# NON-MIGRATION FOR PLANAR ELASTIC FLOWS

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Abstract. We consider a closed planar curve evolving by the free or the length-preserving elastic flow. We prove that the scale-invariant bound $L\int k^2\,ds \le 4\pi^2 + 8/3$ at the initial time prevents the entire curve from crossing any line between two times. For the free flow, the proof combines a monotonicity estimate for the normalized bending energy with a barrier that compares centroid motion to an expanding energy radius. For the length-preserving flow, a quantitative gradient estimate controls the total centroid displacement.

## 1. Introduction

Let $\gamma : \mathbb{T} \to \mathbb{R}^2$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, be a smooth closed immersed curve. We denote by $s$ the arclength parameter, by $T = \partial_s\gamma$ the unit tangent, and by $\nu$ the left unit normal (the counterclockwise rotation of $T$ by $\pi/2$), so that $T_s = k\nu$. Euler's bending energy, the length, and the normalized bending energy of $\gamma$ are

$$B(\gamma) := \int_\gamma k^2\,ds, \qquad L(\gamma) := \int_\gamma ds, \qquad U(\gamma) := L(\gamma)B(\gamma). \tag{1.1}$$

Note that the product $U$ is invariant under dilations.

We consider the $L^2(ds)$-gradient flow of $B$ and its projection onto the fixed-length constraint:

$$\partial_t\gamma = -G\nu, \qquad G = \begin{cases} G_0 := 2k_{ss} + k^3, & \text{free flow,} \\ G_c := 2k_{ss} + k^3 - \lambda_c(t)k, & \text{length-preserving flow,} \end{cases} \tag{1.2}$$

where

$$\lambda_c(t) = \frac{\int_{\gamma_t}(k^4 - 2k_s^2)\,ds}{\int_{\gamma_t} k^2\,ds}. \tag{1.3}$$

Thus $\lambda_c$ is chosen so that $\int_{\gamma_t} kG_c\,ds = 0$. We write the evolution in normal gauge; tangential reparametrizations do not affect the geometric statements below. By [6], both flows exist for all times starting from any smooth closed curve.

The study of elastic flows of closed curves goes back at least to Wen [14] and Polden [12]. Dziuk–Kuwert–Schätzle [6] established global existence for the free, length-penalized, and length-preserving flows of smooth closed curves. The survey of Mantegazza–Pluda–Pozzetta [8] gives an account of the analysis of elastic flows of curves and networks; see also Barrett–Garcke–Nürnberg [2] for numerical approximations. For open curves, Dall'Acqua–Lin–Pozzi studied the fixed-length flow with natural boundary conditions [3]; clamped-end versions were treated in [4, 13], and a free-boundary version in [5]. In the free closed case, circles expand rather than approach a stationary curve. Near-circle stability after rescaling was proved first under a curvature-derivative condition by Miura–Wheeler [11] and then under a small curvature-oscillation condition by Andrews–Wheeler [1].

A question raised by Huisken asks whether a closed planar curve under the length-penalized elastic flow can move entirely from one side of a line to the other [8, 7]. Such

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*migration* has been constructed for open curves with natural boundary conditions, both for the length-preserving flow [7] and for the length-penalized elastic flow with constant multiplier [10]. These examples do not resolve the question for closed curves. Miura–Wheeler [11] also identify the migration question for the free closed flow as open. Our result supplies a quantitative non-migration criterion for the two closed-curve flows in (1.2).

Fenchel's theorem and Cauchy–Schwarz give $U \geq 4\pi^2$ for every closed immersed curve, with equality exactly for a once-covered circle. If $n$ is the turning number of $\gamma$, then

$$L \int_\gamma \left(k - \frac{2\pi n}{L}\right)^2 ds = U - 4\pi^2 n^2. \tag{1.4}$$

The defect on the right is the normalized curvature oscillation used in [1]. Our assumption concerns $U$ alone: it does not impose a turning number, embeddedness, or smallness of curvature derivatives.

**Theorem 1.1** (Non-migration result)**.** *Let $\gamma : \mathbb{T} \times [0, +\infty) \to \mathbb{R}^2$ be a smooth family of closed regular immersed curves solving either equation in (1.2). Suppose that*

$$U(\gamma_0) = L(\gamma_0) \int_{\gamma_0} k^2 \, ds \leq 4\pi^2 + \frac{8}{3}. \tag{1.5}$$

*Then, for all $0 \leq t_0 < t_1$, we have*

$$4\pi^2 \leq U(\gamma_{t_1}) \leq U(\gamma_{t_0}) \leq U(\gamma_0). \tag{1.6}$$

*In particular, every $\gamma_t$ is embedded. Moreover, for every unit vector $e \in S^1$ and $h \in \mathbb{R}$, the following two inclusions cannot both hold:*

$$\gamma_{t_0} \subset \{x : x \cdot e \geq h\}, \qquad \gamma_{t_1} \subset \{x : x \cdot e \leq h\}. \tag{1.7}$$

