ETA 7/10/14

$\mathcal{K}$ simplicial complex : simplices in $\mathbb{R}^n$

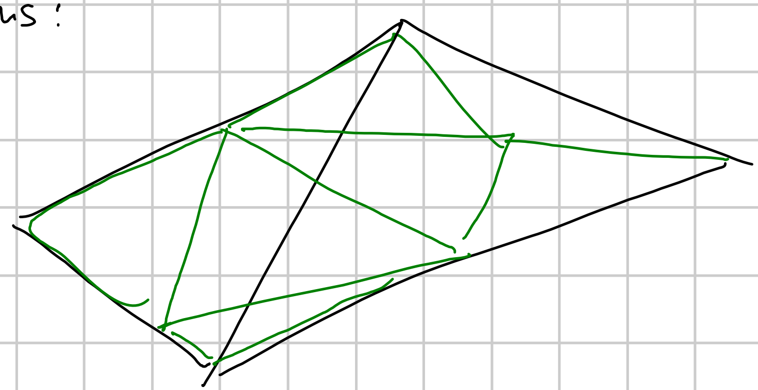
$$\left(C_n(\mathcal{K}), \partial_n\right)_{n=0}^{+\infty} \qquad Z_n(\mathcal{K}), B_n(\mathcal{K}) \rightsquigarrow H_n(\mathcal{K})$$

- indep of orient^t / isomorphism
- $\mathcal{L}$ subdivision of $\mathcal{K}$ then $H_n(\mathcal{K}) \xrightarrow{\cong} H_n(\mathcal{L})$ -

$H_n(K) = H_n(|K|)$ follows from:

Thm: $|K_1| = |K_2| \Rightarrow \exists$ common subdivision.

Not obvious:

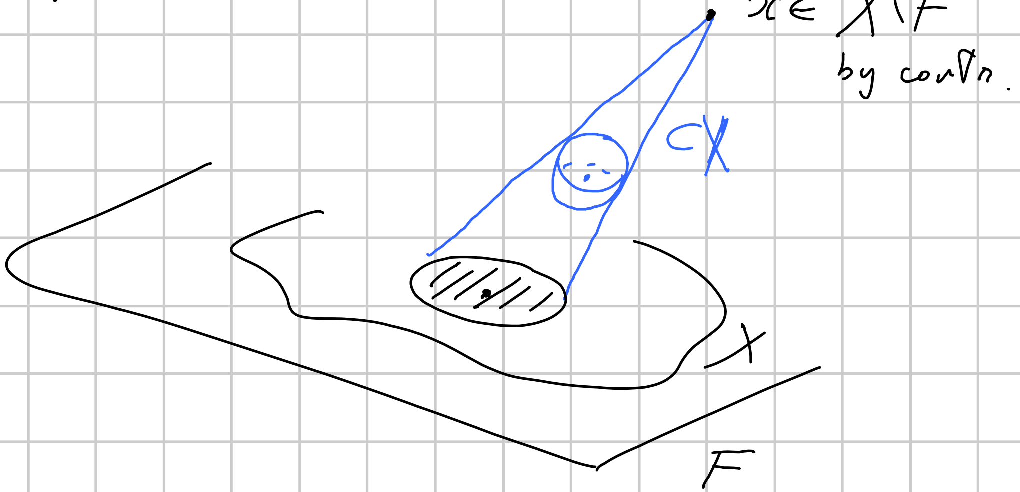


Tool: polytopal complexes.

Lem: $X \subset \mathbb{R}^N$ convex $\Rightarrow \exists! \, \Delta(X)$ affine subspace of maximal dimension among F's such that $\mathrm{int}_F(X \cap F) \neq \emptyset$; moreover $X \subset \Delta(X)$.

Pf: exclude $X = \emptyset$. Take F of max dim. with $\mathrm{int}_F(X \cap F) \neq \emptyset$. Enough to show that $X \subset F$: otherwise if some $F' \neq F$ exists we have $X \subset F \cap F'$ but $\mathrm{int}_F(F \cap F') \neq \emptyset$.

