ON THE TOTAL SURFACE AREA OF POTATO PACKINGS

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Abstract. We prove that if we fill without gaps a bag with infinitely many potatoes, in such a way that they touch each other in few points, then the total surface area of the potatoes must be infinite. In this context potatoes are measurable subsets of the Euclidean space, the bag is any open set of the same space. As we show, this result also holds in the general context of doubling (even locally) metric measure spaces satisfying Poincaré inequality, in particular in smooth Riemannian manifolds and even in some sub-Riemannian spaces.

Introduction

In this short note we prove the following curious fact: if we fill without gaps a bag (modeled, say, by an arbitrary open connected subset of $\mathbb{R}^d$) with potatoes (modeled, in this case, by open sets of finite perimeter), in such a way that the potatoes touch each other in a single point (more generally in a set of zero surface measure), see Figure 1, then the total surface area of the potatoes is infinite. We show that this very simple result is in fact true for a fairly large class of metric measure spaces, where the perimeter of a set can be naturally defined.

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Figure 1. A potato bag.

This result implies in particular that the residual set (the complement of the union of the potatoes) has Hausdorff dimension at least $d-1$. Theorem 1.4(ii) from [7] shows that this statement is sharp: in fact it proves the existence of a packing of a planar convex set by planar strictly convex sets for which the dimension of the residual set is exactly 1.

In this way we generalise the results of [7] dedicated to packing of convex sets which in their turn generalise those from the series of papers [5, 3, 4, 6] and [8] dedicated to packings of spheres.

1. Notation and preliminaries

We work here with metric measure spaces $(X, \mathrm{d}, \mathfrak{m})$ where X is a nonempty set equipped with a distance d and a possibly σ-finite Borel measure $\mathfrak{m}$ with $\mathfrak{m}(X) \neq 0$. For a fairly general class of metric measure spaces, namely those with a doubling measure and satisfying the Poincaré inequality, the perimeter functional has been defined in [2]. This includes, of course, the classical example $X = \mathbb{R}^d$ with the Euclidean distance d, the Lebesgue measure $\mathfrak{m}$ and the Caccioppoli perimeter P.

We say that a function F defined on the borel sets of a metric space X with values in $[0, +\infty]$ is a *perimeter-like evaluation* if it satisfies the following properties:

(0) $F(\emptyset) = 0$;
(T) $F(A) \geq \limsup_n F(A_n)$ whenever $A_n \subset A$ and $F(A \setminus A_n) \to 0$;
(C) $F(X \setminus A) = F(A)$;
(L) $F(A) \leq \liminf_n F(A_n)$ if $\lim_n \mathfrak{m}(A_n \triangle A) = 0$ (one says that F is lower semicontinuous with respect to $L^1(X, \mathfrak{m})$ convergence of sets);
(Z) if $\mathfrak{m}(A \triangle B) = 0$ then $F(A) = F(B)$.

Lemma 1.1. *The property (T) is valid if*

(T') *there exists a $c > 0$ such that $F(A) \geq F(A \setminus B) - cF(B)$ whenever $B \subset A$.*

Proof. Just notice that

$$\begin{aligned} F(A) &= F(A_n \cup (A \setminus A_n)) \\ &\geq F(A_n) - cF(A \setminus A_n) && \text{by } (T') \end{aligned}$$

and if $F(A \setminus A_n) \to 0$, then we have (T). □

2. Abstract results

Proposition 2.1. *If F satisfies properties (0), (C), and the family of sets E_k, $k \in \mathbb{N}$ is such that $\bigcup E_i = X$, and*

$$F\left(\bigcup_j E_j\right) \geq \sum_j F(E_j)$$

then $F(E_i) = 0$ for all $i \in \mathbb{N}$.

If, additionally, F satisfies (Z) then the same holds true if $\mathfrak{m}(X \setminus \bigcup E_i) = 0$ rather than $\bigcup E_i = X$.

Proof. One has

$$\begin{aligned} 0 &= F(\emptyset) && \text{by (0)} \\ &= F(X) && \text{by (C)} \\ &= F\left(\bigcup E_i\right) && \text{by assumption} \\ &\geq \sum_i F(E_i) && \text{by assumption} \end{aligned}$$

hence $F(E_i) = 0$ for all $i \in \mathbb{N}$.

