ANALISI MATEMATICA B
LEZIONE 51 - 4.2.2022

Polinomio di Taylor:

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

Formula di Taylor (Peano):

$$f(x) = P_n(x) + R_n(x) \quad \text{e} \quad \frac{R_n(x)}{(x-x_0)^n} \to 0.$$

Notazione di Landau: o-piccolo e O-grande.

ES $R_1(x)$ $\qquad \dfrac{R_1(x)}{x^3} \to 0$ per $x \to x_0$.

Def diremo che $\boxed{f(x) = o(g(x))}$ se $\dfrac{f(x)}{g(x)} \to 0$

$f(x) << g(x)$

Nell' ES $R_1(x) = o(x^3)$

Formula di Taylor: $\boxed{f(x) = P_n(x) + o((x-x_0)^n)}$

significa $f(x) - P_n(x) = o((x-x_0))^n$

cioè $\displaystyle\lim_{x \to x_0} \frac{f(x) - P_n(x)}{(x-x_0)^n} = 0.$

def $\quad f(x) = O(g(x))$ se $\displaystyle\limsup_{x \to x_0} \left|\frac{f(x)}{g(x)}\right| < +\infty.$

ovvero esiste U intorno di x_0, $\exists M$ t.c.

$$\forall x \in U: \left|\frac{f(x)}{g(x)}\right| \le M$$

Es
$$f(x) = P_n(x) + O\left((x-x_0)^{n+1}\right)$$

infatti:

$$f(x) = P_n(x) + \boxed{\frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1} + o\left((x-x_0)^{n+1}\right)}$$

$$R(x) = \frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1} + o\left((x-x_0)^{n+1}\right)$$

$$\frac{R(x)}{(x-x_0)^{n+1}} = \frac{f^{(n+1)}(x_0)}{(n+1)!} + \underbrace{\cancel{\frac{o\left((x-x_0)^{n+1}\right)}{(x-x_0)^{n+1}}}}_{\to 0}$$

Es
$$\lim_{x\to 0} \frac{\cos x \cdot \sin^2 x - 2 + 2\cos x}{(x \cdot \operatorname{tg} x)^2}$$

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$ per $x \to 0$

$$\cos x = 1 - \frac{x^2}{2} + o(x^3)$$

$$\operatorname{tg} x = x + o(x)$$

$$\boxed{\frac{\cos x \cdot \sin^2 x - 2 + 2\cos x}{(x \cdot \operatorname{tg} x)^2}} =$$

$$\sin^2 x = x^2 - \frac{x^4}{3} + o(x^4)$$

$$(\sin^2)^{IV}(0) = -8$$

$$\boxed{x \to 0}$$

$$= \frac{\left(1 - \frac{x^2}{2} + o(x^3)\right)\cdot\left(x - \frac{x^3}{6} + o(x^3)\right)^2 - 2 + 2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right)}{x^2\left(x + o(x)\right)^2}$$

$$= \frac{\left(1 - \frac{x^2}{2} + o(x^3)\right)\cdot\left(x^2 - \frac{x^4}{3} + o(x^4)\right) \cancel{-2} \cancel{+2} - x^2 + \frac{x^4}{12} + o(x^4)}{x^2 \cdot \left(x^2 + o(x^2)\right)} =$$

$$2x \, o(x^3) = 2 \cdot o(x^4) = o(x^4)$$

$$-\frac{x^3}{3} o(x^3) = o(x^6) \overset{\to}{=} o(x^4)$$

$$\frac{o(x^6)}{x^4} = \frac{o(x^6)}{x^6} \cdot x^2 \to 0$$

$$f(x) \cdot o(g(x)) \overset{?}{=} o(f(x) \cdot g(x))$$

$$\frac{\cancel{f(x)} \cdot o(g(x))}{\cancel{f(x)} \cdot g(x)} \to 0$$

$$c \cdot o(g(x)) = o(g(x))$$

$$c \cdot \frac{o(g(x))}{g(x)} \to 0$$

$$\underbrace{o(x^4)} + o(x^4) = o(x^4)$$

$$\to o(x^4) + o(x^6) = o(x^4)$$

$$o(f(x))^n = o(f(x)^n)$$

$$o(f(x)) \cdot o(g(x)) = o(f(x) g(x))$$

$$o(x^3)^2 = o(x^6)$$

$$\frac{x^6}{36} = o(x^4)$$

$$\frac{\frac{x^6}{36}}{x^4} = \frac{x^2}{36} \to 0$$

$$\sin^2 x = x^2 - \frac{x^4}{3} + o(x^4) = \left(x^2 - \cancel{\frac{x^4}{3}} + o(x^3)\right) = x^2 + o(x^3)$$