The energy threshold in Theorem 1.1 is sufficient, and we do not claim that it is optimal. Note that $4\pi^2 + 8/3 \simeq 42.15$ lies well below Miura's sharp threshold $4\varpi^* \simeq 112.44$: every closed immersed curve with $U < 4\varpi^*$ is embedded, and the figure-eight elastica attains equality [9]. For the length-preserving flow, this threshold also yields embeddedness at all times [9, Theorem 4.2], but embeddedness does not by itself prevent the curve from crossing a line. In our smaller sublevel, embeddedness follows from an elementary argument (Lemma 2.2). The non-migration conclusion is stronger and requires control of the position of the curve.

The geometric argument uses the length centroid

$$c = \frac{1}{L} \int_\gamma \gamma \, ds \tag{1.8}$$

and the directional margins

$$\rho_e^+ = \max_\gamma \gamma \cdot e - c \cdot e, \qquad \rho_e^- = c \cdot e - \min_\gamma \gamma \cdot e. \tag{1.9}$$

For the free flow, $B$ decreases strictly and the auxiliary energy radius $R_E = 2\pi/B$ expands. Dissipation of $B$ alone does not imply monotonicity of $U = LB$, since $L$ varies. We first prove that the sublevel set defined by (1.5) is preserved and that $U$ is nonincreasing on it; then we establish

$$|c'| \leq \frac{3}{4} R_E', \qquad \rho_e^\pm \geq \frac{3}{4} R_E \quad (e \in S^1). \tag{1.10}$$

For the length-preserving flow, $L$ is constant and a quantitative gradient estimate shows that the total displacement of $c$ between any two times is less than $L/(8\pi^2)$; the directional margins are at least $3L/(8\pi)$. Both estimates exclude a complete crossing. We establish the variation formulas and the topological facts in Section 2, the free-flow energy estimate in Section 3, the geometric bounds in Section 4, and the length-preserving estimate in Section 5; they are combined in Section 6.

## 2. Preliminaries

2.1. **Variation formulas.** All integrals are with respect to arclength unless otherwise indicated. Put

$$Y = \int_\gamma G^2\, ds, \qquad S = \int_\gamma (k^4 - 2k_s^2)\, ds. \tag{2.1}$$

**Lemma 2.1.** *Along the free flow in (1.2),*

$$B' = -Y, \qquad L' = S, \qquad U' = BS - LY, \qquad R_E' = \frac{2\pi Y}{B^2} > 0. \tag{2.2}$$

*Furthermore,*

$$Lc' = \int_\gamma G\,[k(\gamma - c) + \nu] ds, \tag{2.3}$$

$$B = -\int_\gamma G\,(\gamma - c)\cdot \nu\, ds, \qquad \sqrt{Y} \geq \frac{2\pi B}{L^{3/2}}. \tag{2.4}$$

*Proof.* For an arbitrary normal variation $\partial_t \gamma = f\nu$, the variation formulas corresponding to $T_s = k\nu$ are

$$\partial_t ds = -kf\, ds, \qquad k_t = f_{ss} + k^2 f.$$

Taking $f = -G_0$, differentiation and periodic integration by parts give

$$B' = \int_\gamma [-2kG_{ss} - k^3 G] ds = -\int_\gamma G^2 ds, \qquad L' = \int_\gamma kG\, ds = S.$$

This yields the first three formulas in (2.2); the fourth follows from $R_E = 2\pi/B$. There is no stationary smooth closed free elastica: by spatial scaling $\gamma \mapsto (1+\varepsilon)\gamma$, the first variation of $B$ equals $-B \neq 0$, so $G$ cannot vanish identically. Thus $Y > 0$.

Translational invariance implies $\int_\gamma G\nu\, ds = 0$. Differentiating $L^{-1}\int \gamma\, ds$ then gives

$$Lc' = \int_\gamma [-G\nu + kG(\gamma - c)] ds = \int_\gamma G[k(\gamma - c) + \nu] ds,$$

where the last equality again uses $\int G\nu\, ds = 0$. The scaling variation gives the first identity in (2.4). The elementary estimate

$$\|\gamma - c\|_{L^\infty} \leq \frac{L}{4} \tag{2.5}$$

results from writing $\gamma(s) - c = L^{-1}\int_0^L (\gamma(s) - \gamma(r))\, dr$ and bounding each difference by the length of the shorter connecting arc. The Wirtinger inequality for $L$-periodic functions with zero mean gives the $L^2$ bound

$$\|\gamma - c\|_2 \leq \frac{L}{2\pi}\|T\|_2 = \frac{L^{3/2}}{2\pi}.$$

Consequently, by Cauchy–Schwarz, $B \leq \sqrt{Y}\|\gamma - c\|_2 \leq L^{3/2}\sqrt{Y}/(2\pi)$, proving (2.4). □

2.2. **Low-energy topology.** The turning number and embeddedness needed below follow directly from the energy bound.

