ETA 9/12/14

Omologia a coeff. in un gruppo G (invece che $\mathbb{Z}$) –
(G abeliano)

$$C_n(X,A;G) := C_n(X,A) \otimes G$$ –

Fatto: $\partial_n^G = \partial_n \otimes id_G$ dà una struttura di complesso di catene

$\Longrightarrow H_*(X,A;G)$ –

Fatto: $H_*^G \not\cong H_*^{\mathbb{Z}} \otimes G$.

Es: $X = \mathbb{P}^2(\mathbb{R})$. Una j-cella c_j $j = 0,1,2$.

$\Rightarrow C_0 \cong C_1 \cong C_2 = \mathbb{Z}$

$\partial_1 c_1 = 0$, $\partial_2 c_2 = 2c_1$ $\Rightarrow$ $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}/_2$, $H_2 = 0$.

A coeff in $\mathbb{Z}/2$: $C_0^{\mathbb{Z}/2} \cong C_1^{\mathbb{Z}/2} \cong C_2^{\mathbb{Z}/2} \cong \mathbb{Z}/2$

$\partial_1^\bullet c_1^\bullet = 0$ $\partial_2^\bullet c_2^\bullet = 2c_1^\bullet = 0$ $(\bullet = \mathbb{Z}/2)$

$\Rightarrow H_0^{\mathbb{Z}/2} \cong H_1^{\mathbb{Z}/2} \cong H_2^{\mathbb{Z}/2} \cong \mathbb{Z}/2$.

Esercizio: Calcolare $H_k(\mathbb{P}^n(\mathbb{R}), R)$ $R = \mathbb{Z}/p, \mathbb{Q}$

(Oss: Quando R è un anello comm. (con 1)
C_n^R è in modo nat. un R-modulo e
tutte le mappe sono R-lineari
$\Longrightarrow H_*^R$ è R-modulo.)

(1) $C_j^{\mathbb{Z}/p} \cong \mathbb{Z}/p = \langle c_j \rangle$; $\partial_j c_j = \begin{cases} 0 & j \text{ dispari} \\ 2c_{j-1} & j \text{ pari} \end{cases}$

$p = 2q$:

$$C_*^{\mathbb{Z}/2q}$$

$$\begin{array}{ccccccccccc} 4 & & 3 & & 2 & & 1 & & 0 & & \\ \mathbb{Z}/2q & \xrightarrow{2\cdot} & \mathbb{Z}/2q & \xrightarrow{0} & \mathbb{Z}/2q & \xrightarrow{2\cdot} & \mathbb{Z}/2q & \xrightarrow{0} & \mathbb{Z}/2q & \longrightarrow & 0 \end{array}$$

$$H_k(\mathbb{P}^n(\mathbb{R}); \mathbb{Z}/2q) = \begin{cases} \mathbb{Z}/2q & k=0 \\ \mathbb{Z}/2 & 1 \leq k < n \\ \mathbb{Z}/2q & k=n \text{ dispari} \\ \mathbb{Z}/2 & k=n \text{ pari} \end{cases}$$

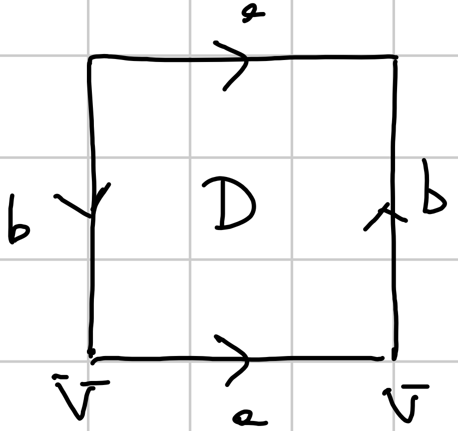
$p = 2q+1$ $\quad (2\cdot)$ è invertibile

$$\Longrightarrow H_k(\mathbb{P}^n(\mathbb{R}); \mathbb{Z}/2q+1) = \begin{cases} \mathbb{Z}/2q+1 & k=0 \\ 0 & 1 \leq k < n \\ 0 & k=n \text{ pari} \\ \mathbb{Z}/2q+1 & k=n \text{ dispari} \end{cases}$$