$$= \frac{\left(1-\frac{x^2}{2}+o(x^3)\right)\cdot\left(x^2-\frac{x^4}{3}+o(x^4)\right)-x^2+\frac{x^4}{12}+o(x^4)}{x^4+o(x^4)}$$

$$= \frac{\cancel{x^2}-\frac{x^4}{3}-\frac{x^4}{2}+o(x^4)\cancel{-x^2}+\frac{x^4}{12}+o(x^4)}{x^4+o(x^4)}$$

$$= \frac{-\frac{3}{4}x^4+o(x^4)}{x^4+o(x^4)} = \frac{\cancel{x^4}\left(-\frac{3}{4}+\underbrace{\frac{o(x^4)}{x^4}}_{0}\right)}{\cancel{x^4}\left(1+\underbrace{\frac{o(x^4)}{x^4}}_{0}\right)} \to -\frac{3}{4}$$

$$\frac{o(x^4)}{x^4} = o(1)$$

o

Es 1 II compitin
anno scorso

$$\lim_{x\to 0} \frac{\operatorname{arctg}(\sin x) - x\cdot\cos x}{\operatorname{tg}\left(x-\frac{x^3}{6}-\sin x\right)\cdot \operatorname{arctg}\cos x}$$

$$\sin x = x - \frac{x^3}{6}+\frac{x^5}{120}+o(x^5)$$

$$\cos x = 1-\frac{x^2}{2}+\frac{x^4}{24}+o(x^4)$$

$$\to \operatorname{arctg} y = \underline{y+o(y)} \quad 4 \quad \text{per } y\to 0$$

$$\operatorname{arctg} \sin x = \sin x + o(\sin x)$$

$$= x \cancel{-\frac{x^3}{6}} \cancel{+o(x^4)} + \underbrace{o\left(x \cancel{-\frac{x^3}{6}} \cancel{+o(x^4)}\right)}$$

$$o\left(x-\frac{x^3}{6}+o(x^4)\right) = o(x+o(x)) = o(x\cdot(1+o(x)))$$

$$= o(x)\cdot o(\cancel{1+o(x)})$$

Verifica:

$$o\left(x-\frac{x^3}{6}+o(x^4)\right) \overset{?}{=} o(x)$$

$$\frac{o\left(x-\frac{x^3}{6}+o(x^4)\right)}{x} = \boxed{\frac{o\left(x-\frac{x^3}{6}+o(x^4)\right)}{x-\frac{x^3}{6}+o(x^4)}} \cdot \frac{x-\frac{x^3}{6}+o(x^4)}{x} \to 0$$

$\to 0$, $\to 1$

Ci serve uno sviluppo più lungo dell'arctg x:

$f(x) = \text{arctg } x$ per $x \to 0$ — $f(0) = 0$

$f'(x) = (1+x^2)^{-1}$ — $f'(0) = 1$

$f''(x) = -2(1+x^2)^{-2} \cdot x$ — $f''(0) = 0$

$f'''(x) = 8(1+x^2)^{-3}x^2 - 2(1+x^2)^{-2}$ — $f'''(0) = -2$

$f^{IV}(x) = -48(1+x^2)^{-4}x^3 + 24(1+x^2)^{-3}x$ — $f^{IV}(0) = 0$

$f^{V}(x) = 384(1+x^2)^{-5}x^4 - 144(1+x^2)^{-4}x^2$

$\quad - 144(1+x^2)^{-4}x^2 + 24(1+x^2)^{-3}$

$f^{V}(0) = 24$

$$\text{arctg } y = y - \frac{1}{3}y^3 + \frac{1}{5}y^5 + o(y^5)$$

$$\text{arctg } \sin x = \sin x - \frac{1}{3}\sin^3 x + \frac{1}{5}\sin^5 x + o(x^5)$$

$f(x) \sim g(x)$ $\quad$ $o(f(x)) = o(g(x))$

$$x^2 + x^3 - 4x^4 + o(x^4) \sim x^2$$