**Lemma 2.2.** *Let $\gamma$ be a smooth closed unit-speed immersed planar curve. If*

$$LB - 4\pi^2 < \frac{8\pi^2}{3}, \tag{2.6}$$

*then its turning number is either $+1$ or $-1$. If, in addition, $LB - 4\pi^2 \leq 8/3$, then $\gamma$ is embedded.*

*Proof.* Write $\theta = 2\pi s/L$ and identify $\tau(\theta) = T(L\theta/(2\pi))$ with a complex number of modulus one. Closure implies $\int_0^{2\pi} \tau\, d\theta = 0$. Expand

$$\tau(\theta) = \sum_{j\in\mathbb{Z}\setminus\{0\}} a_j e^{ij\theta}.$$

Parseval gives

$$\sum_j |a_j|^2 = 1, \qquad \sum_j j^2|a_j|^2 = \frac{LB}{4\pi^2}.$$

With $d = LB - 4\pi^2$ we have

$$\sum_{|j|\geq 2} (j^2-1)|a_j|^2 = \frac{d}{4\pi^2}.$$

The elementary telescoping sum $\sum_{|j|\geq 2}(j^2-1)^{-1} = 3/2$ therefore yields

$$\left\|\tau - (a_1 e^{i\theta} + a_{-1}e^{-i\theta})\right\|_\infty \leq \sqrt{\frac{3d}{8\pi^2}} < 1. \tag{2.7}$$

As $|\tau| = 1$, the first-harmonic curve $\tau_1(\theta) = a_1 e^{i\theta} + a_{-1}e^{-i\theta}$ never passes through the origin; straight-line homotopy between $\tau$ and $\tau_1$ also avoids the origin. If $|a_1| = |a_{-1}|$, the function $\tau_1$ vanishes somewhere, a contradiction. If $|a_1| > |a_{-1}|$, it is homotopic in $\mathbb{C}\setminus\{0\}$ to $a_1 e^{i\theta}$; otherwise it is homotopic to $a_{-1}e^{-i\theta}$. Thus $\deg\tau = \deg\tau_1 = \pm 1$.

For the second assertion reverse the orientation if necessary. Write

$$T(s) = e^{i\alpha}e^{2\pi i s/L}e^{i\varphi(s)}, \qquad \varphi_s = k - \frac{2\pi}{L}, \qquad \int_0^L \varphi\, ds = 0.$$

Fourier series and Cauchy–Schwarz yield

$$\|\varphi\|_\infty \leq \left(\sum_{j\neq 0}\frac{L^2}{4\pi^2 j^2}\right)^{1/2}\left(\frac{1}{L}\int_0^L \varphi_s^2\, ds\right)^{1/2} = \sqrt{\frac{LB-4\pi^2}{12}} \leq \frac{\sqrt{2}}{3}.$$

Let $s \neq s'$ in $[0, L)$. Exchanging them if necessary, we may write $s' = s + \ell$ modulo $L$ with $0 < \ell \leq L/2$. Comparing the displacement along this arc with the corresponding chord of a circle of radius $L/(2\pi)$, and using $|e^{i\varphi} - 1| \leq |\varphi|$ and $\sin x \geq 2x/\pi$ for $x \in [0, \pi/2]$, we get

$$|\gamma(s+\ell) - \gamma(s)| \geq \frac{L}{\pi}\sin\frac{\pi\ell}{L} - \ell\|\varphi\|_\infty \geq \ell\left(\frac{2}{\pi} - \frac{\sqrt{2}}{3}\right) > 0.$$

Hence $\gamma(s) \neq \gamma(s')$, and the curve has no self-intersections. □

## 3. Signed evolution for the free flow

In this section we establish the analytic input to Theorem 1.1 independently of the later geometry. The point is to retain the favorable signs of fourth- and sixth-order curvature terms.

**Proposition 3.1.** *For any smooth closed free elastic flow with turning number $\pm 1$, if $0 < U - 4\pi^2 \leq 8/3$, then $U' < 0$. More precisely, after instantaneous ambient rescaling to $L = 2\pi$ and an orientation choice so that $\int k\, ds = 2\pi$, writing $k = 1 + u$, $\int u\, ds = 0$, one has*

$$\frac{U'}{L} \leq -\frac{1}{250}\|u_{ss}\|_2^2 < 0. \tag{3.1}$$

*Consequently $U(t)$ is nonincreasing as long as $U - 4\pi^2 \leq 8/3$, and the sublevel is forward invariant.*

*Proof. Instantaneous normalization.* Both $U$ and the sign of $U'$ are invariant under ambient dilations (with the flow time rescaled by the fourth power of the spatial factor). Hence we may freeze the rescaling at one time and work with $L = 2\pi$, $k = 1 + u$, $\int u = 0$. Set

$$\begin{aligned} V &= \int u^2, \quad Q = \int u_s^2, \quad X^2 = \int u_{ss}^2, \\ I_j &= \int u^j \quad (3 \le j \le 6), \qquad J_1 = \int u u_s^2, \quad J_2 = \int u^2 u_s^2. \end{aligned} \tag{3.2}$$

Thus $d := U - 4\pi^2 = 2\pi V$, and $V \le 4/(3\pi) < 1/2$.