If, additionally, F satisfies (Z), and $\mathfrak{m}(X \setminus \bigcup E_i) = 0$ one has

$$\begin{aligned} 0 &= F(\emptyset) = F(X) \\ &= F\left(X \setminus \bigcup E_i\right) && \text{by (Z)} \\ &= F\left(\bigcup E_j\right) && \text{by (C)} \\ &\geq \sum F(E_j) && \text{by assumption} \end{aligned}$$

so that $F(E_j) = 0$ for all j. □

Theorem 2.2. *If F satisfies properties (0), (C), (T), (L) and the family of sets E_k, $k \in \mathbb{N}$ is such that $F(E_i \cup E_j) = F(E_i) + F(E_j)$ for $i \neq j$, $\bigcup_{i=0}^{+\infty} E_i = X$, and $\mathfrak{m}\left(\bigcup_{i=1}^{+\infty} E_i\right) < +\infty$, then either $\sum_{i=0}^{+\infty} F(E_i) = +\infty$ or $F(E_i) = 0$ for all $i \in \mathbb{N}$.*

If, additionally, F satisfies (Z) then the same holds true if $\mathfrak{m}(X \setminus \bigcup_{i=0}^{+\infty} E_i) = 0$ for all $i \neq j$, in place of $\bigcup E_i = X$.

Proof. Let

$$T_n := \bigcup_{i=n+1}^{+\infty} E_j, \qquad T_n^m := \bigcup_{i=n+1}^{m} E_j.$$

Clearly $T_n^m \nearrow T_n$ as $m \to +\infty$, hence $\lim_m \mathfrak{m}(T_n^m) = \mathfrak{m}(T_n)$. Since $\mathfrak{m}(T_n) \leq \mathfrak{m}\left(\bigcup_{i=1}^{+\infty} E_i\right) < +\infty$ then we have that $\mathfrak{m}(T_n^m \triangle T_n) \to 0$ as $m \to +\infty$. Therefore

$$\begin{aligned} F(T_n) &\leq \liminf_m F(T_n^m) && \text{by (L)} \\ &= \liminf_m \sum_{i=n+1}^{m} F(E_i) && \text{by assumption} \\ &= \sum_{i=n+1}^{+\infty} F(E_i). \end{aligned} \tag{1}$$

If $\sum_{i=1}^{+\infty} F(E_i) = +\infty$ the proof is concluded. Otherwise the righthand side of (1) is the remainder of a convergent number series and hence $F(T_n) \to 0$ as $n \to +\infty$.

We have then

$$\begin{aligned} F\left(\bigcup_{i=0}^{+\infty} E_i\right) &\geq \limsup_n F\left(\bigcup_{i=0}^{n} E_i\right) && \text{by (T)} \\ &= \lim_n \sum_{i=0}^{n} F(E_i) && \text{by assumption} \\ &= \sum_{i=0}^{+\infty} F(E_i). \end{aligned}$$

The proof is then concluded using the previous proposition. □

3. Applications

In [1, 2] the perimeter functional P has been defined for a fairly general class of metric measure spaces $(X, \mathrm{d}, \mathfrak{m})$ satisfying doubling condition and the weak $(1,1)$-Poincaré inequality. It has been further shown there that for a Borel set $E \subset X$ the perimeter of E is given by

$$P(E) = \int_{\partial^e E} \theta_E(x)\, d\mathcal{H}^{-1}(x)$$

where $\mathcal{H}^{-1}$ is the so called codimension 1 Hausdorff measure, $\partial^e E$ is the essential boundary of E, and θ_E is a positive Borel function concentrated on $\partial^e E$ bounded from below and from above. The precise definitions of all these objects can be looked up in the cited paper [2]. We are interested here in a subclass of PI spaces, called *isotropic* in [2], where one has $\theta_E(x) = \theta_F(x)$ for H^{-1}-a.e. $x \in \partial^e E \cap \partial^e F$, whenever $F \subset E$. It is important to know that this includes a lot of classical cases, with the usual perimeter functionals, like for instance

(1) the Euclidean space $X = \mathbb{R}^d$ (or X open subset of $\mathbb{R}^d$) with the Lebesgue measure (in this case $\mathcal{H}^{-1}$ is the classical $(d-1)$-dimensional Hausdorff measure and $\partial^e E$ is equivalent to the reduced boundary of E as defined by De Giorgi),
(2) X a finite dimensional space equipped with any anisotropic norm and the Lebesgue measure, or even some of the so-called *RCD* metric measure spaces,
(3) X a Heisenberg group of any dimension, or even some of the more general Carnot groups.

For the definition and properties of the perimeter, $\mathcal{H}^{-1}$ and the essential boundary $\partial^e E$ in metric measure spaces we refer to [1, 2].

