ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 12 - 9.4.2024

Def (<u>minimo debole</u>) $\mathcal{L}(u) = \int_a^b L(x, u(x), u'(x))\, dx$

diremo che $u \in W^{1,1}$ è un <u>minimo debole</u> per $\mathcal{L}$

se $\exists \delta > 0$ t.c. $\forall \varphi \in C_0^1(a,b)$ con $\underline{\|\varphi\|_{C^1} = \|\varphi\|_{C^0} + \|\varphi'\|_{C^0}} < \delta$

$$\mathcal{L}(u) \le \mathcal{L}(u+\varphi)$$

<u>Oss 1</u>. Se u è minimo debole allora $\forall \varphi \in C_c^\infty(a,b)$ $\exists \delta > 0$ (*)

$\forall \varepsilon \in (-\delta, \delta)$ $\quad \mathcal{L}(u) \le \mathcal{L}(u + \varepsilon\varphi)$

ovvero $\varepsilon \mapsto \mathcal{L}(u+\varepsilon\varphi)$ ha un minimo locale per $\varepsilon = 0$

<u>Oss 2</u>. Se u è minimo assoluto $\Rightarrow$ u è <u>minimo debole</u>.

<u>Teorema</u> (EL in $W^{1,p}$ ver. 2)

$L = L(x,y,z)$ $\quad x \in [a,b]$, $y \in \mathbb{R}$, $z \in \mathbb{R}$.

$\tilde{\forall} x$ $\forall (y,z)$ $\quad \exists \frac{\partial L}{\partial y}(x,y,z)$ e $\exists \frac{\partial L}{\partial z}(x,y,z)$

- L, $\frac{\partial L}{\partial y}$, $\frac{\partial L}{\partial z}$ sono di Carathéodory — Hyp. Struttura

- $\exists C > 0$ $\quad \tilde{\forall} x$ $\forall y, \forall z$ — Hyp crescita

$$\left|\frac{\partial L}{\partial y}(x,y,z)\right| + \left|\frac{\partial L}{\partial z}(x,y,z)\right| \le C\left(1 + |y|^p + |z|^p\right)$$

Sia $u \in W^{1,p}(a,b)$ tale che $\mathcal{L}(u) = \int_a^b L(x, u(x), u'(x))\, dx \in \mathbb{R}$

e u minimo debole (ma basta (*))

Allora $\frac{\partial L}{\partial z}(x, u(x), u'(x)) \in W^{1,1}$ e $\boxed{\frac{d}{dx}\frac{\partial L}{\partial z}(x,u,u') = \frac{\partial L}{\partial y}(x,u,u')}$

derivata debole

dim Sia $\varphi \in C_c^\infty$, $\varepsilon > 0$

$\exists\ |\theta| < \varepsilon, |\tau| < \varepsilon$

$$(*) = \frac{L(x, u+\varepsilon\varphi, u'+\varepsilon\varphi') - L(x,u,u')}{\varepsilon} =$$

$$= \underbrace{\frac{\partial L}{\partial y}(x, u+\theta\varphi, u'+\varepsilon\varphi')}_{L^1} \cdot \underbrace{\varphi(x)}_{L^\infty} + \underbrace{\frac{\partial L}{\partial z}(x, u, u'+\tau\varphi')}_{L^1} \underbrace{\varphi'(x)}_{L^\infty}$$

$$\left|\frac{\partial L}{\partial y}(x, u+\theta\varphi, u'+\varepsilon\varphi')\right| \leq C\left(1 + |u+\theta\varphi|^p + |u'+\varepsilon\varphi'|^p\right)$$

$$\leq C\Big(1 + 2^p\big[|u|^p + \underbrace{|\theta|^p}_{<1}\underbrace{|\varphi|^p}_{L^\infty} + |u'|^p + \underbrace{|\varepsilon|^p}_{<1}\underbrace{|\varphi'|^p}_{L^\infty}\big]\Big)$$

$\varepsilon < 1$

$$\leq C'\Big(1 + \underbrace{|u|^p}_{L^1} + \underbrace{|u'|^p}_{L^1}\Big) = h \in L^1$$

non dipende da ε

$u \in W^{1,p}$
$u \in L^p$ $\quad u' \in L^p$
$|u|^p \in L^1$ $\quad |u'|^p \in L^1$

$$\left|\frac{\partial L}{\partial z}(x, u, u'+\tau\varphi')\right| \leq C\left(1 + |u|^p + |u'+\tau\varphi'|^p\right)$$

$$\dots \leq C''\left(1 + |u|^p + |u'|^p\right) = \tilde{h} \in L^1$$

$(*)$ è dominato da una funzione L^1!