Proof that $X \subset F$:



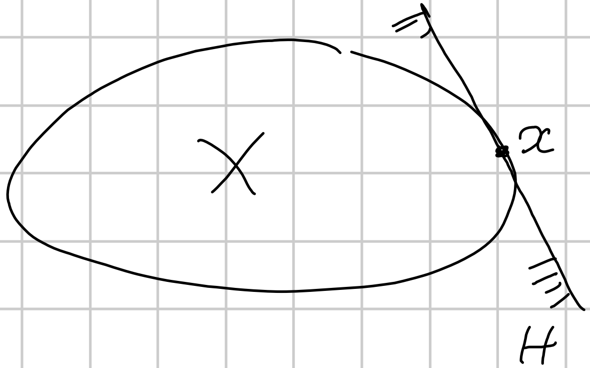
contradicts maximality of $\dim(F)$. $\boxtimes$

Def: if $X \subset \mathbb{R}^N$ convex a half-space H is a support half-space for X if $X \subset H$.

Lem: $X \subset \mathbb{R}^N$ convex + closed, $\mathrm{int}(X) \neq \emptyset$

$\Rightarrow \partial X = \bigcup \{ X \cap \partial H : H$ support half-space $\}$.

Proof: $\supset$ obvious:

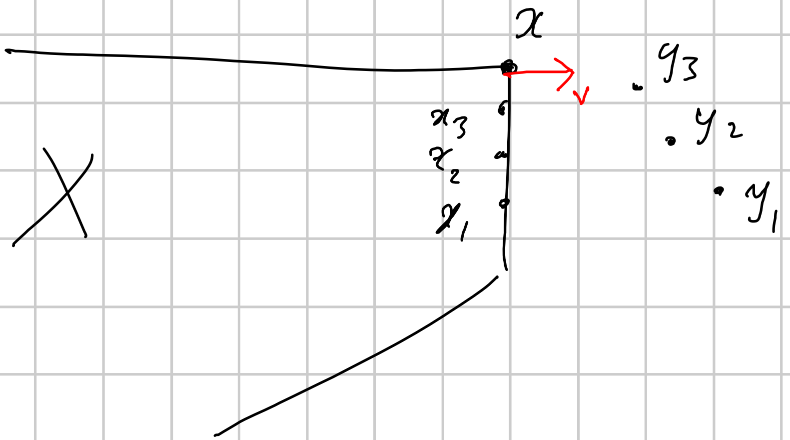


$\Rightarrow x \in \partial X.$

$\subset$ take $x \in \partial X$; $\exists\, y_n$ s.t.
$y_n \to x$ $y_n \notin X$. Take
x_n = closest point to y_n of X

Up to taking subsequences we have $\dfrac{y_n - x_n}{\|y_n - x_n\|} \longrightarrow v$

and one sees that $x + v^{\perp}$ is the boundary of a support half-space for X:



∎

Def: A convex polytope is $\mathrm{Conv}(p_1, \dots, p_k) \subset \mathbb{R}^n$;

A face of a polytope X with $\mathrm{int}(X) \neq \emptyset$

is $X \cap \partial H$ for H support half-space.

Lem: $\mathrm{Conv}(p_1, \dots, p_k) = \left\{ \sum_{i=1}^{k} t_i p_i : \sum_{i=1}^{k} t_i = 1, \ t_i \geq 0 \right\}$

(Pf as for simplices: $\subset$ because RHS is convex & contains $p_1, \dots, p_k$

$\supset$ by induction on $\#\{i : t_i \neq 0\}$.)

