ELEMENTI di CALCOLO delle VARIAZIONI   047AA

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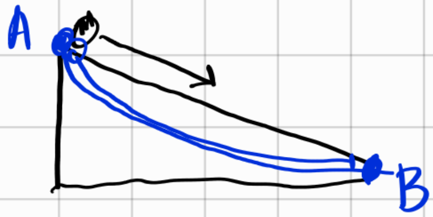
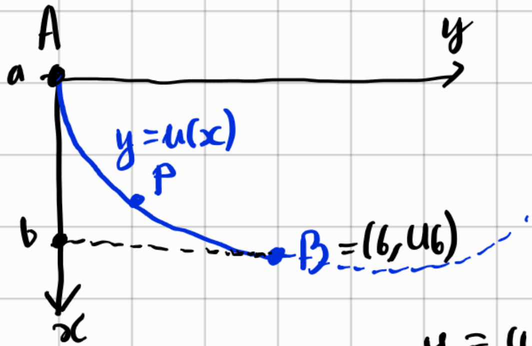
LEZIONE 1   (26.2.2024)

Problema della Brachistocrona



Trovare la curva che congiunge A a B per cui un grave impiega il tempo minimo per raggiungere B



$$\frac{1}{2} m v^2 - m g x = 0$$

E. cinetica   E. potenziale

$$\boxed{\frac{1}{2} v^2 = g \cdot x}$$

$u(a) = u(0) = u_a = 0$

$u(b) = u_b$

$y = u(x)$

$$T = \int_0^T 1 \cdot dt = \int_a^b \frac{1}{\dot{x}}\, dx$$

$$v^2 = 2 g x$$

$$\dot{x}^2 + \dot{y}^2 = 2 g x$$

$$\dot{x}^2 \left(1 + \left(\frac{\dot{y}}{\dot{x}}\right)^2\right) = 2 g x$$

$$\frac{\dot{y}}{\dot{x}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = u'(x).$$

$y = u(x)$

$y(t) = u(x(t))$

$\dot{y} = u'(x) \cdot \dot{x}$

$$\dot{x}^2 = \frac{2 g x}{1 + (u'(x))^2}$$

$$T = \int_0^b \boxed{\frac{\sqrt{1 + (u'(x))^2}}{\sqrt{2 g x}}}\, dx$$

Funzionale classico del calcolo delle variazioni:

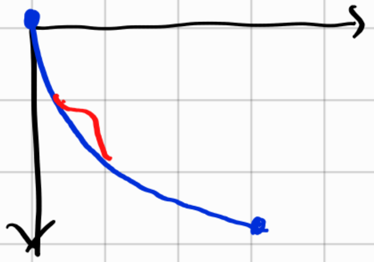
$$\mathcal{L}(u) = \int_a^b L(x, u(x), u'(x))\, dx \qquad a < b$$

$$\begin{cases} \text{Trovare } u \text{ che rende minimo } \mathcal{L} \quad u \in C^1([a,b]) \qquad L = L(x,y,z) \\ \text{e } u(a) = u_a, \quad u(b) = u_b \end{cases}$$

Nel nostro caso: $a=0$, $b>0$, $u_a=0$, $u_b>0$

$$L(x,y,z) = \frac{\sqrt{1+z^2}}{\sqrt{2gx}}$$

<u>Idea</u>:

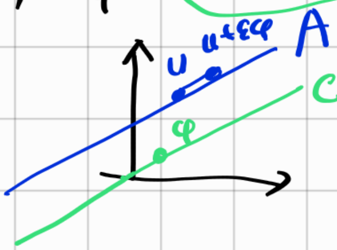


$$\left.\begin{array}{l}\varphi : C^1([a,b]) \\ \varphi(a) = \varphi(b) = 0\end{array}\right\} \varphi \in C^1_0([a,b])$$

$$A = \{u \in C^1([a,b]) : u(a) = u_a, \ u(b) = u_b\}$$

u_a, u_b fissati

$u \in A \Rightarrow \underbrace{u + \varepsilon \cdot \varphi}_{} \in A$ se $\varepsilon \in \mathbb{R}$, $\varphi \in C^1_0([a,b])$.



