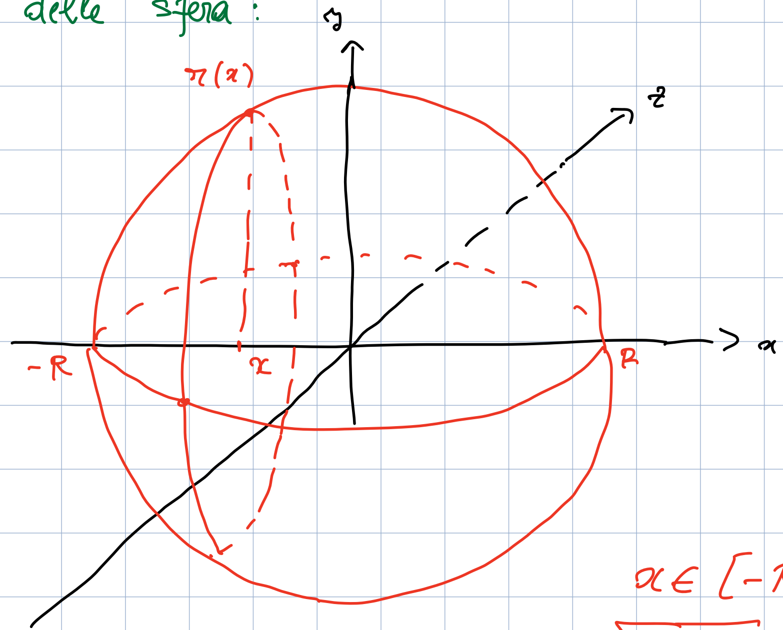
Solido di rotazione: $r : [a,b] \to \mathbb{R}$, $\quad r(x) > 0 \quad \forall x$

$$S = \left\{ (x,y,z) : \ x \in [a,b], \ \sqrt{y^2+z^2} \le r(x) \right\}$$



Da sopra: $\mathrm{Vol}(S) = \int_a^b \pi \cdot r(x)^2 \, dx$

Es: volume della sfera:



$x \in [-R, R]$

$r(x) = \sqrt{R^2 - x^2}$

$$\mathrm{Vol}(S_R) = \int_{-R}^{R} \pi \cdot (R^2 - x^2)\, dx = \pi \left(R^2 \cdot x - \frac{1}{3} x^3 \right) \Big|_{-R}^{R}$$

$$= \pi \cdot \left(R^3 - \frac{1}{3} R^3 - \left(-R^3 + \frac{1}{3} R^3 \right) \right) = \frac{4}{3} \pi R^3 .$$

Integrali impropri.

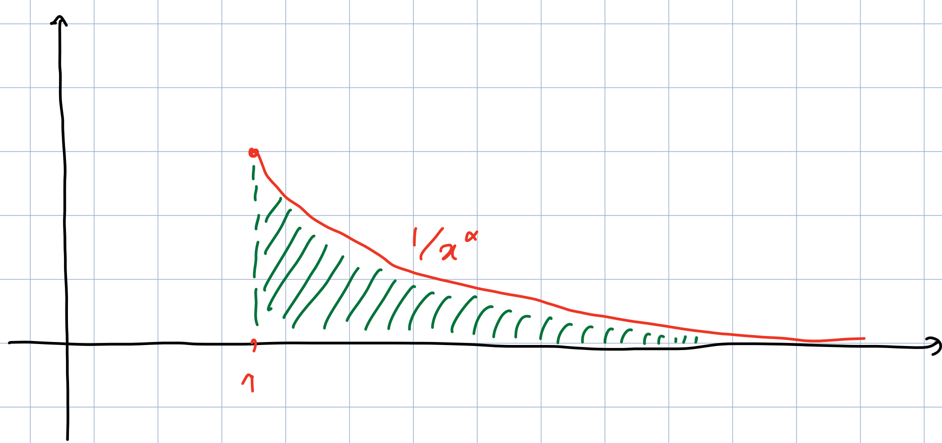
$f : [a, b) \to \mathbb{R} \qquad b > a \qquad (b = +\infty)$

