LEMMA (FONDAMENTALE DEL CALC. DELLE VARIAZIONI)

$u \in L^1_{loc}(a,b)$ $\quad$ [cioè $u \in L^1([c,d])$ $\forall [c,d] \subseteq (a,b)$]

$\int_a^b u\varphi = 0 \quad \forall \varphi \in C_c^\infty(a,b) \Rightarrow u = 0$

DIM $\quad \forall n$ SIA $A_n = \left\{ x \in \left[a+\frac{1}{n}, b-\frac{1}{n}\right] : u(x) > 0 \right\}$

SE $|A_n| = 0$ $\forall n \Rightarrow u \leq 0$.

SE INVECE $|A_n| > 0$ PER UN CERTO $n \Rightarrow$

$\exists K \subseteq A_n$ COMPATTO CON $|K| > 0$, IN PART. $\int_K u > 0$.

SCELGO $\varphi_n \in C_c^\infty$ T.C. $\varphi_n \to \chi_K = \begin{cases} 1 & x \in K \\ 0 & x \notin K \end{cases}$ IN L^1 E q.o.

[$\varphi_n = \chi_K * \rho_n$ APPROSSIM. PER CONVOLUZIONE]

$$0 < \int_K u = \int_a^b u \chi_K = \lim_n \int_a^b u \varphi_n = 0 \qquad \text{ASSURDO}$$

$$\Rightarrow u \leq 0.$$

FACENDO LO STESSO RAGIONAMENTO CON $B_n = \left\{ x \in \left[a+\frac{1}{n}, b-\frac{1}{n}\right] : u(x) < 0 \right\}$

OTTENIAMO $u = 0$.

LEMMA (DU BOIS-REYMOND)

$$u \in L^1_c(a,b) \qquad \int_a^b u\varphi' = 0 \quad \forall \varphi \in C_c^\infty(a,b)$$

$$\Rightarrow u = c \in \mathbb{R}.$$

OSS: $\varphi \in C_c^\infty$ T.C. $\int_a^b \varphi = 0 \Rightarrow \exists\, v = \int_a^x \varphi(t)\,dt \in C_c^\infty(a,b)$ T.C. $\varphi = v'$.

DIM SIANO $w, \varphi \in C_c^\infty(a,b)$ CON $\int \varphi = 1$

CONSIDERIAMO $\sigma = w - \left(\int_a^b w\right)\varphi \in C_c^\infty(a,b)$

$$\int_a^b \sigma = \int_a^b w - \int_a^b w \cdot \int_a^b \varphi = 0 \overset{\text{OSS}}{\Longrightarrow} \exists\, v \in C_c^\infty \text{ T.C. } \sigma = v'$$

$$\Rightarrow \quad 0 = \int_a^b u v' = \int_a^b u \sigma = \int_a^b u \cdot w - \left(\int_a^b w\right) u \cdot \varphi = \int_a^b \left(u - \int_a^b u \cdot \varphi\right) w$$

PER IL LEMMA FOND. HO CHE $u - \int_a^b u \cdot \varphi = 0$ CIOÈ $u = \int_a^b u \cdot \varphi \in \mathbb{R}$.

LEMMA $\quad v \in L^1(a,b) \qquad u = \int_a^x v(t)\,dt$

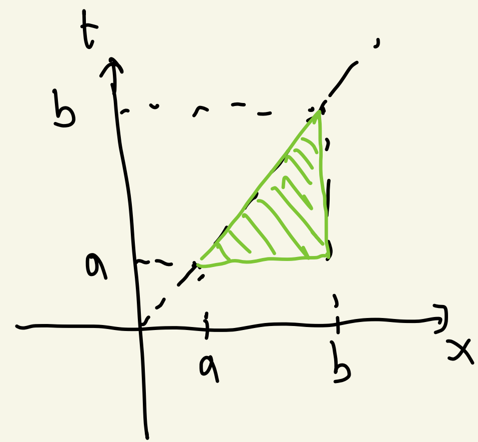
$$\Rightarrow \int_a^b u\varphi' = -\int_a^b v\varphi \qquad \forall \varphi \in C_c^\infty(a,b)$$

DIM.

$$\int_a^b u(x)\varphi'(x)\,dx = \int_a^b \left(\int_a^x v(t)\varphi'(x)\,dt \right) dx$$

$$\underset{\text{F.T.}}{=} \int_a^b \left(\int_t^b v(t)\varphi'(x)\,dx \right) dt$$

$$= \int_a^b \left(\int_t^b \varphi'(x)\,dx \right) v(t)\,dt = -\int_a^b v(t)\varphi(t)\,dt$$