Following [2] we call the *PI* metric measure space $(X, \mathrm{d}, \mathfrak{m})$ *isotropic* if for all Borel sets $F \subset E$ of finite perimeter in X, one has

$$\theta_E(x) = \theta_F(x) \quad \mathcal{H}^{-1}\text{-a.e. on } \partial^e E \cap \partial^e F.$$

Lemma 3.1. *Let $(X, \mathrm{d}, \mathfrak{m})$ be an isotropic PI space. Then the perimeter functional P is a perimeter-like evaluation, i.e. it satisfies all the properties (0), (T), (C), (L), and (Z).*

Proof. Property (0) follows from $\partial^e \emptyset = \emptyset$. Proposition 1.7 (i), (ii), and (vi) in [2] shows (Z), (L), and (C) respectively.

Let us prove (T') which by Lemma 1.1 implies (T). By theorem 1.23 in [2] we have

$$P(E) = \int_{\partial^e E} \theta_E(x)\, d\mathcal{H}^{-1}(x)$$

and $\theta_E(x) \in [\alpha, \beta]$ where $0 < \alpha \leq \beta$ are constants depending only on the metric measure space $(X, \mathrm{d}, \mathfrak{m})$ but not on the set E. Then, for Borel sets $B \subset A \subset X$, one has

$$\begin{aligned}
\partial^e(A \setminus B) &= \emptyset \cup \partial^e(A \setminus B) = \partial^e X \cup \partial^e(A \setminus B) \\
&= \partial^e(A \cup B^c) \cup \partial^e(A \cap B^c) \\
&\subset \partial^e A \cup \partial^e B^c && \text{by proposition 1.16(ii) in [2]} \\
&= \partial^e A \cup \partial^e B && \text{by proposition 1.16(i) in [2].}
\end{aligned}$$

Notice also that $\theta_{A\setminus B}(x) = \theta_A(x)$ for $\mathcal{H}^{-1}$-a.e. $x \in \partial^e A \cap \partial^e(A \setminus B)$ since X is isotropic. Hence

$$\begin{aligned}
P(A \setminus B) &= \int_{\partial^e(A\setminus B)} \theta_{A\setminus B}(x)\, d\mathcal{H}^{-1}(x) \\
&= \int_{\partial^e A \cap \partial^e(A\setminus B)} \theta_{A\setminus B}(x)\, d\mathcal{H}^{-1}(x) + \int_{(\partial^e B\setminus \partial^e A)\cap \partial^e(A\setminus B)} \theta_{A\setminus B}(x)\, d\mathcal{H}^{-1}(x) \\
&\leq \int_{\partial^e A \cap \partial^e(A\setminus B)} \theta_A(x)\, d\mathcal{H}^{-1}(x) + \int_{\partial^e B} \beta\, d\mathcal{H}^{-1}(x) \\
&\leq P(A) + \frac{\beta}{\alpha} P(B).
\end{aligned}$$

showing (T') and thus concluding the proof. $\square$

Combining Theorem 2.2 with Lemma 3.1 we obtain the following result.

Theorem 3.2. *Let $(X, \mathrm{d}, \mathfrak{m})$ be an isotropic PI space. Let E_k, $k \in \mathbb{N}$, be a sequence of Borel subsets of X such that*

(i) $\mathfrak{m}(E_k \cap E_j) = 0$ *for* $k \neq j$*;*
(ii) $\mathfrak{m}\left(X \setminus \bigcup E_k\right) = 0$*;*
(iii) $\mathcal{H}^{-1}(\partial^e E_k \cap \partial^e E_j) = 0$ *for all* $k \neq j$*.*
(iv) $\mathfrak{m}(E_0) > 0$ *and* $\mathfrak{m}(E_1) > 0$*.*

Then

$$\sum_k P(E_k) = +\infty.$$

Proof. Conditions (i) and (iii) imply

$$P(E_k \cup E_j) = P(E_k) + P(E_j)$$

in view of Lemma 2.3(ii) in [2]. The assumptions of Theorem 2.2 are therefore satisfied by P in view of Lemma 3.1. Hence by Theorem 2.2 we conclude that either $\sum P(E_k) = +\infty$ or $P(E_k) = 0$ for all k. But the latter option is excluded by (iv) in view of the relative isoperimetric inequality for PI spaces, provided in theorem 1.17 of [2]. $\square$

Corollary 3.3. *Let $\mathfrak{m}$ be the Lebesgue measure on $\mathbb{R}^d$. Let $\mathcal{F}$ be a family of at least two measurable nonempty subsets of $\mathbb{R}^d$, each with positive volume $\mathfrak{m}$ and kissing each other on sets of zero $\mathcal{H}^{d-1}$ measure, i.e.*

$$\mathcal{H}^{d-1}(\bar{A} \cap \bar{B}) = 0, \qquad \text{for every } A, B \in \mathcal{F}, A \neq B. \tag{2}$$

In particular this assumption is satisfied if the sets in $\mathcal{F}$ are strictly convex.