*Exact identity.* Since $G = 2u_{ss} + (1+u)^3$, direct expansion and periodic integration by parts yield

$$\begin{aligned} B &= 2\pi + V, \\ S &= 2\pi + 6V + 4I_3 + I_4 - 2Q, \\ Y &= 4X^2 - 12Q - 24J_1 - 12J_2 + 2\pi + 15V + 20I_3 + 15I_4 + 6I_5 + I_6. \end{aligned} \tag{3.3}$$

Substituting into $U' = BS - LY$ gives

$$\begin{aligned} \frac{U'}{L} = &-4X^2 + 10Q - 8V - 16I_3 - 14I_4 - 6I_5 - I_6 \\ &+ 24J_1 + 12J_2 + \frac{V}{2\pi}(6V + 4I_3 + I_4 - 2Q). \end{aligned} \tag{3.4}$$

By $J_1 = -\frac{1}{2}\int u^2 u_{ss}$ and $J_2 = -\frac{1}{3}\int u^3 u_{ss}$, the following identity is *exact*:

$$\begin{aligned} &-4X^2 + 24J_1 + 12J_2 - 14I_4 - 6I_5 - I_6 \\ &\qquad = -4\int \left(u_{ss} + \frac{3}{2}u^2 + \frac{1}{2}u^3\right)^2 ds - 5I_4. \end{aligned} \tag{3.5}$$

It follows that

$$\begin{aligned} \frac{U'}{L} = &-4\left\|u_{ss} + \frac{3}{2}u^2 + \frac{1}{2}u^3\right\|_2^2 + 10Q - 8V - 16I_3 - 5I_4 \\ &+ \frac{V}{2\pi}(6V + 4I_3 + I_4 - 2Q). \end{aligned} \tag{3.6}$$

*Fourier closure.* With $T = e^{is + i\varphi(s) + i\alpha}$, $\varphi_s = u$ and $\int \varphi = 0$, the closure $\int T = 0$ and $|e^{ix} - 1 - ix| \le x^2/2$ imply

$$\left|\int_0^{2\pi} e^{is}\varphi(s)ds\right| \le \frac{1}{2}\|\varphi\|_2^2 \le \frac{1}{2}V.$$

Let $V_1$ be the squared $L^2$ norm of the $j = \pm 1$ Fourier modes of $u$. Parseval gives

$$V_1 \le \frac{V^2}{4\pi}, \qquad \frac{X^2}{V} \ge 16 - \frac{15V}{4\pi}, \qquad \frac{Q}{V} \ge 4 - \frac{3V}{4\pi}. \tag{3.7}$$

For $V > 0$, define $z = X/\sqrt{V}$. Since $d \le 8/3$ and $\pi > 157/50$, formula (3.7) gives $z > 492/125$. Modewise Fourier estimates yield

$$\|u\|_\infty \le C_2\sqrt{Q} + W, \qquad \|u\|_\infty \le C_4 X + W, \qquad W := \frac{V}{2\pi}, \tag{3.8}$$

where

$$C_2 = \sqrt{\frac{\pi}{6} - \frac{1}{\pi}} < \frac{227}{500}, \qquad C_4 = \sqrt{\frac{\pi^4/45 - 2}{2\pi}} < \frac{81}{500}.$$

Indeed, the first mode satisfies $\|P_1 u\|_\infty \le W$ by (3.7); for $|j| \ge 2$ apply Cauchy–Schwarz to the Fourier series, using respectively $\sum_{|j|\ge 2} j^{-2} = \pi^2/3 - 2$ and $\sum_{|j|\ge 2} j^{-4} = \pi^4/45 - 2$. The stated bounds on $C_2, C_4$ follow from $\pi < 355/113$.