Lem: $p_0 \in \mathrm{Conv}(p_1, \dots, p_k) \iff \mathrm{Conv}(p_0, \dots, p_k) = \mathrm{Conv}(p_1, \dots, p_k)$

Pf: $\Leftarrow$: obvious

$\Rightarrow$ $\quad p_0 = \sum_{i=1}^{k} s_i p_i$

$$\Rightarrow \sum_{j=0}^{k} t_j p_j = \underbrace{\sum_{j=1}^{k} (t_0 s_j + t_j)\, p_j}_{\text{convex combination}}$$

$\blacksquare$

Prop: given $p_1, \dots, p_k \in \mathbb{R}^N$ $\exists !$
$\{v_1, \dots, v_h\} \subset \{p_1, \dots, p_k\}$ minimal
and such that $Conv(v_1, \dots, v_h) = Conv(p_1, \dots, p_k)$

Pf: Using Lm 4 remove one after each other
p_j's that belong to Conv(pts not yet removed).
Eventually we get minimal set $v_1, \dots, v_h$
with $Conv(v_1, \dots, v_h) = Conv(p_1, \dots, p_k)$.
Uniqueness: follows from

Claim: the v_j's are precisely the pts of $X := \mathrm{Conv}(p_1, \dots, p_k) = \mathrm{Conv}(v_1, \dots, v_\ell)$ that do not belong to the interior of any segment with ends in X.

Pf of claim: suppose by contradiction that $v_1 \in \mathrm{int}(x, x')$ $x, x' \in X$ i.e.

$$v_1 = \lambda \cdot x + (1-\lambda) \cdot x' = \lambda \cdot \sum_{i=1}^{h} t_i v_i + (1-\lambda) \sum_{i=1}^{h} t_i' v_i$$

$0 < \lambda < 1$, $t_1 \neq v_1$, $t_1' \neq v_1$

$$\Rightarrow t_1 < 1 \qquad t_1' < 1$$

$$\Rightarrow s t_1 + (1-s) t_1' < 1$$

moving $(s t_1 + (1-s) t_1') \cdot v_1$ to LHS we find

$v_1 \in \operatorname{Conv}(v_2, \dots, v_n)$: contradiction.

Conversely : must show that any $y \in X$ that is not one of the v_i's belongs to $\operatorname{int}(x, x')$:

$y = \sum_{i=1}^{n} t_i v_i$; $y \notin \{v_1, \dots, v_n\} \Rightarrow$ wlog $0 < t_1 < 1$

$$\Rightarrow \quad y = t_1 \cdot v_1 + (1-t_1) \cdot \sum_{i=2}^{n} \frac{t_i}{1-t_1} \cdot v_i$$

$$\overset{\cap}{X} \qquad \underbrace{}_{\text{convex comb.}} \Rightarrow \in X. \qquad \blacksquare$$

Def: X convex polytope; face of X is
$X \cap \partial H \neq \emptyset$ with H support half-space.

Lem: any face of X is a convex polytope.

Pf: Let $v_1, \dots, v_n$ be the vertices of X;

suppose $H = \{x \in \mathbb{R}^N : x_1 \geq 0\}$; wlog

$$x_1(v_j) \begin{cases} = 0 & j = 1 \dots p \\ > 0 & j = p+1 \dots h \end{cases}$$

Claim $X \cap \partial H = \mathrm{Conv}(v_1, \dots, v_p)$
(whence conclusion) —

$\supset$: obvious

$\subset$: if $x \in X$, $x = \sum_{i=1}^{h} t_i v_i$

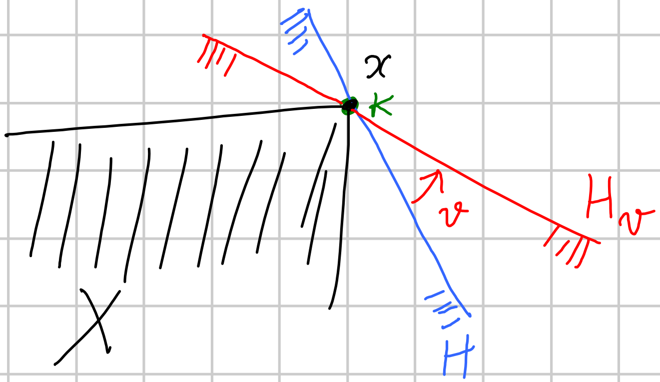
$x_i = 0 \iff t_i = 0$ for $i > p$. $\blacksquare$

Def: $\dim(X) := \dim(A(X))$.

