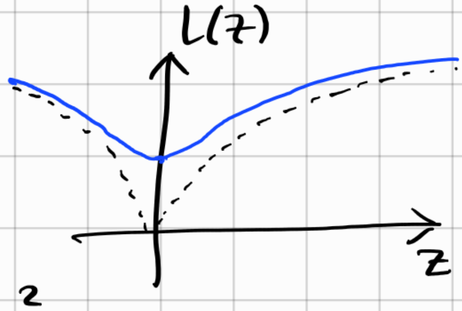
ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 6 – 12.3.2024

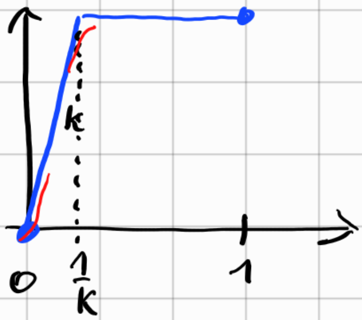
ESEMPI NEGATIVI

Esempio

$$\begin{cases} \mathcal{L}(u) = \int_0^1 \sqrt[3]{1+(u'(x))^2}\,dx \\ u(0) = 0 \\ u(1) = 1 \end{cases}$$



$$L(x,y,z) = \sqrt[3]{1+z^2} \sim |z|^{\frac{2}{3}} \quad \text{per } |z| \to \infty$$



$$u_k(x) = \begin{cases} kx & \text{se } 0 \le x \le \frac{1}{k} \\ 1 & \text{se } \frac{1}{k} \le x \le 1 \end{cases}$$

$$u_k(x) = \begin{cases} k & \text{se } 0 \le x < \frac{1}{k} \\ 0 & \text{se } \frac{1}{k} < x \le 1 \end{cases}$$

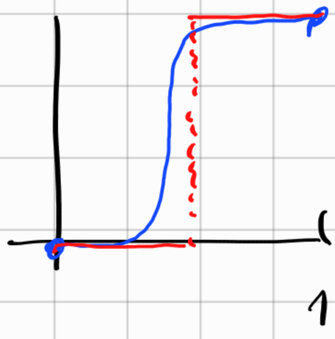
$$\mathcal{L}(u_k) = \int_0^{1/k} \sqrt[3]{1+k^2} + \int_{1/k}^1 \sqrt[3]{1+0^2}$$

$$= \frac{\sqrt[3]{1+k^2}}{k} + \left(1-\frac{1}{k}\right) \xrightarrow{k\to\infty} 0 + 1 = 1$$

$$\mathcal{L}(u) = \int_0^1 \sqrt{1+(u')^2}\,dx \ge \int_0^1 1 = 1.$$

$$\inf_{\substack{C^1_p \\ u(0)=0 \\ u(1)=1}} \mathcal{L} = 1.$$

C^∞_c



$\mathcal{L}$ ha minimo su funzioni discontinue

Esempio (Weierstraß)

$$\mathcal{L}(u) = \frac{1}{2}\int_0^1 x \cdot |u'(x)|^2\,dx \quad , \; u(0)=1,\; u(1)=0$$

$$L(x,y,z) = x \cdot z^2$$

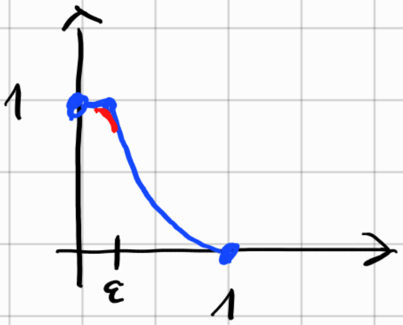
$$\frac{\partial L}{\partial y} = 0 \qquad \frac{\partial L}{\partial z} = 2xz \qquad \text{E.L.} \quad \frac{\partial L}{\partial z} = \text{cost}$$