$$(\#) = \int_a^b \underbrace{\frac{L(x, u+\varepsilon\varphi, u'+\varepsilon\varphi') - L(x,u,u')}{\varepsilon}}_{L^1} dx \quad \text{esiste finito}$$

Per ipotesi $\mathcal{L}(u)$ è finito $\Rightarrow$ anche $\mathcal{L}(u+\varepsilon\varphi)$ è finito $\forall\varepsilon$

$$\frac{\mathcal{L}(u+\varepsilon\varphi) - \mathcal{L}(u)}{\varepsilon} = (\#) \xrightarrow[\varepsilon \to 0]{\text{convergenza dominata}} \int_a^b \Big[\underbrace{\frac{\partial L}{\partial y}(x,u,u')}_{v \in L^1}\varphi + \underbrace{\frac{\partial L}{\partial z}(x,u,u')}_{w \in L^1}\varphi'\Big] dx$$

$$= 0$$

u minimo debole

Si conclude come ieri. Ovvero:

Lemma. Se $u, v \in L^1(a,b)$ e $\forall \varphi \in C_c^\infty(a,b)$ $\int_a^b (v\varphi + u\varphi') = 0$

Allora $u \in W^{1,1}(a,b)$ e $u' = v$.

dim

$$V(x) = \int_a^x v \qquad v \in L^1 \Rightarrow V \in W^{1,1} \text{ e } V' = v$$

$$\int_a^b (V'\varphi + u\varphi') = 0 \qquad \forall \varphi \in C_c^\infty$$

$$\int [-V + u]\varphi' = 0 \overset{\text{D.B.R}}{\Rightarrow} -V + u = c$$

$$u = c + V \in W^{1,1}$$
$$u' = V' = v \qquad \square$$

Nel nostro caso $v = \frac{\partial L}{\partial y}$ $u = \frac{\partial L}{\partial z}$ $\Rightarrow$ $\boxed{\frac{d}{dx}\frac{\partial L}{\partial z} = \frac{\partial L}{\partial y}}$

REGOLARITA'

Se $u \in W^{1,p}$ risolve EL in senso debole e se L soddisfa opportune ipotesi allora $u \in C^1 \dots C^k \dots C^\infty$ e soddisfa EL in senso classico.

Vari step: $W^{1,p} \to \underset{\text{Lip}}{W^{1,\infty}} \to C^1 \to C^k \to C^\infty$

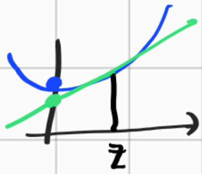
Teo (Regolarita lipschitz) Sia $L = L(x,y,z)$ $\overset{L \in C^0}{\forall} u \in W^{1,1}(a,b)$, $p > 1$

$\boxed{x \mapsto \frac{\partial L}{\partial z}(x, u(x), u'(x))} \in W^{1,1}$, $z \mapsto L(x,y,z)$ sia convessa,

$L(x,y,z) \geq \alpha \cdot |z|^p - \varphi(y)$ $\quad \exists \alpha > 0$, $\exists \varphi \in C^0$

Allora $u \in W^{1,\infty} \subseteq \mathrm{Lip}$

dim $L(x,y,\cdot)$ convessa: $\overbrace{L(x,y,0) \geq L(x,y,z) - z \cdot \frac{\partial L}{\partial z}(x,y,z)}^{\text{retta tangente}}$



$$z \cdot \frac{\partial L}{\partial z}(x,y,z) \geq L(x,y,z) - L(x,y,0)$$

$$\geq \alpha |z|^p - \varphi(y) - L(x,y,0)$$

$y = u(x)$, $z = u'(x)$ $\quad u \in W^{1,1} \subseteq C^0 \subseteq L^\infty$ $\quad M = \|u\|_{L^\infty} < +\infty$

$\varphi \in C^0$, $L \in C^0 \Rightarrow \underbrace{\varphi(u(x)) - L(x,u(x),0)}_{\text{è limitata}}$

$$u'(x) \cdot \frac{\partial L}{\partial z}(x,u(x),u'(x)) \geq \alpha \cdot |u'(x)|^p - C$$

$$|u'(x)| \cdot \underbrace{\left|\frac{\partial L}{\partial z}(x,u(x),u'(x))\right|}_{W^{1,1} \subseteq L^\infty} \geq \alpha |u'(x)|^p - C$$

$$|u'(x)|^p \leq \frac{|u'(x)| \cdot \left|\frac{\partial L}{\partial z} \cdots\right| + C}{\alpha} \leq c' \cdot (|u'(x)| + 1)$$