*Retaining the quartic sign.* Put $M_u = \|u\|_\infty$, $s_4 = \sqrt{I_4}/X$, and $a = (3+M_u)/2$. Cauchy–Schwarz gives

$$0 \le s_4 \le \frac{M_u}{z}, \quad \frac{|I_3|}{X^2} \le \frac{s_4}{z}, \quad \left\| \frac{3}{2}u^2 + \frac{1}{2}u^3 \right\|_2 \le aXs_4. \tag{3.9}$$

Fourier series of $u$ (which has zero mean) also give the elementary inequality

$$10Q - 8V \le 2X^2, \tag{3.10}$$

indeed $2j^4 - 10j^2 + 8 = 2(j^2-1)(j^2-4) \ge 0$ for every nonzero integer $j$. When $as_4 < 1$, using (3.9) and discarding the nonpositive term $-2WQ$ in (3.6) gives

$$\frac{U'}{LX^2} \le -4(1-as_4)^2 + 2 + \frac{16+4W}{z}s_4 + (-5+W)s_4^2 + \frac{6W}{z^2}. \tag{3.11}$$

We now bound its right side in three ranges. In all of them,

$$x := \sqrt{V} < x_* := \frac{163}{250}, \qquad W < w_0 := \frac{17}{250}, \qquad z > z_0 := \frac{492}{125},$$

as follows from $V \le 4/(3\pi)$ and $\pi > 157/50$.

If $z_0 < z \le 7$, the second estimate in (3.8) and $C_4 < 81/500$ give $M_u \le c_* xz + W$, where $c_* := 81/500$. Set

$$s_*(z) = c_*x_* + \frac{w_0}{z}, \qquad a_*(z) = \frac{1}{2}(3 + c_*x_*z + w_0), \qquad r_*(z) = a_*(z)s_*(z).$$

Then $s_4 \le s_*(z)$ and $a \le a_*(z)$. The function $r_*(z)$ is convex, so its maximum on either of the intervals below is attained at an endpoint. Rational evaluation there gives

| range of $z$ | bound on $r_*(z)$ | bounds on $s_*(z)$ |
|---|---|---|
| $[z_0, 5]$ | $r_* < 134/625$ | $149/1250 < s_* < 12291/100000$ |
| $[5, 7]$ | $r_* < 11/50$ | $23/200 < s_* < 3/25$ |

In particular $r_* < 11/50$ and $s_* < 123/1000$. The right side of (3.11) increases in $a, s_4, W$ here: its $s_4$ derivative is at least $12(1-11/50) - 10(123/1000) > 0$, and the other derivatives are nonnegative. Substituting the bounds of the two rows into (3.11) gives, respectively,

$$\frac{U'}{LX^2} < -\frac{67489765177}{15759375000000} < -\frac{1}{250}, \qquad \frac{U'}{LX^2} < -\frac{919777}{10000000} < -\frac{1}{250}. \tag{3.12}$$

For example, the first fraction is the exact value of

$$-4\left(1 - \frac{134}{625}\right)^2 + 2 + \frac{(16+4w_0)(12291/100000)}{z_0} + \frac{6w_0}{z_0^2} - (5-w_0)\left(\frac{149}{1250}\right)^2.$$

If $z \ge 7$, then $Q \le Vz$ and the first estimate in (3.8) imply

$$M_u \le \left(C_2 + \frac{\sqrt{V}}{2\pi\sqrt{z}}\right)\sqrt{Vz} < \left(\frac{227}{500} + \frac{1}{25}\right)\sqrt{Vz} < \frac{1}{2}\sqrt{Vz}.$$

Here $\sqrt{V}/(2\pi\sqrt{z}) < 1/25$ follows from $x < x_*$, $z \ge 7$, and $\pi > 157/50$. Thus $s_4 \le \frac{1}{2}\sqrt{V/z}$ and $a \le \widehat{a} := (3 + \frac{1}{2}\sqrt{Vz})/2$. Write $y := z^{-1/2}$, so that $0 < y \le 7^{-1/2} < y_* := 189/500$. Then $\widehat{a}s_4 \le \frac{3}{4}xy + \frac{1}{8}x^2 < 1/4$ and $s_4 \le \frac{1}{2}xy < 1/8$, so, exactly as in the previous range, the right side of (3.11) increases with $a, s_4, W$. We may therefore set $s_4 = \frac{1}{2}xy$, $a = \widehat{a}$, and $W = w_0$, and we obtain the polynomial bound

$$\begin{aligned}\frac{U'}{LX^2} \le \mathcal{F}(x,y) := {} & -2 + 6xy + x^2 + (8+2w_0)xy^3 \\ & - \frac{14-w_0}{4}x^2y^2 - \frac{3}{4}x^3y - \frac{1}{16}x^4 + 6w_0y^4.\end{aligned} \tag{3.13}$$

The polynomial $\mathcal{F}$ is increasing in both $x$ and $y$ on the rectangle $(0, x_*) \times (0, y_*)$. Indeed,

$$\partial_x \mathcal{F} = y\left(6 - \frac{14 - w_0}{2}xy - \frac{9}{4}x^2\right) + x\left(2 - \frac{1}{4}x^2\right) + (8 + 2w_0)y^3 > 0,$$
$$\partial_y \mathcal{F} = x\left(6 - \frac{14 - w_0}{2}xy - \frac{3}{4}x^2\right) + 3(8 + 2w_0)xy^2 + 24w_0 y^3 > 0$$

there, since $xy < 0.247$ and $x^2 < 0.426$. Thus

$$\mathcal{F}(x, y) \le \mathcal{F}\left(\frac{163}{250}, \frac{189}{500}\right) = -\frac{1605550457639}{15625000000000} < -\frac{1}{250}. \tag{3.14}$$

This proves (3.1) when $V > 0$.