E.L. $\quad x\,u'(x) = c$

$u'(x) = \frac{c}{x} \qquad u(x) = c\ln x + d$

$u(1) = 0 \qquad d = 0$

$u(0) = 1$ impossibile.



$$u_\varepsilon(x) = \begin{cases} 1 & \text{se } x \le \varepsilon \\ \frac{\ln x}{\ln \varepsilon} & \text{se } x \ge \varepsilon \end{cases}$$

$u_\varepsilon(0) = 1,\; u_\varepsilon(1) = 0 \qquad u_\varepsilon'(x) = \begin{cases} 0 & \text{se } x < \varepsilon \\ \frac{1}{x\ln\varepsilon} & \text{se } x > \varepsilon \end{cases}$

$$\mathcal{L}(u_\varepsilon) = \int_0^\varepsilon x \cdot 0 + \int_\varepsilon^1 x \cdot \left| \frac{1}{x^2 \ln^2 \varepsilon} \right| dx$$

$$= \frac{1}{\ln^2\varepsilon}\int_\varepsilon^1 \frac{1}{x}\,dx = \frac{1}{\ln^2\varepsilon}\left[\ln x\right]_\varepsilon^1 = \frac{-\ln\varepsilon}{\ln^2\varepsilon} = -\frac{1}{\ln\varepsilon} \xrightarrow{\varepsilon\to 0} 0^+$$

$\mathcal{L}(u) \geq 0 \qquad \mathcal{L}(u) = 0 \Rightarrow u'(x) = 0 \quad \forall x > 0$

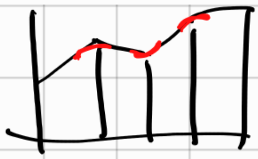
$u \in C^1_p \Rightarrow u$ costante

$\inf_{C^1_p} \mathcal{L} = 0 \qquad \Rightarrow u(0) = u(1)$ non è ammissibile.

anche C^∞

non ha minimo

Lemma [di "allisciamento")



$\forall u \in C^1_p([a,b])$ $\forall \varepsilon > 0$ $\exists u_\varepsilon \in C^1([a,b])$ tale che:

$$\|u_\varepsilon - u\|_\infty \le \varepsilon, \quad \|u_\varepsilon'\|_\infty \le 2(\|u'\|_\infty + 1)$$

$$|\{x \in [a,b] : u_\varepsilon(x) \neq u(x)\}| < \varepsilon.$$

Se L è "abbastanza regolare" $\mathcal{I}(u_\varepsilon) \to \mathcal{I}(u)$.

Esempio

$$\begin{cases} \mathcal{I}(u) = \int_0^1 W(u'(x))\,dx \\ u(0) = 0, \ u(1) = 3 \\ \mathcal{L}(u) \longrightarrow \min. \end{cases}$$

$W(z) = (1-z^2)^2$

$$\frac{\partial L}{\partial z} = W'(z) = 2(1-z^2)(-2z) = -4z(1-z^2)$$

$$\frac{\partial L}{\partial y} = 0$$

(E.L.) $W'(u'(x)) = \text{cost.}$

$u \in C^1$

$\Downarrow$

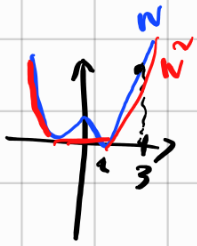
$u'(x) = \text{cost.}$

$u(x) = mx + q$

$u(0) = 0 \Rightarrow q = 0$

$u(1) = 3 \Rightarrow m = 3$

$u(x) = 3x$.



Posso dire che $u(x) = 3x$ è minimo?

$$\tilde{W}(x) = \begin{cases} (1-x^2)^2 & \text{se } |x| \ge 1 \\ 0 & \text{se } |x| \le 1 \end{cases}$$

$\tilde{W}$ è convessa.