PROP $u \in W^{1,1}(a,b) \Rightarrow \exists$ RAPPR. $\tilde{u}$ T.C.

$$\tilde{u}(y) - \tilde{u}(x) = \int_x^y u'(t)\,dt \quad \forall \; x<y \in [a,b].$$

DIM SIA $\tilde{u}(x) = \int_a^x u'(t)\,dt$

CONSIDERO LA DIFFERENZA $u - \tilde{u}$ E VALUTO PER $\varphi \in C_c^\infty(a,b)$

$$\int_a^b (u-\tilde{u})\varphi' = \int_a^b u\varphi' - \int_a^b \tilde{u}\varphi' \underset{\text{LEMMA}}{=} -\int_a^b u'\varphi + \int_a^b u'\varphi = 0$$

$\Rightarrow$ $u - \tilde{u} = c \in \mathbb{R}$ PER q.o. x, CIOÈ

LEMMA DU BOIS-R.

$$u(x) = c + \tilde{u}(x) = c + \int_a^x u'(t)\,dt$$

$$\Rightarrow u(y) - u(x) = \tilde{u}(y) - \tilde{u}(x) = \int_x^y u'.$$

COR: $u \in W^{1,1} \Rightarrow u \in AC$ (IN REALTÀ VALE $\Leftarrow$)

$$\sum_k |\tilde{u}(b_k) - \tilde{u}(a_k)| = \left| \int_{\cup I_k} u' \right| \leq \int_{\cup I_k} |u'|^{\in L^1}$$

$I_k = [a_k, b_k]$

$\Rightarrow$ PER L'ASS. CONT. DI $\int$, $\int_E |u'| \leq \varepsilon$ SE $|E| \leq \delta$.

IN PART. $u \in W^{1,p} \Rightarrow \tilde{u}$ CONTINUA.

PROP $u \in W^{1,p}(a,b)$ CON $1 < p < +\infty \Rightarrow \tilde{u} \in C^{\alpha}(a,b)$ $\alpha = 1 - \frac{1}{p}$.

CIOÈ $\tilde{u}$ È HÖLDERIANA.

DIM $|\tilde{u}(y) - \tilde{u}(x)| = \left| \int_x^y u' \cdot 1 \right| \underset{\text{HÖLDER}}{\leq} \|u'\|_{L^p(x,y)} \cdot \|1\|_{L^{p'}(x,y)} \leq \|u'\|_{L^p(a,b)} \cdot (y-x)^{\frac{1}{p'}}$ ($\alpha = \frac{1}{p'}$)

$x < y \in [a,b]$

$\frac{1}{p} + \frac{1}{p'} = 1$, $p' = \frac{p}{p-1}$ ESPONENTE CONIUGATO

OSS: NON VALE $\Leftarrow$

PROP (COMPATTEZZA IN $W^{1,p}$)

$u_n \in W^{1,p}(a,b)$, $\|u_n\|_{W^{1,p}} \leq C \Rightarrow \exists\, u \in C^0([a,b])$ E n_k T.C. $\tilde{u}_{n_k} \to u$ UNIF. IN $[a,b]$.

RICORDIAMO IL TEO DI ASCOLI-ARZELÀ.

TEO $u_n \in C^0([a,b])$ SONO T.C. $\|u_n\|_{C^0} \leq C$ E

u_n EQUICONTINUE, CIOÈ $\forall \varepsilon\ \exists\, \delta$ T.C. $|x-y| < \delta \Rightarrow |u_n(x) - u_n(y)| < \varepsilon$

$\Rightarrow \exists\, u \in C^0$ E n_k T.C. $u_{n_k} \to u$ IN $C^0([a,b])$.