Analog:

$$H_k(\mathbb{P}^n(\mathbb{R}); \mathbb{Q}) = \begin{cases} \mathbb{Q} & k=0,\ k=n \text{ dispari} \\ 0 & \text{altrimenti} \end{cases}$$

<u>Es</u>: $H_k(\text{Klein}; R)$ $\qquad R = \mathbb{Z}/_p, \mathbb{Q}$



$\partial_1 a = \partial_1 b = 0 \qquad \partial_2 D = 2b$

$\mathbb{Z}/_{2q}$:

$$\begin{array}{ccccccccc} 3 & & 2 & & 1 & & 0 & & 0 \\ 0 & \longrightarrow & \mathbb{Z}/_{2q} & \xrightarrow{1 \mapsto \binom{0}{2}} & (\mathbb{Z}/_{2q})^2 & \xrightarrow{0} & \mathbb{Z}/_{2q} & \longrightarrow & 0 \end{array}$$

$\Rightarrow H_0 = \mathbb{Z}/2q \,,\ H_1 = (\mathbb{Z}/2q) \times \mathbb{Z}/2 \,,\ H_2 = \mathbb{Z}/2$

$\mathbb{Z}/2q+1 \ / \ \mathbb{Q}$ (2·) invertibile :

$H_0 = R \qquad H_1 = R \qquad H_2 = 0$

Assiomi che caratterizzano $H_*(\cdot,\cdot;G)$:

omotopia, escissione, LES :

$$H_k(\{pt\};G) = \begin{cases} G & k=0 \\ 0 & \text{altrimenti} \end{cases}$$

$H_0(X;G) = G$ se X è connesso

(vale Mayer-Vietoris)

Relazione tra H_*^G e $H_* \otimes G$

A, B R-moduli

$$A \otimes_R B = \langle a \otimes b \rangle_R \Big/ \begin{matrix} (r_1 a_1 + r_2 a_2) \otimes b = \dots \\ \dots \end{matrix}$$

Prop: (1) $A \otimes_R B \cong B \otimes_R A$ $\quad (a \otimes b \longmapsto b \otimes a)$

estesa...

(2) $(A_1 \oplus A_2) \otimes B = (A_1 \otimes B) \oplus (A_2 \otimes B)$

$\big((a_1, a_2) \otimes b \longmapsto (a_1 \otimes b, a_2 \otimes b)\big)$

estesa...

(3) $A \otimes_R R \cong A$ $\quad \left(\begin{array}{ccc} a \otimes r & \longmapsto & r \cdot a \\ a \otimes 1 & \longleftarrow & a \end{array} \right)$

(4) $G \otimes (\mathbb{Z}/m) \cong G/m \cdot G$ $\quad$ (5) $\mathbb{Z}/k \otimes \mathbb{Z}/h = \mathbb{Z}/\mathrm{MCD}(h,k)$.

Dim: (4) Definisco $\varphi : G \otimes \mathbb{Z}/m \longrightarrow G/m \cdot G$

estendo $\quad g \otimes [k] \longmapsto [k \cdot g] = k \cdot g + m \cdot G$.

Ben def: $[k'] = [k] \Rightarrow k' = k + p \cdot m$

$$\Rightarrow k'g + m \cdot G = k \cdot g + \underbrace{p \cdot m \cdot g + m \cdot G}_{m \cdot G}.$$

Surj: $[g] = \varphi(g \otimes [1])$.