Se $|u'(x)| \geq 1 \Rightarrow |u'(x)|^p \leq c'(|u'(x)| + |u'(x)|) = 2c'|u'(x)|$

$$\Rightarrow |u'(x)|^{p-1} \leq 2c' \Rightarrow |u'(x)| \leq (2c')^{\frac{1}{p-1}} = c''$$

Altrimenti $|u'(x)| \leq 1$

$\Rightarrow |u'(x)| \leq c''' \quad \tilde{\forall} x \Rightarrow u' \in L^\infty \quad (u \in L^\infty$

$\Downarrow$

$u \in W^{1,\infty}$ $\square$

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Teorema (regolarità C^1)

Sia $L = L(x,y,z)$, $L \in C^0$, $\exists \frac{\partial L}{\partial z} \in C^0$,

$$z \longmapsto \frac{\partial L}{\partial z}(x,y,z) \quad \text{sia iniettiva} \quad \forall (x,y)$$

Sia $u \in Lip$ t.c. $\underbrace{\frac{\partial L}{\partial z}(x,u(x),u'(x))} \in W^{1,1}$

Allora $u \in C^1$.

dim $\quad \forall x \quad \frac{\partial L}{\partial z}(x,u(x),u'(x)) = g(x)$ con $g \in C^0$.

Sia $E = \left\{ x \in [a,b] : \exists u'(x),\ \frac{\partial L}{\partial z}(x,u(x),u'(x)) = g(x) \right\}$

[Rademacher: $u \in Lip \Rightarrow \forall x \ \exists u'(x) \leftarrow$ derivata classica]

$\Rightarrow |[a,b] \setminus E| = 0$

Claim 1 se $\begin{cases} x_k \in E, & x_k \to x, & u'(x_k) \to v \\ x_k' \in E, & x_k' \to x, & u'(x_k') \to w \end{cases}$ allora $v = w$.

$x \in [a,b]$

dim claim 1

$$\frac{\partial L}{\partial z}(x_k, u(x_k), u'(x_k)) = g(x_k)$$

$x_k \to x$, $u(x_k) \to u(x)$, $u'(x_k) \to v$, $g(x_k) \to g(x)$; $\frac{\partial L}{\partial z} \in C^0$

$$\frac{\partial L}{\partial z}(x, u(x), v) = g(x) \quad (*)$$

Analogamente $\frac{\partial L}{\partial z}(x,u(x),w) = g(x)$

$\frac{\partial L}{\partial z}$ iniettiva $\Rightarrow v = w$. $\square$

Dato $x \in [a,b]$ $\exists x_k \to x$, $x_k \in E$, $u'(x_k)$ è limitata ($\Leftarrow u \in Lip$)

$\exists x_{k_j} \to x$ t.c. $u'(x_{k_j}) \to v$

Posso definire $v(x) = v$

- Ovviamente se $x \in E$ prendo $x_k = x \Rightarrow v(x) = u'(x)$

$(*)$ $u'(x) = v(x)$ $\tilde{\forall} x$

- Inoltre $\frac{\partial L}{\partial z}(x, u(x), v(x)) = g(x) \quad \forall x \in [a,b]$.

<u>Claim</u> $v \in C^0$.

Sia $x \in [a,b]$, sia $x_k \in [a,b)$ $x_k \to x$

devo mostrare che $v(x_k) \to v(x)$.

$$v(x_{k_j}) \to w \qquad \left(v(x_k) = \lim_j u'(x_{k,j}) \right)$$

$$\frac{\partial L}{\partial z}(x_{k_j}, u(x_{k_j}), v(x_{k_j})) = g(x_{k_j})$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\frac{\partial L}{\partial z}(x, u(x), w) = g(x)$$

ma anche $\frac{\partial L}{\partial z}(x, u(x), v(x)) = g(x)$

iniettività $\Rightarrow$ $w = v(x)$. $\square$

$u'(x) = v(x)$ $\tilde{\forall} x$, $v \in C^0$

$\tilde{u}(x) = \int_a^x v \qquad \tilde{u} \in C^1$

$$u'(x) = \tilde{u}'(x) \quad \tilde{\forall} x \Rightarrow u(x) = \underbrace{\tilde{u}(x) + c}_{C^1} \quad \tilde{\forall} x$$

$\square$