Def: $\displaystyle\int_a^b f(x)\, dx = \lim_{c \to b^-} \int_a^c f(x)\, dx$ se esiste

$f : (a, b] \to \mathbb{R} \qquad a < b \qquad (a = -\infty)$

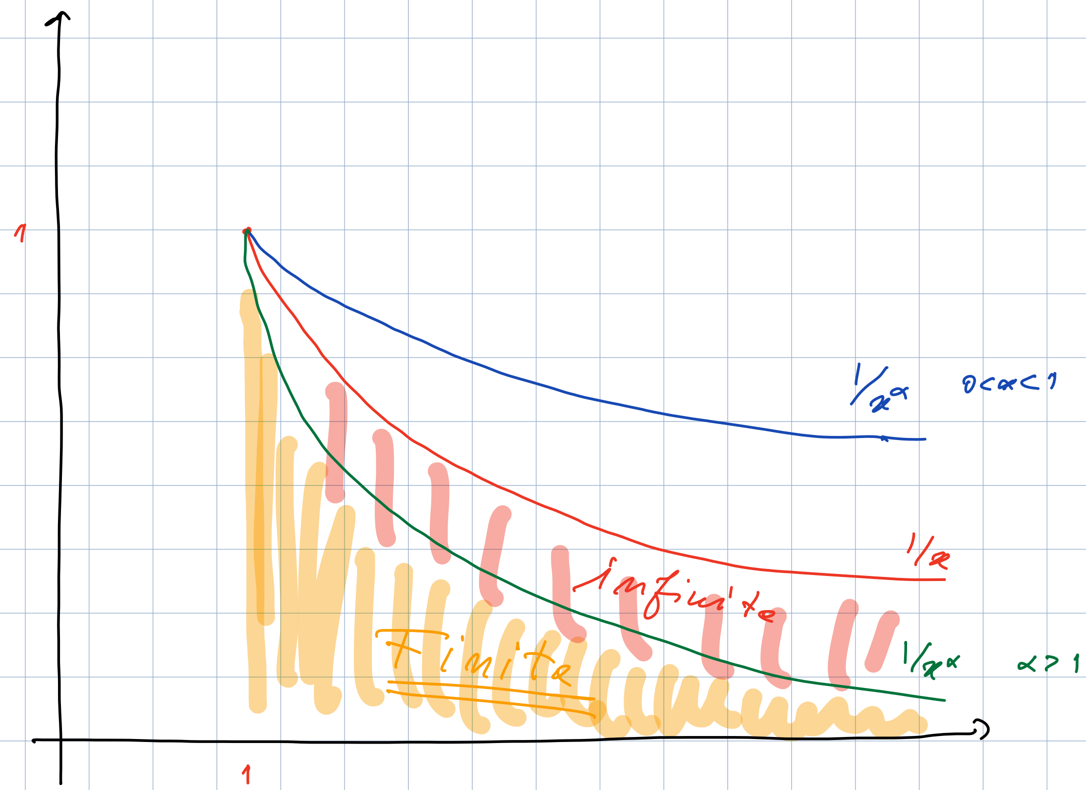
$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$$

<u>Es</u>: $\displaystyle\int_1^\infty \frac{1}{x^\alpha}\,dx \qquad \alpha > 0$



$$\int_1^\infty \frac{1}{x^\alpha}\,dx = \lim_{b \to +\infty} \int_1^b \frac{1}{x^\alpha}\,dx = \lim_{b \to \infty} \begin{cases} \log(b) & \alpha = 1 \\ \frac{1}{1-\alpha} x^{1-\alpha} \Big|_1^b & \alpha \neq 1 \end{cases}$$

$$= \lim_{b \to \infty} \begin{cases} \log(b) & \alpha = 1 \\ \frac{1}{1-\alpha}\left(b^{1-\alpha} - 1\right) & \alpha \neq 1 \end{cases} = \begin{cases} +\infty & \alpha = 1 \\ \frac{1}{\alpha - 1} & \alpha > 1 \\ +\infty & 0 < \alpha < 1 \end{cases}$$