Let $B \subset \mathbb{R}^d$ be an open ball. If $\sum_{E\in\mathcal{F}} \mathcal{H}^{d-1}(\partial E \cap B) < +\infty$, then either there exists a set $E \in \mathcal{F}$ such that

$$\mathfrak{m}(B \setminus E) = 0$$

or

$$\mathfrak{m}\left(B \setminus \bigcup_{E\in\mathcal{F}} E\right) > 0,$$

i.e. there is a subset of B with positive measure which is not covered by the union of $\mathcal{F}$.

In particular if $\Omega \subset \mathbb{R}^d$ is an open set (a "bag of potatos"), and all $E \in \mathcal{F}$ are open nonempty subsets of Ω, regular in the sense that they coincide with the interior of their closure, satisfying (2) and $\mathfrak{m}(\Omega \setminus \bigcup_{E\in\mathcal{F}} E) = 0$, then given any $x \in \Omega \cap \bigcup_{E\in\mathcal{F}} \partial E$, and B an open ball centered at x, one has

$$\sum_{E\in\mathcal{F}} \mathcal{H}^{d-1}(\partial E \cap B) = +\infty. \tag{3}$$

If, moreover, Ω is also connected, this implies in particular

$$\sum_{E\in\mathcal{F}} \mathcal{H}^{d-1}(\partial E \cap \Omega) = +\infty. \tag{4}$$

If $d = 2$ and all $E \in \mathcal{F}$ are convex, this also implies

$$\sum_{E\in\mathcal{F}} \mathrm{diam} E = +\infty. \tag{5}$$

Proof. We apply Theorem 3.2 with $X := B$, equipped with the Euclidean distance, and P is the usual Euclidean (Caccioppoli) perimeter relative to B. Clearly, the doubling condition holds for B and, since B is connected, the Poincaré inequality is satisfied and $X = B$ is an isotropic PI space. Then, the De Giorgi reduced boundary $\partial^* E$ of a Borel set $E \subset \mathbb{R}^d$ satisfies $\partial^* E \subset \partial^e E$ and $\mathcal{H}^{d-1}(\partial^e E) = \mathcal{H}^{d-1}(\partial^* E)$. Notice that $\mathcal{H}^{d-1}$ coincides with the spherical Hausdorff measure $\mathcal{S}^{d-1}$ on rectifiable sets, and (see the Example "weighted spaces" in Section 7 of [1]) $\mathcal{H}^{-1}$ coincides, up to a multiplicative constant, with $\mathcal{S}^{d-1}$.

If there exists $E \in \mathcal{F}$ such that $\mathfrak{m}(B \setminus E) = 0$ or, if $\mathfrak{m}(B \setminus \bigcup_{E\in\mathcal{F}} E) > 0$ there is nothing to prove. Otherwise there should be at least two different sets E_0, E_1 in $\mathcal{F}$ such that $\mathfrak{m}(B \cap E_0) > 0$ and $\mathfrak{m}(B \cap E_1) > 0$. Enumerate now all the sets of $\mathcal{F}$ as E_k, $k \in \mathbb{N}$ (clearly $\mathcal{F}$ is at most countable since each set is assumed to have positive measure, and in the case that $\mathcal{F}$ is finite we can complete the sequence with empty sets). Notice that $\mathcal{H}^{-1}(\partial^e E_k \cap \partial^e E_j) = 0$ for all $k \neq j$ and $\mathfrak{m}(E_k \cap E_j) = 0$ since $\mathcal{H}^{d-1}(\bar{E}_k \cap \bar{E}_j) = 0$.

Therefore, we can apply Theorem 3.2 to the sequence $E_k \cap B$ to get $\sum_k P(E_k, B) = +\infty$ hence

$$\sum_k \mathcal{H}^{d-1}(\partial E_k \cap B) \geq \sum_k \mathcal{H}^{d-1}(\partial^* E_k \cap B) = \sum_k P(E_k, B) = +\infty,$$

showing the first claim.