DIM $\|u_n\|_{W^{1,p}} \leq C \Rightarrow |\tilde{u}_n(x) - \tilde{u}_n(y)| \leq \|u'\|_{L^p} |x-y|^\alpha \leq C |x-y|^\alpha$

$\forall x, y \qquad \alpha = 1 - \frac{1}{p}$

CIOÈ LE $\tilde{u}_n$ SONO EQUICONTINUE

INOLTRE $|\tilde{u}_n(x)| = c_n + \int_a^x u_n' \Rightarrow$

$$|c_n| \leq |u_n(x)| + \int_a^x |u_n'|$$

$$(b-a)|c_n| \leq \int_a^b |u_n| + (b-a) \int_a^b |u_n'|$$

$$\leq \|u_n\|_{L^p} (b-a)^{\frac{1}{p'}} + (b-a)(b-a)^{\frac{1}{p'}} \|u_n'\|_{L^p} \leq C \|u_n\|_{W^{1,p}} \leq C$$

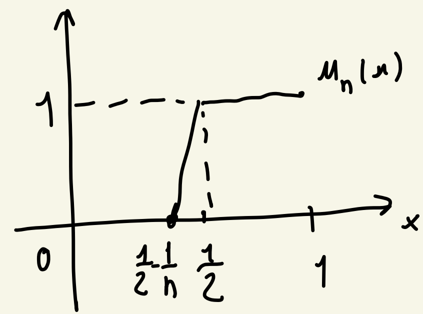
$\Rightarrow \|\tilde{u}_n\|_{L^\infty} \leq C \quad \forall n$. LA TESI SEGUE DA A.A.

OSS: LA PROP. NON VALE IN $W^{1,1}$.

$u_n \in W^{1,1}$ CON $\|u_n\|_{W^{1,1}} \leq C$ SI HA CHE $\|u_n\|_{C^\infty} \leq C$

MA IN GENERALE NON SONO EQUICONTINUE

ES: $u_n(x) = \begin{cases} 0 & x \in [0, \frac{1}{2} - \frac{1}{n}] \\ nx + 1 - \frac{n}{2} & x \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2}] \\ 1 & x \in [\frac{1}{2}, 1] \end{cases}$



$u_n \to u(x) = \begin{cases} 0 & x \in [0, \frac{1}{2}] \\ 1 & x \in [\frac{1}{2}, 1] \end{cases}$ NON CONTINUA

$$\|u_n\|_{W^{1,1}} = \int_0^1 |u_n| + \int_0^1 |u_n'| = \frac{1}{2} + \frac{1}{n} + \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} n \, dx = \frac{3}{2} + \frac{1}{n} \leq C$$

PROP (DIS. DI POINCARÉ)

$$u \in W^{1,p}_0(a,b) = \{u \in W^{1,p}(a,b) : \tilde{u}(a) = \tilde{u}(b) = 0\}$$

$$\Rightarrow \quad \|u\|_{L^p} \leq C \|u'\|_{L^p} \quad \text{DOVE } C = \frac{b-a}{p^{1/p}}.$$

IN PART. $\|u\|_{W^{1,p}_0} := \|u'\|_{L^p}$ È UNA NORMA EQUIV. SU $W^{1,p}_0$

$$\|u\|_{W^{1,p}_0} \leq \|u\|_{W^{1,p}} \leq (C+1) \|u\|_{W^{1,p}_0}$$

DIM. $u(x) = \int_a^x u'(t)\, dt \quad \tilde{\forall} x \in [a,b]$

$$|u(x)|^p \leq \left(\int_a^x |u'(t)|\, dt \right)^p$$

DIS JENSEN

Φ CONVESSA

$$\Phi\left(⨍ u\right) \leq ⨍ \Phi(u)$$

SE APPLICHIAMO LA DIS. DI JENSEN CON $\Phi(x) = x^p$

$$\Phi\left(⨍_a^x |u'| \right) = \left[\frac{1}{x-a} \int_a^x |u'| \right]^p \leq \frac{1}{x-a} \int_a^x |u'|^p dt$$

$$\Rightarrow \quad |u(x)|^p \leq \left[\int_a^x |u'| \right]^p \leq (x-a)^{p-1} \int_a^x |u'|^p \leq \|u'\|_{L^p}^p (x-a)^{p-1} \quad \tilde{\forall} x$$

$$\Rightarrow \quad \|u\|_{L^p}^p = \int |u(x)|^p \leq \|u'\|_{L^p}^p \int_a^b (x-a)^{p-1} = \|u'\|_{L^p}^p \frac{(b-a)^p}{p}.$$

PROP $\quad u \in W^{1,p}(a,b) \Rightarrow$ PER Q.O. $x \ \exists \lim_{h \to 0} \frac{\tilde{u}(x+h) - \tilde{u}(x)}{h} = u'(x)$

DERIVATA DEBOLE DI u

$1 \leq p \leq +\infty$

[NO DIMOSTRAZIONE]