Se u è un minimo di $\mathcal{L}$ su A allora $\forall \varepsilon$

$$\mathcal{L}(u) \leq \mathcal{L}(u + \varepsilon \varphi) \quad \text{ovvero}$$

$$\varepsilon \longmapsto \mathcal{L}(u + \varepsilon\varphi) \quad \text{ha minimo per } \varepsilon = 0$$
$$\mathbb{R} \longrightarrow \mathbb{R}$$

Questa funzione, se è derivabile, avrà quindi derivata nulla in $\varepsilon = 0$.

$$\left[\frac{d}{d\varepsilon}\mathcal{L}(\underbrace{u+\varepsilon\varphi}_{})\right]_{\varepsilon=0} = \left[\frac{d}{d\varepsilon}\int_a^b L(x, u(x)+\varepsilon\varphi(x), u'(x)+\varepsilon\varphi'(x))\,dx\right]_{\varepsilon=0}$$

$$= \left[\int_a^b \frac{d}{d\varepsilon} L(x, u+\varepsilon\varphi, u'+\varepsilon\varphi')\, dx \right]_{\varepsilon=0}$$

$$= \left(\int_a^b \left[\frac{\partial L}{\partial y}(x, u+\varepsilon\varphi, u'+\varepsilon\varphi') \cdot \varphi(x) + \frac{\partial L}{\partial z}(x, u+\varepsilon\varphi, \underbrace{u'+\varepsilon\varphi'}) \cdot \varphi'(x) \right] dx \right)_{\varepsilon=0}$$

$\varepsilon=0$

$$= \int_a^b \left[\frac{\partial L}{\partial y}(x, u, u') \cdot \varphi(x) + \frac{\partial L}{\partial z}(x, u, u') \cdot \varphi'(x) \right] dx.$$

derivo — integro

integro per parti

$$= \int_a^b \left[\frac{\partial L}{\partial y}(x, u, u') \cdot \varphi(x) - \frac{d}{dx}\left(\frac{\partial L}{\partial z}(x, u, u') \right) \cdot \varphi(x) \right] dx$$

$$+ \cancel{\left[\frac{\partial L}{\partial z}(x, u, u') \cdot \varphi(x) \right]_a^b} \qquad \left| \begin{array}{l} \varphi(b) = 0 \\ \varphi(a) = 0 \end{array} \right.$$

$$= \int_a^b \underbrace{\left[\frac{\partial L}{\partial y}(x, u, u') - \frac{d}{dx} \frac{\partial L}{\partial z}(x, u(x), u'(x)) \right]} \cdot \varphi(x)\, dx$$

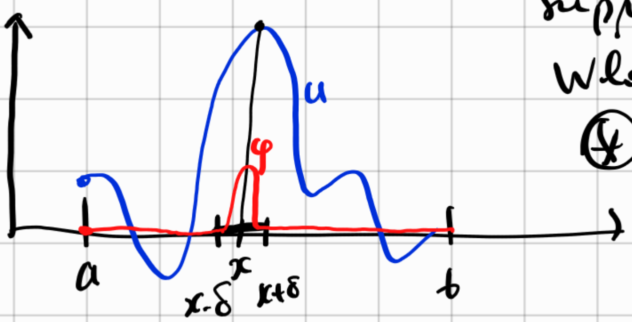
$$= 0 \quad (\text{se } u \text{ è un minimo}) \qquad \forall \varphi \in \underline{C_0^1(a,b)}$$

Lemma (fondamentale del Calcolo delle variazioni)

Se $u \in C^0(a,b)$ e $\int_a^b u \cdot \varphi = 0 \quad \forall \varphi \in C_c^\infty((a,b)) \subseteq C_0^1(a,b)$

Allora $u(x) = 0 \quad \forall x \in [a,b]$.