*Conclusion.* If $V = 0$, the curvature is constant, the curve is a circle, and $U' = 0$. Thus, along the original free flow, $U' \le 0$ whenever $U - 4\pi^2 \le 8/3$, with strict inequality for noncircular curves. The dilation used above is frozen at the time of evaluation; it does not modify the evolution.

Set $C = 4\pi^2 + 8/3$ and assume $U(0) \le C$. Letting $t^*$ be the supremum of the times $t$ for which $U(\tau) \le C$ for every $\tau \in [0, t]$, continuity and maximality imply $U(t^*) = C$. The curve at $t^*$ is not a circle, so $U'(t^*) < 0$. Hence $U < C$ on a right neighbourhood of $t^*$, contradicting its definition. The sublevel is therefore forward invariant, and $U$ is nonincreasing on it. □

## 4. Centroid mobility

We retain the explicit dependence on the energy defect

$$d = U - 4\pi^2, \qquad a(d) = \sqrt{\frac{1}{4} + \frac{3d}{32\pi^2}}, \qquad A(d) = \frac{1}{4} + \frac{a(d)}{\pi}. \tag{4.1}$$

**Lemma 4.1.** *Let $\gamma$ be a smooth closed unit-speed planar curve of turning number $+1$. Write*

$$k = \kappa + u, \quad \kappa = \frac{2\pi}{L}, \quad \int_0^L u\, ds = 0, \quad T = e^{i\alpha} e^{i\kappa s} e^{i\varphi(s)}, \quad \varphi_s = u, \quad \int_0^L \varphi = 0. \tag{4.2}$$

*With $V = \int_0^L u^2\, ds$ and $d = LV$, the $j = \pm 1$ curvature Fourier component satisfies*

$$V_1 \le \frac{L}{8\pi^2} V^2, \tag{4.3}$$

*and*

$$\|\varphi\|_2 \le \frac{L}{2\pi}\sqrt{V}\, a(d), \qquad \frac{\|\varphi\|_2}{\sqrt{L}} \le \frac{\sqrt{d}}{2\pi} a(d). \tag{4.4}$$

*Proof.* Closure gives $0 = \int_0^L e^{i\kappa s + i\varphi(s)} ds$. Using $\int_0^L e^{i\kappa s} ds = 0$, $|e^{ix} - 1 - ix| \le x^2/2$ and Wirtinger's inequality $\|\varphi\|_2 \le \frac{L}{2\pi}\|\varphi_s\|_2$,

$$\left| \int_0^L e^{i\kappa s} \varphi(s) ds \right| \le \frac{1}{2}\|\varphi\|_2^2 \le \frac{L^2}{8\pi^2} V.$$

Writing $\varphi = \sum_{j \ne 0} \varphi_j e^{i\kappa j s}$ and $u = \sum_{j \ne 0} u_j e^{i\kappa j s}$, so that $u_j = i\kappa j \varphi_j$ and $\int_0^L e^{i\kappa s} \varphi\, ds = L\varphi_{-1}$, this gives $|u_{-1}| \le V/(4\pi)$ and $V_1 = 2L|u_{-1}|^2 \le LV^2/(8\pi^2)$. Finally, by Parseval's identity and $j^2 \ge 4$ for $|j| \ge 2$,

$$\|\varphi\|_2^2 \le \left(\frac{L}{2\pi}\right)^2 \left(V_1 + \frac{1}{4}(V - V_1)\right) \le \left(\frac{L}{2\pi}\right)^2 V \left(\frac{1}{4} + \frac{3d}{32\pi^2}\right).$$

This is (4.4). □

**Lemma 4.2.** *Let $H = k(\gamma - c) + \nu$. Under the assumptions of Lemma 4.1,*

$$\|H\|_2 \le A(d) L \sqrt{V}. \tag{4.5}$$

*Along the free elastic flow one consequently has*

$$|c'| \le b(d) R'_E, \qquad b(d) = \frac{A(d)(4\pi^2 + d)\sqrt{d}}{4\pi^2}. \tag{4.6}$$