Iniettiva: sia $\varphi\left(\sum_i g_i \otimes [k_i]\right) = 0$ cioè

$$\sum_i k_i g_i \in m \cdot G \Rightarrow \sum_i k_i g_i = m \cdot \gamma$$

$$\Rightarrow \sum_i g_i \otimes [k_i] = \left(\sum_i k_i g_i\right) \otimes [1] = m \cdot \gamma \otimes [1]$$

$$= \gamma \otimes [m] = \gamma \otimes 0 = 0.$$

(4) $\Rightarrow$ (5) aritmetica facile. $\boxed{}$

Def: se A e B sono gruppi abeliani e

$$0 \longrightarrow K \xrightarrow{i} F \xrightarrow{p} A \longrightarrow 0$$

è una risoluzione libera di A (esatta e F, K sono liberi: realizzano $A \cong F/K$) pongo

$$\mathrm{Tor}(A,B) = \mathrm{Ker}\left(K \otimes B \xrightarrow{i \otimes Id_B} F \otimes B\right).$$

Oss: anche se per ipotesi i è iniettiva, $i \otimes Id_B$ può non esserlo $\Rightarrow Tor(A,B)$ può essere non banale: se $A = B = \mathbb{Z}/p$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i : n \mapsto p \cdot n} \mathbb{Z} \xrightarrow{m \mapsto [m]_p} A = \mathbb{Z}/p \longrightarrow 0$$

$$\begin{array}{ccc}
\mathbb{Z} \otimes B & \xrightarrow{i \otimes Id_B} & \mathbb{Z} \otimes B \\
\shortparallel & & \\
\mathbb{Z} \otimes \mathbb{Z}/p & \xrightarrow{n \otimes [\ell] \mapsto (p\, n) \otimes [\ell]} & \mathbb{Z} \otimes \mathbb{Z}/p \\
\cong & & \cong \\
\mathbb{Z}/p & \xrightarrow{[1] \mapsto [p]} & \mathbb{Z}/p
\end{array}$$

$\Rightarrow i \otimes id_B = 0$

$\Rightarrow \mathrm{Tor}(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p$.

* non è chiaro che $\mathrm{Tor}(A,B)$ sia ben def
perché ho usato $0 \to K \to F \to A \to 0$.

<u>Teo</u>: (1) $\mathrm{Tor}(A,B)$ è ben def.
(2) $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(B,A)$.
(3) $\mathrm{Tor}(A_1 \oplus A_2, B) = \mathrm{Tor}(A_1, B) \oplus \mathrm{Tor}(A_2, B)$

(4) $\mathrm{Tor}(A, \mathbb{Z}) = 0$

(5) $\mathrm{Tor}(A, \mathbb{Z}/p) = \{a \in A : p \cdot a = 0\}$

(6) $\mathrm{Tor}(\mathbb{Z}/p, \mathbb{Z}/q) \cong \mathbb{Z}/_{\mathrm{MCD}(p,q)}$ —

<u>Cor</u>: usando (3), (4), (6) posso calcolare $\mathrm{Tor}(A, B)$ $\forall A, B$ abeliani (fin. gen):

$$\mathrm{Tor}\Big(\underbrace{\mathbb{Z}^k \oplus \bigoplus_{i=1}^{n} \mathbb{Z}/p_i}_{A}, \underbrace{\mathbb{Z}^h \oplus \bigoplus_{j=1}^{m} \mathbb{Z}/q_j}_{B}\Big) =$$

$$= \bigoplus_{i,j=\dots} \mathbb{Z}/_{MCD(p_i, q_j)} .$$

<u>Oss</u>: (4) + (5) dicono che $Tor(A, B)$ = l'insieme degli elementi di torsione comuni.