Lem: X convex polytope in $\mathbb{R}^n$; $\operatorname{int}(X) \neq \emptyset$
$\Rightarrow \partial X = \bigcup(\text{codimension-1 faces})$.

Pf: we know: ∂X is a union of faces.
Must show: codim-1 faces suffice.

Let $x \in \partial X$; $\exists H$ support with
$x \in X \cap \partial H$; if $\dim(X \cap \partial H) = N-1$ ok;
otherwise $\exists W$ codim-2 subspace of $\mathbb{R}^N$
with $X \cap \partial H \subset W$.



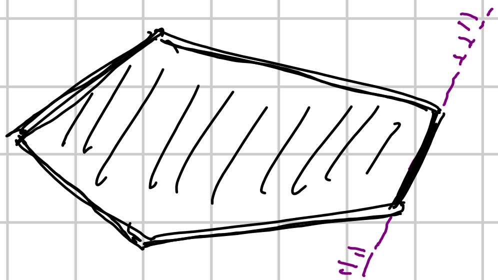
Easy: H_v support half-space for $|v| << 1$.

Take $v_0 = \min \{ v > 0 : \partial H_v \cap X \not\subset K \}$.

We can replace H by H_{v_0},

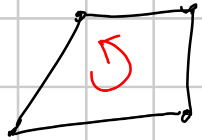
$\dim(X \cap \partial H_{v_0}) \geq 1 + \dim(X \cap \partial H)$

$\Rightarrow$ eventually we get to $N-1$. $\square$



Now we can:

- Define an orientation of X as one of $\mathfrak{o}(X)$
- Define induced orientation on codim-1 faces (outer normal first)



- Show that $Y \subset X$ face of codim 2 $\Longrightarrow \exists Z_1, Z_2$ codim 1 faces with $Y = Z_1 \cap Z_2$ and

$X \rightsquigarrow Z_1 \longrightarrow Y$ opposite orientations

$X \rightsquigarrow Z_2 \longrightarrow Y$

- Define polytopal complex in $\mathbb{R}^N$: finite collection of polytopes "closed under taking faces and intersections";

- $C_n(K) = \mathbb{Z}\langle K^{[n]} \rangle$ all X's oriented

$$\partial_n : C_n(K) \to C_{n-1}(K)$$

$$\partial_n X = \sum_{Y \in K^{(n-1)}} \varepsilon(X,Y) \cdot Y$$

$$\rightsquigarrow Z_n(K),\ B_n(K) \longrightarrow H_n(K)$$

- Independence of orientations

- Define subdivision + show that
  if L subdivides K then

$\exists\ H_n(K) \xrightarrow{\cong} H_n(L)$.

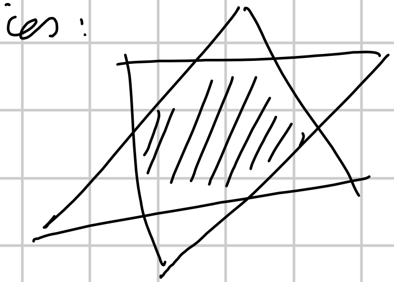
Left: Thm 2^*: K_1, K_2 polytopal complexes, $|K_1| = |K_2|$
$\Longrightarrow$ $\exists$ common subdivision.

(Actually: Thm 2^* also implies Thm 2.)

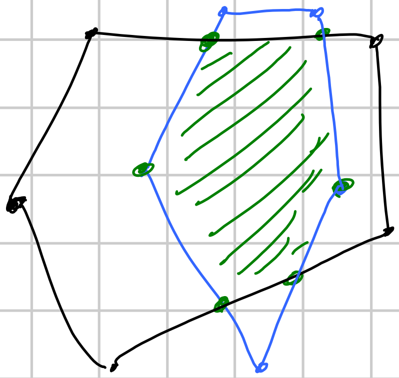
To prove Thm 2^* it is enough to show:

Prop: X, Y polytopes $\Rightarrow X \cap Y$ also is.

Rem: false for simplices:



Rem: "not obvious": what are the vertices?