$\tilde{W}(z) \le W(z)$

$\forall u$: $\tilde{\mathcal{L}}(u) = \int_0^1 \tilde W(u'(x))\,dx \le \int_0^1 W(u'(x))\,dx = \mathcal{L}(u)$

$u_0(x) = 3x$. $\qquad \tilde{\mathcal{L}}(u_0) = \mathcal{L}(u_0)$ $\quad (*)$

$u_0'(x) = 3 \qquad \tilde W(3) = W(3)$.

u_0 è minimo per $\tilde{\mathcal{L}}$ in quanto $\tilde W$ è convessa

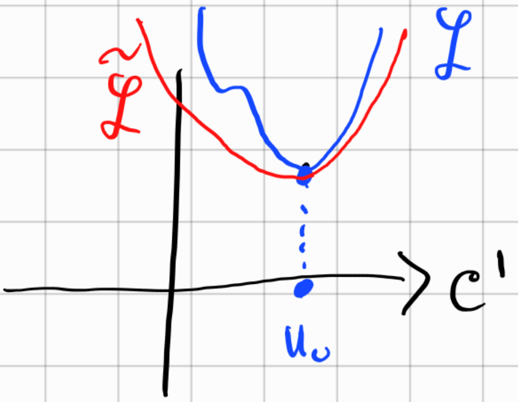
$\Downarrow$

$\forall x \qquad (y,z) \mapsto L(x,y,z)$ è convessa.

$\forall u$ competitore ($u \in C^1$, $u(0)=0$, $u(1)=3$)

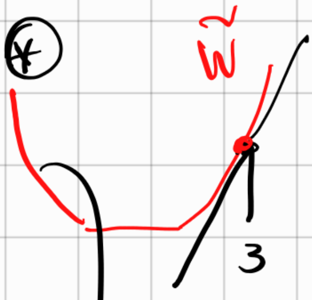
$$\underset{\substack{\uparrow\\ W \ge \tilde W}}{\mathcal{L}(u) \ge} \underset{\substack{\uparrow\\ u_0 \text{ è minimo per } \tilde{\mathcal{L}}}}{\tilde{\mathcal{L}}(u) \ge} \underset{\substack{\uparrow\\ (*)}}{\tilde{\mathcal{L}}(u_0) =} \mathcal{L}(u_0)$$

$\Rightarrow u_0$ è minimo anche per $\mathcal{L}$.



perché u_0 è minimo per $\tilde{\mathcal{L}}$?

$$\tilde{\mathcal{L}}(u) - \tilde{\mathcal{L}}(u_0) = \int_0^1 [\tilde W(u'(x)) - \tilde W(u_0')] = (*)$$



$u_0'(x) = 3$

$$\tilde W(z) \ge \underbrace{\tilde W(3) + \tilde W'(3)\cdot(z-3)}_{\text{retta tangente}}$$

$$(*) \ge \int_0^1 \left[\cancel{\tilde W(3)} + \tilde W'(3)(u'(x)-3) - \cancel{\tilde W(3)}\right]$$

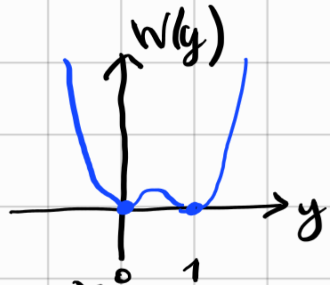
$$\geq \tilde{W}'(3)\left[\int_0^1 u'(x) - \int_0^1 3\right]$$

$$= \tilde{W}'(3)\left(\left[u(1)-u(0)\right] - 3\right)$$

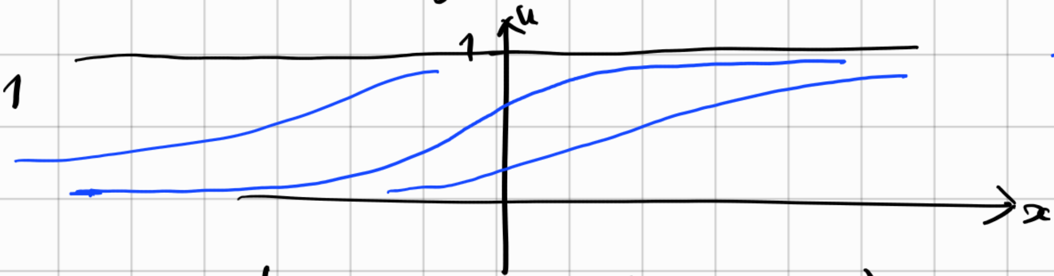
$$= \tilde{W}'(3)\left((3-0)-3\right) = 0$$