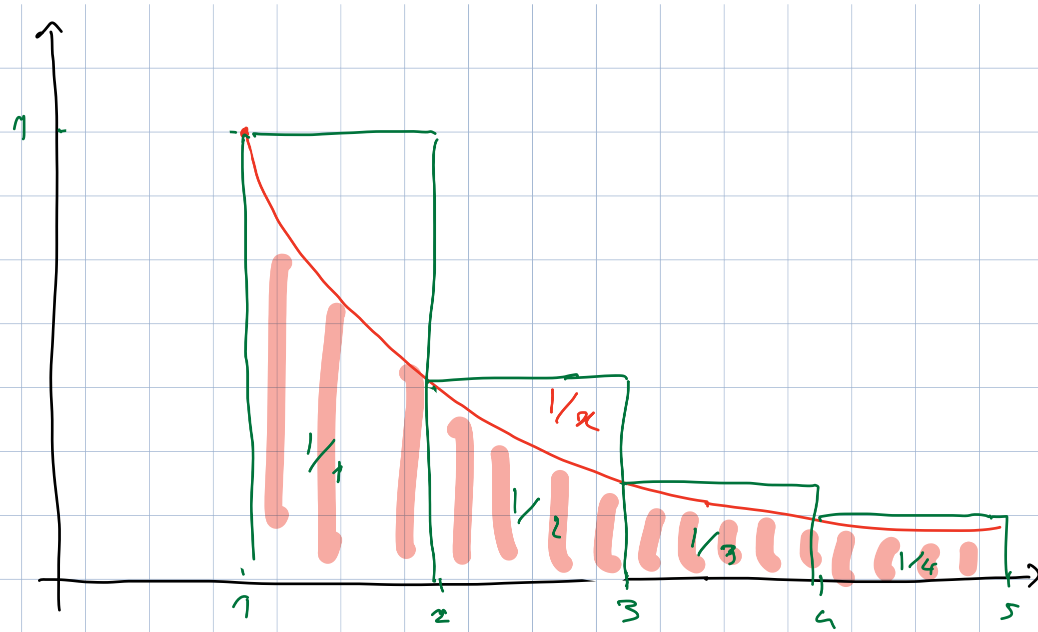


Esercizio: $\displaystyle\int_0^1 \frac{1}{x^\alpha}\,dx = \begin{cases} +\infty & \alpha \geq 1 \\ \dfrac{1}{1-\alpha} & 0<\alpha<1 \end{cases}$

Fatto: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^\alpha} = \begin{cases} +\infty & 0<\alpha\leq 1 \\ \text{finita} & \alpha>1 \end{cases}$

Idea: confrontare $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ con $\displaystyle\int_1^{\infty} \frac{1}{x^\alpha}\,dx$

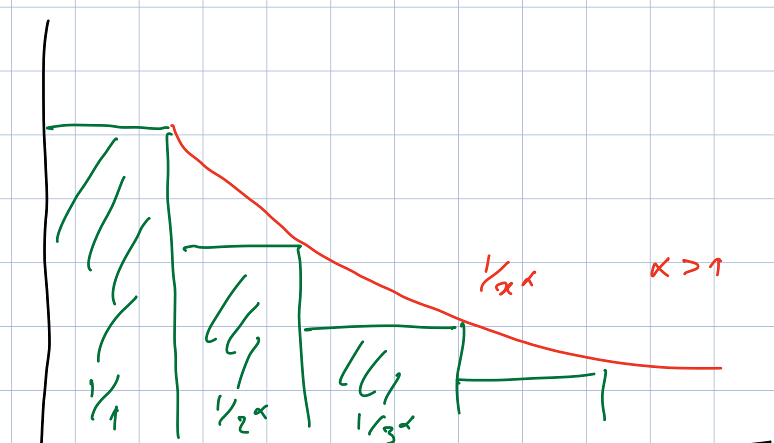
$\alpha = 1$:



$$\sum_{n=1}^{M} \frac{1}{n} > \int_{1}^{n} \frac{1}{x}\,dx \xrightarrow[M\to+\infty]{} +\infty$$