<u>Lem</u>: se $X \xrightarrow{f} Y \xrightarrow{g} Z$ è esatta e T è libero allora

$$X \otimes T \xrightarrow{f \otimes id_T} Y \otimes T \xrightarrow{g \otimes id_T} Z \otimes T \text{ è esatta.}$$

<u>Dim</u>: $T \cong \mathbb{Z}^t$

$$X \otimes T = X^{\oplus t}, \quad Y \otimes T = Y^{\oplus t}, \quad Z \otimes T = Z^{\oplus t}$$

$$f = (\underbrace{f, \dots, f}_{t \text{ volte}}) \qquad g = (\underbrace{g, \dots, g}_{t \text{ volte}}) \quad . \quad \boxed{}$$

<u>Dim</u>: Da (2) segue (1). Prendo risoluz. libere:

$$0 \longrightarrow K \xrightarrow{i} F \xrightarrow{p} A \longrightarrow 0$$
$$0 \longrightarrow H \xrightarrow{j} G \xrightarrow{q} B \longrightarrow 0$$

Per il <u>Lem</u> la prima resta esatta se tensorizzo con H, G

e le seconde svite esatte se tensorizzo con K, F.

Cerco un isomorfismo:

$$\mathrm{Tor}(A,B) = \mathrm{Ker}(i \otimes id_B) \longrightarrow \mathrm{Ker}(id_A \otimes j) \cong \mathrm{Tor}(B,A)$$

(isomorf canonico indotto da $g \otimes a \longmapsto a \otimes g$)

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K\otimes H \arrow[r, "i\otimes id_H"] \arrow[d, "id_K\otimes j"] & {}_{z}F\otimes H \arrow[r, "p\otimes id_H", "6"'] \arrow[d, "id_F\otimes j", "5"'] & {}_{x}A\otimes H \arrow[r] \arrow[d, "id_A\otimes j\ 7"] & 0 \\
0 \arrow[r] & {}_{v}K\otimes G \arrow[r, "i\otimes id_G", "3"'] \arrow[d, "id_K\otimes q", "2"'] & {}_{w}F\otimes G \arrow[r, "p\otimes id_G", "8"'] \arrow[d, "id_F\otimes q", "4"'] & A\otimes G \arrow[r] \arrow[d, "id_A\otimes q"] & 0 \\
 & {}^{u}K\otimes B \arrow[r, "i\otimes id_B", "1"'] \arrow[d] & F\otimes B \arrow[r, "p\otimes id_B"] \arrow[d] & A\otimes B & \\
 & 0 & 0 & &
\end{tikzcd}
$$

$$u \in \operatorname{Ker}\underbrace{(i\otimes id_B)}_{1} \Longrightarrow u = \underbrace{(id_K\otimes q)}_{2}(v)$$

Ora $\underset{4}{(id_F \otimes q)}\,\underset{3}{(i \otimes id_G)}(v) = \underbrace{i \otimes id_B\big(\underbrace{(id_K \otimes q)(v)}_{u}\big)}_{0}$

$\Rightarrow$ posto $w = (i \otimes id_G)(v)$ esiste z t.c.

$$w = \underset{5}{(id_F \otimes j)}\, z$$

Poniamo $x = (p \otimes id_H)(z) \in A \otimes H$.

Affermo che $x \in \ker \underset{7}{(id_A \otimes j)}$: infatti

$$\underset{7}{(id_A \otimes j)}(x) = (7 \circ 6)(z) = 8 \circ 5\,(z) = 8(w)$$
$$= 8 \circ 3\,(v) = 0$$

Dunque posso definire $u \longmapsto x$
E' omomorfismo ben def. perché l'inverso
$x \mapsto u$ si fa con le stesse doppie intermedie _

(3) facile

(4) $0 \to 0 \xrightarrow{i} \mathbb{Z} \xrightarrow{id} \mathbb{Z} \to 0$ ris. lib. di $\mathbb{Z}$

$$\Rightarrow \mathrm{Tor}(A,\mathbb{Z}) = \mathrm{Ker}\,(\underbrace{0 \otimes A}_{0} \to \mathbb{Z} \otimes A) = 0$$ _