Why enough for Thm 2*: $|\mathcal{K}_1| = |\mathcal{K}_2|$

$$\mathcal{L} = \{ \underbrace{X_1 \cap X_2}_{\text{by Prop. this is polytope}} : X_j \in \mathcal{K}_j \}$$

$\rightsquigarrow$ easy to see that $\mathcal{L}$ is polytopal complex and subdivides $\mathcal{K}_1$ & $\mathcal{K}_2$.

Prop. easily follows from:

__Lem:__ $X \subset \mathbb{R}^n$

X convex polytope $\Longleftrightarrow$ X bounded & a finite intersection of closed half-spaces.

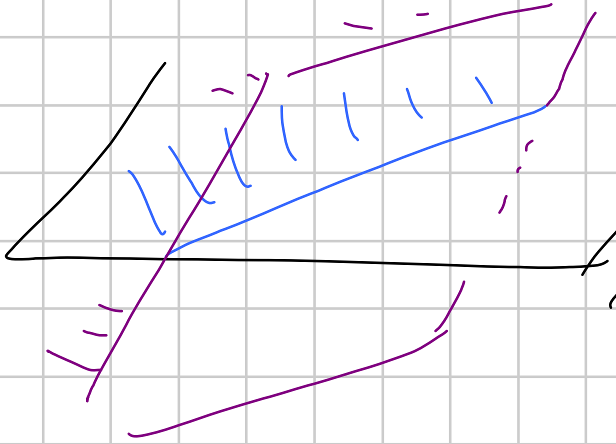
(Enough for Prop because closed under $\bigcap$).

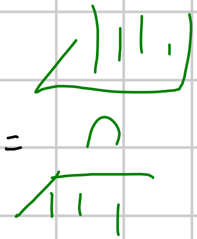
__Pf.__ __$\Longrightarrow$:__ bounded ✓

Wlog I can assume $s(X) = \mathbb{R}^N$ i.e. $\mathrm{int}_{\mathbb{R}^N}(X) \neq \emptyset$:

because if I know it's true in this case, then it's true in $s(X)$ but

- $s(X)$ is an intersection of closed half-spaces
- $H \subset s(X)$ closed half-space is $s(X) \cap H'$ with $H' \subset \mathbb{R}^N$ closed half-space



$\partial(X) =$ 

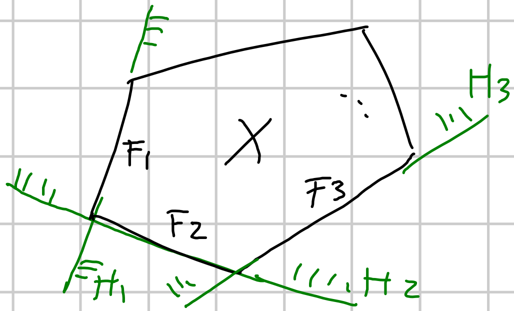
So: suppose $\mathrm{int}_{\mathbb{R}^N}(X) \neq \emptyset$, $X = \mathrm{Conv}(v_1, \dots, v_n)$

vertices. We know each face is

$\mathrm{Conv}(v_{i_1}, \dots, v_{i_p})$ for some $\{i_1, \dots, i_p\} \subset \{1, \dots, h\}$
(not conversely)

so X has a finite number q of codimension 1 faces $X \cap \partial H_j$ $j = 1, \dots, q$.

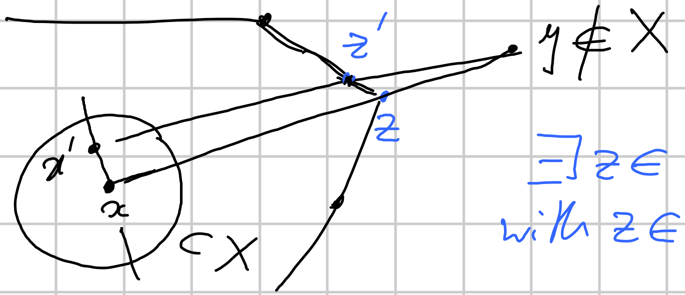
Claim: $X = H_1 \cap \dots \cap H_q$. This concludes the proof of $\underbrace{\Rightarrow}$.