*Proof.* Set $T_0 = e^{i\alpha} e^{i\kappa s}$, $\nu_0 = iT_0$, and $\Gamma_0 = -\nu_0/\kappa$. Then $\Gamma_0' = T_0$, and $\int_0^L \Gamma_0 \, ds = 0$. With $w = (\gamma - c) - \Gamma_0$, the function $w$ has zero mean and $w_s = T - T_0$. By Wirtinger's inequality and $|e^{ix} - 1| \le |x|$,

$$\|w\|_2 \le \frac{L}{2\pi} \|T - T_0\|_2 \le \frac{L}{2\pi} \|\varphi\|_2, \qquad \|\nu - \nu_0\|_2 \le \|\varphi\|_2.$$

Since $\nu_0 = -\kappa \Gamma_0$, we have $H = u(\gamma - c) + \kappa w + (\nu - \nu_0)$, and the centered bound (2.5) together with (4.4) gives

$$\|H\|_2 \le \frac{L}{4} \sqrt{V} + 2\|\varphi\|_2 \le A(d) L \sqrt{V}.$$

By (2.3) and Cauchy–Schwarz, $|c'| \le L^{-1} \|G\|_2 \|H\|_2 \le A(d) \sqrt{d/L} \sqrt{Y}$. Since $R'_E = 2\pi Y / B^2$ and (2.4) gives $\sqrt{Y} \ge 2\pi B / L^{3/2}$,

$$\frac{|c'|}{R'_E} \le \frac{A(d) B^2 \sqrt{d}}{2\pi \sqrt{LY}} \le \frac{A(d) B L \sqrt{d}}{4\pi^2} = \frac{A(d)(4\pi^2 + d)\sqrt{d}}{4\pi^2}.$$

□

**Lemma 4.3.** *Under the assumptions of Lemma 4.1, let $S_* := \pi^2/3 - 1$. Then for every unit vector $e$,*

$$\rho_e^{\pm} \ge \frac{L}{2\pi} \left[ 1 - \frac{d \, a(d)^2}{8\pi^2} - \sqrt{S_*} \, \frac{\sqrt{d} \, a(d)}{2\pi} \right]. \tag{4.7}$$

*Proof.* Identify $\mathbb{R}^2$ with $\mathbb{C}$ and expand $\gamma - c = \sum_{j \ne 0} \hat{\gamma}_j e^{i\kappa j s}$ and $T = \sum_j \hat{T}_j e^{i\kappa j s}$ in complex Fourier series, so that $\hat{\gamma}_j = \hat{T}_j / (i\kappa j)$ for $j \ne 0$. The coefficient $\hat{\gamma}_1$ satisfies

$$|\hat{\gamma}_1| = \frac{L}{2\pi} \left| \frac{1}{L} \int_0^L e^{i\varphi} \, ds \right| \ge \frac{L}{2\pi} \left( 1 - \frac{\|\varphi\|_2^2}{2L} \right),$$

using $\cos x \ge 1 - x^2/2$. For $j \ne 0, 1$, the coefficient $\hat{T}_j$ coincides with the $j$-th coefficient of $T - T_0$, since $T_0$ is a pure first harmonic. By Cauchy–Schwarz and Parseval,

$$\| \sum_{j \ne 0,1} \hat{\gamma}_j e^{i\kappa j s} \|_\infty \le \frac{L}{2\pi} \sqrt{\sum_{j \ne 0,1} \frac{1}{j^2}} \, \frac{\|T - T_0\|_2}{\sqrt{L}} \le \frac{L}{2\pi} \sqrt{S_*} \, \frac{\|\varphi\|_2}{\sqrt{L}}.$$

The leading harmonic $\hat{\gamma}_1 e^{i\kappa s}$ describes a circle of radius $|\hat{\gamma}_1|$ centered at the origin. In each direction its extreme values are $\pm|\hat{\gamma}_1|$; the remainder changes either extremum by at most its $L^\infty$ norm. Substituting (4.4) proves (4.7). □

**Lemma 4.4.** *For every $0 \le d \le 8/3$,*

$$b(d) < \frac{3}{4}, \qquad 1 - \frac{d \, a(d)^2}{8\pi^2} - \sqrt{\frac{\pi^2}{3} - 1} \, \frac{\sqrt{d} \, a(d)}{2\pi} > \frac{3}{4}. \tag{4.8}$$

*Proof.* The first expression increases and the second decreases with $d$, so it suffices to verify the endpoint $d = 8/3$. The elementary bounds

$$\frac{157}{50} < \pi < \frac{22}{7}, \quad \sqrt{\frac{8}{3}} < \frac{5}{3}, \quad a(8/3) < \frac{53}{100}, \quad \sqrt{\frac{\pi^2}{3} - 1} < \frac{303}{200}, \quad 4\pi^2 + \frac{8}{3} < \frac{169}{4}$$

lead to

$$b(d) < \frac{(21/50)(169/4)(5/3)}{4(157/50)^2} = \frac{147875}{197192} < \frac{3}{4}.$$