Esempio (transizione di fase)

$$\begin{cases} \mathcal{L}[u] = \int_{-\infty}^{+\infty} W(u(x)) + \frac{1}{2}|u'(x)|^2 \\ \lim_{x\to-\infty} u(x) = 0 \\ \lim_{x\to+\infty} u(x) = 1 \end{cases}$$



$$W(y) = \frac{1}{2}\left(y(1-y)\right)^2$$



Eq. Beltrami $\quad L - u'\frac{\partial L}{\partial z} = \text{cost.} \quad L = L(y,z)$

$$L(y,z) = \frac{1}{2}y^2(1-y)^2 + \frac{1}{2}z^2 \qquad \frac{\partial L}{\partial z} = z$$

$$\frac{1}{2}u^2(1-u)^2 - \frac{1}{2}u'\cdot u' = \text{cost}$$

$$\frac{1}{2}u'^2 = \frac{1}{2}u^2(1-u)^2 + \cancel{c}$$

$u \to 1$ per $x \to +\infty$

$(u')^2 \to c \Rightarrow c = 0$

$$u' = |u|\cdot|1-u| \qquad \frac{u'}{u(1-u)} = \pm 1$$

$$\int \frac{du}{u(1-u)} = \pm(x-x_0)$$

$$\frac{1}{u(1-u)} = \frac{1}{u} + \frac{1}{1-u}$$

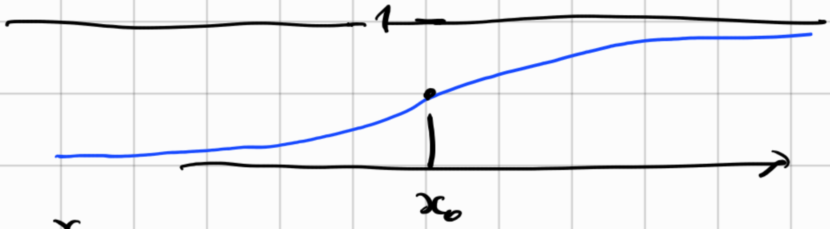
$$\int \frac{du}{u(1-u)} = \ln u - \ln(1-u) = \ln \frac{u}{1-u} = \pm(x-x_0)$$

$$\frac{u}{1-u} = e^{\cancel{\pm}(x-x_0)}$$

$x \to +\infty$ $\quad u \to 1$ $\qquad$ $x \to -\infty$ $\quad u \to 0$

$$u = e^{x-x_0}(1-u)$$

$$u\left(1+e^{x-x_0}\right) = e^{x-x_0} \qquad u(x) = \frac{e^{x-x_0}}{1+e^{x-x_0}}$$



$$u_0(x) = \frac{e^x}{1+e^x} \qquad (\text{wlog } x_0 = 0.)$$

u_0 è minimo ?

$$\mathcal{I}(u) \overset{?}{\geq} \mathcal{I}(u_0)$$

$$\mathcal{I}(u) = \frac{1}{2}\int_{-\infty}^{+\infty} (u(1-u))^2 + (u')^2 \geq \int_{-\infty}^{+\infty} u(x)(1-u(x))u'(x)\,dx$$

$(a-b)^2 \geq 0 \Rightarrow a^2+b^2 \geq 2ab \qquad a^2+b^2 = ab \qquad a=b$

$$= \int_0^1 u(1-u)\,du = \left[\frac{u^2}{2} - \frac{u^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