dim



Supponiamo $u(x_0) \neq 0 \quad x_0 \in (a,b)$

wlog: $u(x_0) > 0$

(*) Se trovo φ tale che

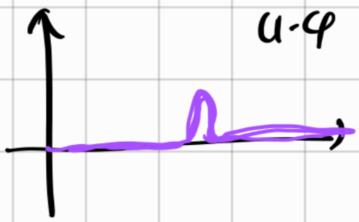
$\{\varphi(x) \neq 0\} \subseteq [x_0 - \delta, x_0 + \delta]$

$u(x) > 0$ su tutto $[x_0 - \delta, x_0 + \delta]$

$\varphi(x) \geq 0 \;\; \forall x \in [a,b)$, $\varphi(x_0) > 0$

Allora $\int_a^b u(x)\cdot\varphi(x)\,dx = \int_{x_0-\delta}^{x_0+\delta} u\cdot\varphi \, dx > 0$

$\uparrow$ continua, ≥ 0, > 0 in x_0

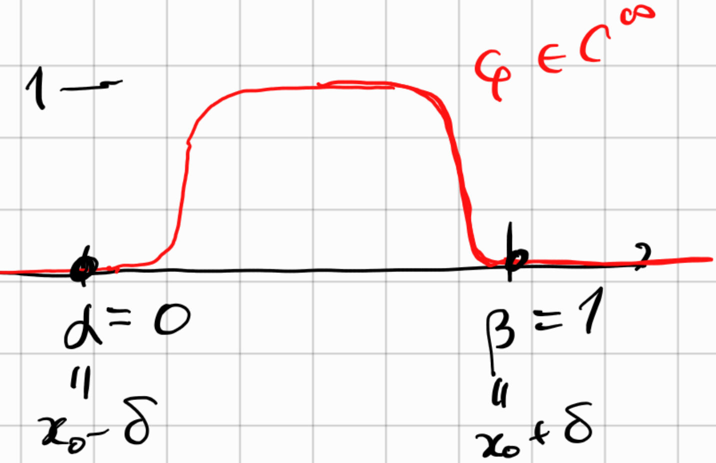


assurdo!

cioè $u(x_0) = 0 \quad \forall x_0 \in (a,b)$

$u \in C^0 \Rightarrow u(x) = 0 \quad \forall x \in [a,b]$.

(*) Trovare $\varphi \in C_c^\infty(a,b)$



$\varphi \in C^\infty$ ma non può essere analitica.

$$\varphi(x) = \begin{cases} e^{-\frac{1}{x(1-x)}} & \text{se } x \in (0,1) \\ 0 & \text{se } x \notin (0,1) \end{cases}$$

$\varphi(x) = e^{R(x)}$ $\qquad R(x) = -\frac{1}{x(1-x)}$ è razionale

$\varphi'(x) = R'(x)\, e^{R(x)} \longrightarrow 0$ per $x \to 0^+$ e per $x \to 1^-$

$\varphi''(x) = \left(R''(x) + R'(x)^2\right) e^{R(x)} \longrightarrow 0$

$\vdots$

φ è C^∞.

ζ

Torniamo alla nostra variazione:

$$\int_a^b \left[\frac{\partial L}{\partial y}(x,u,u') - \frac{d}{dx} \frac{\partial L}{\partial z}(x,u,u') \right] \cdot \varphi(x)\, dx = 0 \qquad \forall \varphi \in C_0^1(a,b)$$

lemma fondamentale $\Rightarrow$ se L è C^2, $u \in C^1$ $\Rightarrow x \mapsto [\text{-------}]$ è C^0.

EQUAZIONE di EULERO-LAGRANGE:

$$\boxed{\frac{\partial L}{\partial y}(x,u,u') = \frac{d}{dx} \frac{\partial L}{\partial z}(x,u,u')}$$

[In astratto $\frac{\partial \mathcal{L}}{\partial \varphi}(u) = \lim_{\varepsilon \to 0} \left. \frac{\mathcal{L}(u+\varepsilon\varphi) - \mathcal{L}(u)}{\varepsilon} \right|_{\varepsilon=0} = 0$ ↑ derivate direzionali

variazione di $\mathcal{L} \longrightarrow \delta\mathcal{L}(u) = 0$.]