In the particular case when $\mathfrak{m}(\Omega \setminus \bigcup_{E\in\mathcal{F}} E) = 0$ for some open $\Omega \subset \mathbb{R}^d$, taking x and B as in the statement, we consider a ball $B' \subset B$ centered at x such that $B' \subset \Omega$. Then, for every $E \in \mathcal{F}$ either $E \cap B' = \emptyset$ or $E \cap B' \neq \emptyset$. In the latter case one has $\partial E \cap B' \neq \emptyset$ since otherwise we would have $B' \subset E$ which contradicts the choice of the center x and the fact that all the sets in $\mathcal{F}$ are disjoint. The regularity assumption implies that the interior of $B' \setminus E$ is nonempty, hence $\mathfrak{m}(B' \setminus E) > 0$. Therefore we have shown that $\mathfrak{m}(B' \setminus E) > 0$ for every $E \in \mathcal{F}$ and hence by the previous claim with B' in place of B, one has

$$\sum_{E\in\mathcal{F}} \mathcal{H}^{d-1}(\partial E \cap B) \geq \sum_{E\in\mathcal{F}} \mathcal{H}^{d-1}(\partial E \cap B') = +\infty$$

as claimed.

If Ω is an open and connected set, and $\mathcal{F}$ has at least two elements, we obtain that $\bigcup_{E\in\mathcal{F}} \partial E \cap \Omega$ is not empty. Finally, from (3) one has

$$\sum_{E\in\mathcal{F}} \mathcal{H}^{d-1}(\partial E) = +\infty,$$

and hence, if $d = 2$ and all $E \in \mathcal{F}$ are convex, then $\mathrm{diam} E \leq \mathcal{H}^1(\partial E)$ so that also (5) follows.

$\square$

Remark 3.4. It is clear from the proof that the statement of Corollary 3.3 is valid under the slightly weaker assumption that $\mathcal{H}^{d-1}(\partial^* A \cap \partial^* B) = 0$ for all $A, B \in \mathcal{F}$ instead of $\mathcal{H}^{d-1}(\bar{A} \cap \bar{B}) = 0$, where ∂^* stands for the reduced boundary in the sense of De Giorgi.

A possible alternative proof of the corollary for the family $\mathcal{F}$ of strictly convex sets could proceed as follows:

- For the planar case $d = 2$ one uses the coarea inequality to state that if the total perimeter of the sets in the family is finite then almost every line (in fact every except a countable number of lines), say horizontal, intersects the boundary of the sets in a set of finite $\mathcal{H}^0$ measure (i.e. in a finite set). On the other hand if the sets of $\mathcal{F}$ are assumed to be strictly convex then the lines intersecting only a finite number of sets in the family should be at most countable: in fact, the intersection of a convex set with a line is a line segment, and hence if the line intersects only a finite number of sets, then each point of its intersection with the boundary of some set of $\mathcal{F}$ is a point of intersection between two different sets of $\mathcal{F}$, which are countably many in total. This contradiction shows that the total perimeter of the sets in the family is infinite for planar packing of strictly convex sets (even in the case when the ambient set is not convex).
- For the general space dimension $d \geq 2$, again assuming that the total perimeter of the packing is convex, we have again by coarea inequality that the intersection of almost every hyperplane, say, horizontal, with the boundary of the sets in the family, has finite Hausdorff measure $\mathcal{H}^{d-2}$. If the ambient set is convex, then inside every such plane we have a packing of a convex set by strictly convex sets. Proceeding by backward induction on the dimension we arrive at a contradiction. Convexity of the ambient set is clearly essential for the argument to work in the case $d > 2$.

Corollary 3.5. *Corollary 3.3 is valid in any metric measure space* $(X, \mathrm{d}, \mathfrak{m})$ *satisfying the Poincaré inequality where for every* $x \in X$ *there is an open ball* $U \subset X$ *containing* x *such that* $(U, \mathrm{d}, \mathfrak{m})$ *is doubling. In particular it is valid in any* C^2 *smooth Riemannian manifold.*

Proof. It is enough to rewrite word-to-word the proof of Corollary 3.3 with balls $B \subset U$. In the case when X a C^2 smooth Riemannian manifold it is enough to note that over every ball U the curvature of X is bounded hence $(U, \mathrm{d}, \mathfrak{m})$ is doubling (in fact it is enough for this purpose for the curvature to be bounded from below). □

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