(5) $0 \longrightarrow \mathbb{Z} \xrightarrow{n \mapsto p\cdot n} \mathbb{Z} \longrightarrow \mathbb{Z}/p \longrightarrow 0$

$$\Rightarrow \mathrm{Tor}(A, \mathbb{Z}/p) = \mathrm{Ker}\left(A \otimes \mathbb{Z} \xrightarrow{i \otimes id_{\mathbb{Z}}} A \otimes \mathbb{Z}\right)$$

$$\begin{array}{ccc} A \otimes \mathbb{Z} & \xrightarrow{i \otimes id_{\mathbb{Z}}} & A \otimes \mathbb{Z} \\ \| \wr & & \| \wr \\ A & \xrightarrow{a \mapsto p\cdot a} & A \end{array} .$$

(6) Segue de (5).

$\boxed{\mathbb{V}}$

Def: Una succ. esatta di gruppi abeliani

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

splitta se $\exists\, s : C \to B$ omomorf. sezione, cioè tale che $p \circ s = id_C$.

Oss: ($0 \to A \to B \to C \to 0$ esatta) $\Longleftrightarrow$ $C = B/A$

splitta $\Longrightarrow$ $B \cong A \oplus C$.

Infatti se ho $s : C \to B$ definisco

$$\phi : A \oplus C \longrightarrow B$$
$$(a, c) \longmapsto i(a) + s(c)$$

Omomorfismo: ok (s omomorfismo) _

Iniettiva:

$$i(a) + s(c) = 0 \Rightarrow p(\underbrace{i(a)}_{0} + \underbrace{s(c)}_{c}) = 0 \Rightarrow c = 0$$

$$\Rightarrow i(a) = 0 \Rightarrow a = 0$$ _

Surgettiva: $b \in B$; noto che

$$p(b - s(p(b))) = p(b) - \underbrace{(p \circ s)}_{= id_C}(p(b)) = 0$$

$$\Rightarrow b - s(p(b)) = i(a)$$

$$\Rightarrow b = i(a) + s(p(b)) = \phi(a, p(b))$$

__Prop__: $\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "p"] \arrow[l, dashed, bend left] & C \arrow[r] & 0 \end{tikzcd}$

splitta se e solo se esiste $q: B \to A$ t.c. $q \circ i = id_A$.

Dim: provo che se q esiste allora ho un isomorfismo $\psi : B \longrightarrow A \oplus C$
$b \longmapsto (q(b), p(b))$.

(Esercizio).

$\exists s \Longrightarrow \phi = i + s : A \oplus C \to B$ isomorfismo
$\exists q \Longrightarrow \psi = q + p : B \longrightarrow A \oplus C$ isomorfismo

$s \rightsquigarrow q = \phi^{-1} - p \qquad q \rightsquigarrow \psi^{-1} - i$. $\blacksquare$

<u>Oss</u>: se C è libero allora ogni $0 \to A \to B \to C \to 0$ splitta (basta definire $s : C \to B$ sui generatori).

<u>Teo</u>: se $(C_n, \partial_n)_{n=0}^{+\infty}$ è complesso con omologia H_*, posto $C_n^R = C_n \otimes R$, $\partial_n^R = \partial_n \otimes id_R$ si ha un complesso con omologia H_*^R. Inoltre è canonicamente definita una successione esatta

$$0 \longrightarrow H_n \otimes R \xrightarrow{g_n} H_n^R \xrightarrow{f_n} Tor(H_{n-1}, R) \to 0$$

e tale successione splitta ma non canonicamente.

<u>Cor</u>: $H_n^R \cong (H_n \otimes R) \oplus \mathrm{Tor}(H_{n-1}, R)$

ma <u>non</u> canonicamente.

<u>Cor</u>: $H_n(X,A;R) \cong \left(H(X,A;\mathbb{Z}) \otimes R\right) \oplus$

$$\oplus \ \mathrm{Tor}\left(H_{n-1}(X,A;\mathbb{Z}), R\right)$$

ma questo isomorfismo non è funtoriale, cioè non è naturale rispetto a mappe indotte da

$f: (X, A) \to (Y, B)$ __