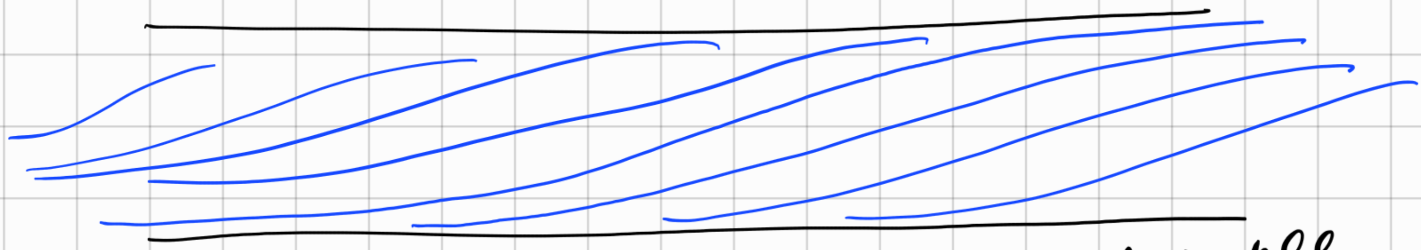
$u = u(x)$
$du = u'(x)\,dx$

$\mathcal{I}(u_0) = \dots$ ma c'è l'uguaglianza $\dots = \frac{1}{6}$

$a = b$

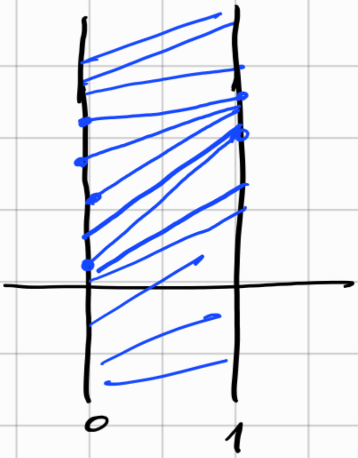
$u(1-u) = u' \leftarrow$ (EC).

Prossima settimana: CALIBRAZIONI



idea: se riesco a "fibrare" lo spazio con i grafici delle soluzioni dell'equazione di E.L. (non imposto il dato al bordo)
e se $\forall\, x, y$ $\quad z \mapsto L(x,y,z)$ è convessa

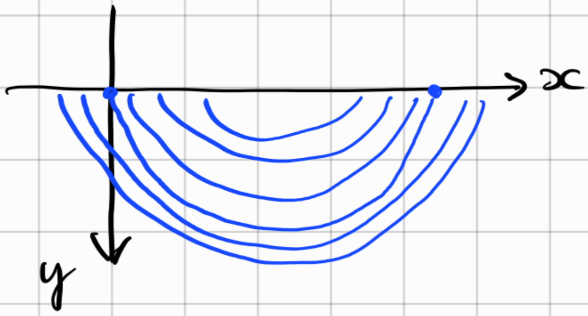
allora ogni tale funzione è minimo con il suo dato al bordo.



ES Geodetiche

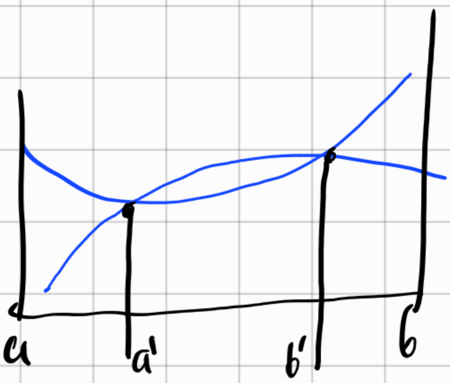
$$\int \sqrt{1+u'^2}$$

ES brachistocrona



$$\int \frac{\sqrt{1+u'^2}}{\sqrt{u}}$$

OSS



Se c'è unicità dei minimi mi aspetto che due minimi non si attraversano
$\Rightarrow$ è sensato pensare che le fibrazioni esistono.