$\alpha < 1$ stesso

$\alpha > 1$



$$\int_{1}^{M} \frac{1}{x^\alpha}\,dx > \sum_{n=2}^{\infty} \frac{1}{n^\alpha}$$

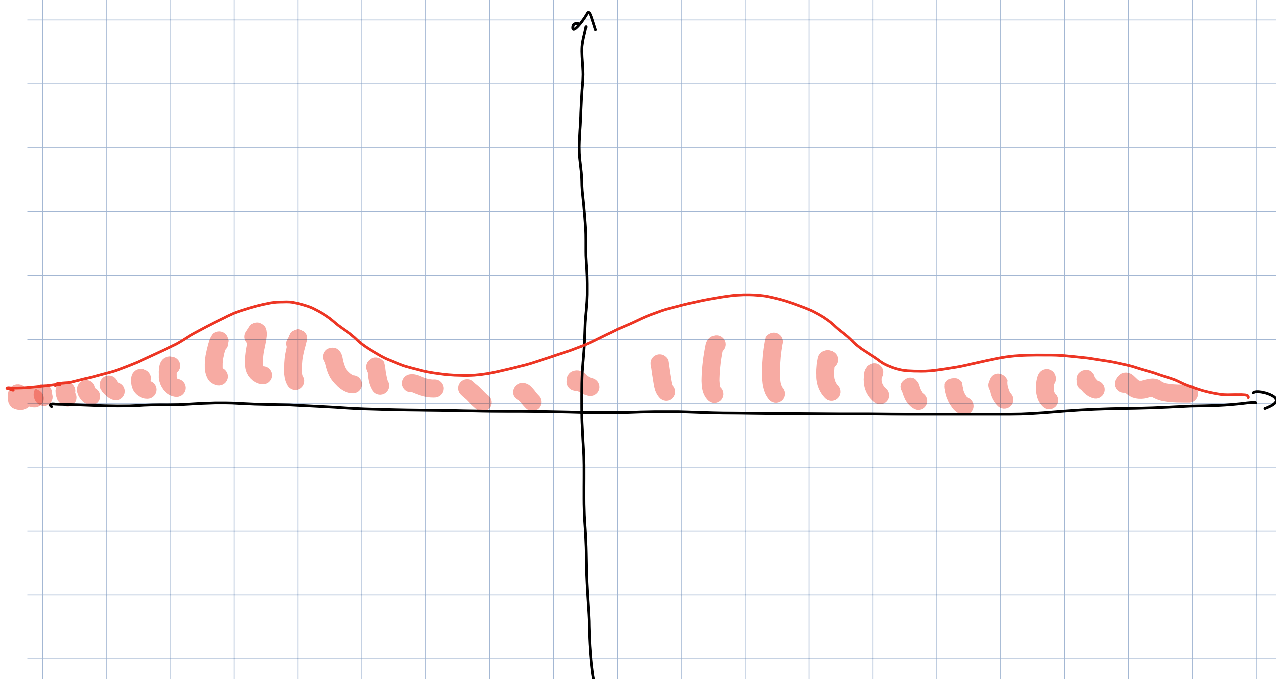
$\downarrow$

finito

Att: se $f:(a,b)\to\mathbb{R}$ $\quad \int_a^b f(x)\,dx$

esiste (per def) se esistono

$$\int_a^c f(x)\,dx \quad e \quad \int_c^b f(x)\,dx$$



Es: $\displaystyle\int_{-\infty}^{+\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{+\infty} x\,dx$

$$\left.\frac{x^2}{2}\right|_{-\infty}^{0} \qquad \left.\frac{x^2}{2}\right|_{0}^{+\infty}$$

$$\begin{array}{cc} \shortparallel & \\ -\infty & +\infty \end{array}$$

$$\int_{-\infty}^{+\infty} x\,dx = \int_{-a}^{a} x \cdot dx = 0$$