Torniamo alla brachistocrona

$$L(x,y,z) = \frac{\sqrt{1+z^2}}{\sqrt{2gx}} \qquad \frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial z} = \frac{1}{\sqrt{2gx}} \cdot \frac{z}{\sqrt{1+z^2}}$$

E.L.

$$0 = \frac{d}{dx} \frac{1}{\sqrt{2gx}} \cdot \frac{u'(x)}{\sqrt{1+u'(x)^2}}$$

$$\frac{1}{\sqrt{\cancel{2g}\cdot x}} \cdot \frac{u'(x)}{\sqrt{1+(u'(x))^2}} = c$$

$$(u'(x))^2 = c^2 x (1+(u'(x))^2)$$

$$(u'(x))^2 \cdot (1-c^2 x) = c^2 x$$

$$u'(x) = \sqrt{\frac{c^2 x}{1-c^2 x}}$$

$$u(x) = c\int \frac{\sqrt{x}}{\sqrt{1-c^2x}}\, dx$$

$$= c\int \frac{x}{\sqrt{x-c^2x^2}}\, dx = -\frac{1}{c}\int \frac{-c^2x}{\sqrt{x-c^2x^2}}\, dx =$$

$$\left[D\left(\sqrt{x-c^2x^2}\right) = \frac{1}{2}\frac{1-2c^2x}{\sqrt{x-c^2x^2}} = \frac{1}{2}\frac{1}{\sqrt{x-c^2x^2}} \boxed{-\frac{c^2x}{\sqrt{x-c^2x^2}}} \right]$$

$$= -\frac{1}{c}\sqrt{x-c^2x^2} + \frac{1}{2c}\int \frac{1}{\sqrt{x-c^2x^2}}\, dx =$$

$$\rightsquigarrow \frac{1}{\sqrt{1-s^2}}$$

$$\left[\sqrt{x-c^2x^2} = \sqrt{\frac{1}{4c^2} - \left(\frac{1}{2c} - cx\right)^2} = \frac{1}{2c}\cdot\sqrt{1-(1-2c^2x)^2} \right.$$

doppio prodotto — quadrato — completamento del quadrato

$$\left. \boxed{D \arcsin(1-2c^2x) = \frac{-2c^2}{\sqrt{1-(1-2c^2x)^2}}} \right]$$

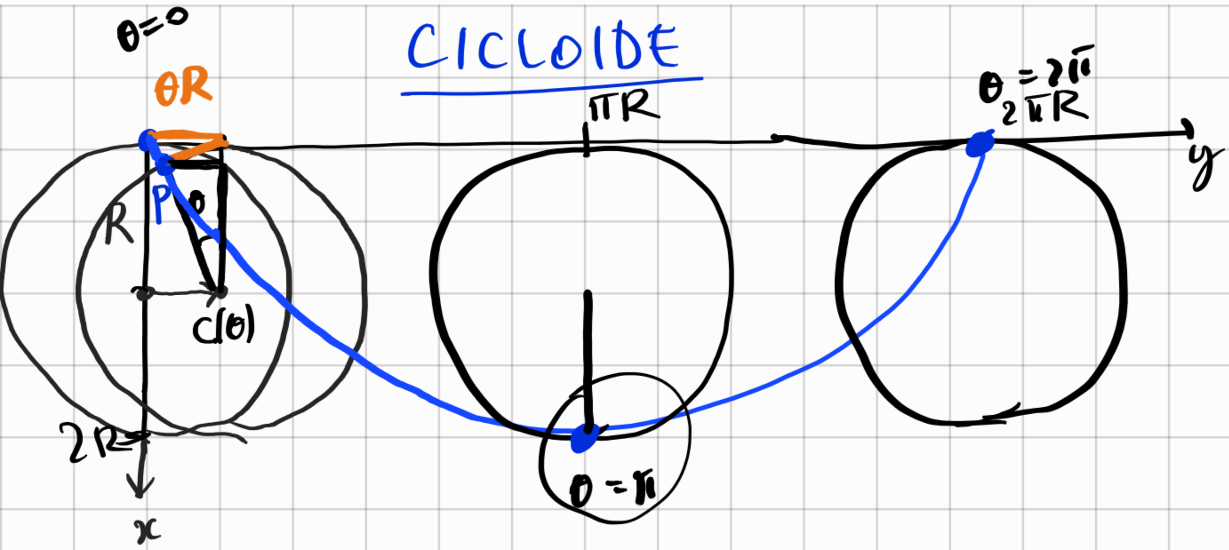
$$= -\frac{1}{c}\frac{1}{2c}\sqrt{1-(1-2c^2x)^2} - \frac{1}{2c^2}\int \frac{-2c^2}{\sqrt{1-(1-2c^2x)^2}}\, dx$$