Claim : $X = H_1 \cap \ldots \cap H_q$.

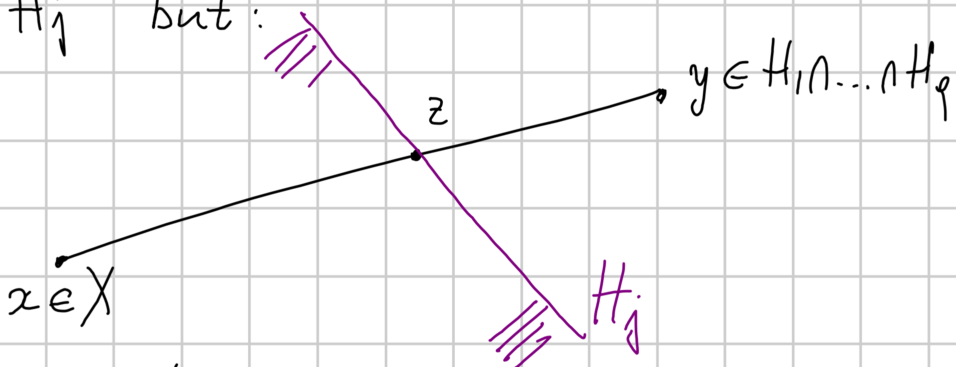
$\subset$: obvious because $X \subset H_j$.

$\supset$: by contradiction let $y \in H_1 \cap \ldots \cap H_q$, $y \notin X$. We know $\mathrm{int}(X) \neq \emptyset$; take $x \in \mathrm{int}(X)$ and $[x,y]$.



$\exists\, z \in [x,y]$ with $z \in \partial X$

Actually: up to perturbing x on $X \cap (x + (y-x)^{\perp})$
we can assume $z \notin$ any codim-2 face
$\Rightarrow z$ belongs to a unique codim-1 face
$X \cap \partial H_j$ but:



contradiction to $y \in H_j$.

$\underline{\Leftarrow}$ X bounded finite intersection of half-spaces $\Longrightarrow$ X convex polytope.

Wlog: can assume $\text{int}(X) \neq \emptyset$. (Exercise)

Now: by induction on N.

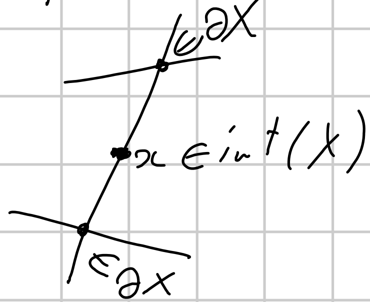
$N = 1$: $X = [a, b] = \text{Conv}(a, b)$.

$N > 1$: Suppose $X = H_1 \cap \dots \cap H_q$;

we claim $\partial X = (X \cap \partial H_1) \cup \dots \cup (X \cap \partial H_q)$.

This is enough because:

- by induction $X \cap \partial H_j$ is convex polytope
- $X = \mathrm{Conv}(\partial X)$:

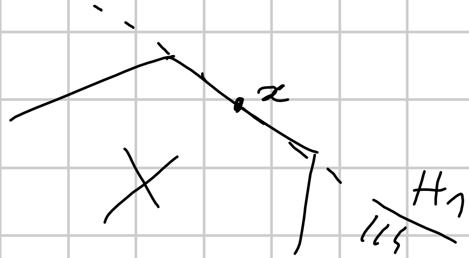


(uses boundedness)

Pf of claim: $\supset$ obvious:

if $x \in X \cap \partial H_j$

$\Longrightarrow x \in \partial X$



$\subset$: Suppose

$x \in X$, $x \notin \partial H_j$ for all j

$\Longrightarrow x \in \mathrm{int}(H_j)$ $j = 1 \dots q$

$x \in \operatorname{int}(H_1 \cap \dots \cap H_q) = \operatorname{int}(X)$

$\Longrightarrow x \notin \partial X$. $\blacksquare$