$$= -\frac{1}{2c^2}\sqrt{1-(1-2c^2x)^2} - \frac{1}{2c^2}\cdot\underbrace{\left(\arcsin(1-2c^2x) - \frac{\pi}{2}\right)}_{-\arccos}$$

$u(0) = 0$

$$= \frac{1}{2c^2}\left[\arccos(1-2c^2x) - \sqrt{1-(1-2c^2x)^2}\right]$$

CICLOIDE



$$P = P(\theta) = (x(\theta), y(\theta))$$

$$C(\theta) = (R, \theta R)$$

$$P = (R - R\cos\theta,\ \theta R - R\sin\theta)$$
$$= (x,\ u(x))$$

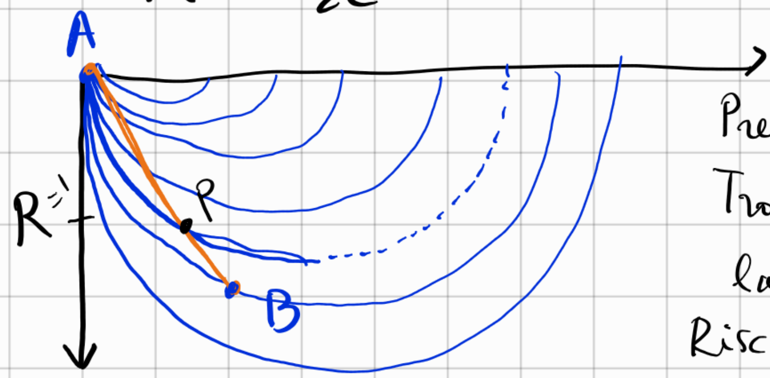
$$\begin{cases} x = R - R\cos\theta \\ y = \theta R - R\sin\theta \end{cases} \qquad \begin{cases} \theta = \arccos\dfrac{R-x}{R} = \arccos\left(1-\dfrac{x}{R}\right) \\ y = R\cdot\arccos\left(1-\dfrac{x}{R}\right) - R\sqrt{1-\left(1-\dfrac{x}{R}\right)^2} \end{cases}$$

$$u(x) = R\arccos\left(1-\frac{x}{R}\right) - R\sqrt{1-\left(1-\frac{x}{R}\right)^2}$$

$$= \frac{1}{2c^2}\left[\arccos(1-2c^2x) - \sqrt{1-(1-2c^2x)^2}\right]$$

Con: $R = \dfrac{1}{2c^2}$.

Come trovo R in modo che $u(b) = u_b$



Es

Prendo una cicloide $(R=1)$

Trovo P intersezione tra la cicloide e $\overline{AB}$

Riscalo $R = \dfrac{\overline{AB}}{\overline{AP}}$.

□

<u>Questioni</u>: esiste il minimo?
regolarità di u e di L?