Similarly,

$$1 - \frac{d\,a(d)^2}{8\pi^2} - \sqrt{\frac{\pi^2}{3} - 1}\frac{\sqrt{d}\,a(d)}{2\pi}$$
$$> 1 - \frac{(8/3)(53/100)^2}{8(157/50)^2} - \frac{(303/200)(5/3)(53/100)}{2(157/50)} = \frac{9197897}{11831520} > \frac{3}{4}.$$

□

## 5. The length-preserving flow

In this section $G = G_c$ and $\lambda_c$ is given by (1.3). Thus $G_c$ is the $L^2(ds)$-orthogonal projection of $G_0$ onto the orthogonal complement of $k$, which is the tangent space of the fixed-length constraint; the multiplier $\lambda_c$ is not assumed to be positive. Set

$$Y_c = \int_\gamma G_c^2\, ds. \tag{5.1}$$

**Lemma 5.1.** *Along the length-preserving flow,*

$$L' = 0, \qquad B' = -Y_c, \qquad U' = -LY_c, \tag{5.2}$$

*and*

$$Lc' = \int_\gamma G_c\,[k(\gamma - c) + \nu] ds. \tag{5.3}$$

*Proof.* The first variation of $B$ is $\int G_0 f\, ds$, whereas that of $L$ is $-\int kf\, ds$. By definition $\int kG_c\, ds = 0$. Thus, for $f = -G_c$, the length is fixed and

$$B' = -\int G_0 G_c\, ds = -\int G_c^2\, ds - \lambda_c \int kG_c\, ds = -Y_c.$$

Translation invariance of $B$ and $L$ gives $\int G_0 \nu\, ds = \int k\nu\, ds = 0$; hence also $\int G_c \nu\, ds = 0$. Differentiating the centroid proves (5.3), exactly as in Lemma 2.1. □

**Proposition 5.2.** *Suppose that $\gamma$ has turning number $\pm 1$ and $0 < d := LB - 4\pi^2 \le 8/3$. Then*

$$Y_c \ge \frac{25}{4}\frac{(2\pi)^4}{L^5}\, d. \tag{5.4}$$

*In particular, a noncircular curve in this sublevel cannot be a stationary point of the length-preserving flow.*

*Proof.* By reversal of orientation if needed, we assume turning number $+1$. Normalize the *curve at the time under consideration* to $L = 2\pi$ and write $k = 1 + u$, $\int u\, ds = 0$. Put

$$V = \int u^2\, ds = \frac{d}{2\pi}, \qquad Q = \int u_s^2\, ds, \qquad I_j = \int u^j\, ds \quad (j = 3, 4).$$

Since $d \le 8/3$, we have $0 < V \le 4/(3\pi) < 1/2$. Formula (3.7), which depends only on closure and not on the evolution, yields

$$q := \frac{Q}{V} \ge 4 - \frac{3V}{4\pi} > \frac{31}{8}. \tag{5.5}$$

Since $B = 2\pi + V$ and

$$\lambda_c = \frac{2\pi + 6V + 4I_3 + I_4 - 2Q}{2\pi + V}, \qquad -\int G_c u\, ds = 2Q - 3V - 3I_3 - I_4 + \lambda_c V,$$

we obtain the exact identity

$$-\int G_c u\,ds = \frac{4\pi}{2\pi+V}(Q-V) + \frac{3V^2}{2\pi+V} + \left(\frac{4V}{2\pi+V} - 3\right) I_3 - \frac{2\pi}{2\pi+V} I_4. \tag{5.6}$$

The first Fourier bound in (3.8) is geometric and applies to the constrained flow. Together with $Q = Vq$, it gives

$$M := \|u\|_\infty \le C_2\sqrt{Vq} + \frac{V}{2\pi} < m(q) := \frac{\sqrt{q}}{3} + \frac{2}{25}. \tag{5.7}$$

Indeed, since $V \le 4/(3\pi)$ and $3 < \pi < 22/7$, we have $C_2^2 V \le 2/9 - 4/(3\pi^2) < 1/9$ and $V/(2\pi) \le 2/(3\pi^2) < 2/25$. Now $|I_3| \le MV$ and $I_4 \le M^2 V$. Since $d \le 8/3$ and $\pi > 3$, we have $4\pi/(2\pi+V) > 24/13$; also the coefficient of $I_3$ in (5.6) has absolute value at most 3. Discarding its positive term $3V^2/(2\pi+V)$, we obtain

$$-\frac{1}{V}\int G_c u\,ds \ge h(q) := \frac{24}{13}(q-1) - 3m(q) - m(q)^2. \tag{5.8}$$

The function $h$ is strictly increasing on $[31/8, \infty)$ since

$$h'(q) = \frac{24}{13} - \frac{1}{9} - \frac{79}{150\sqrt{q}} > 0.$$

Because $\sqrt{31/8} < 2$, we have $m(31/8) < 56/75$. Hence

$$h(q) > h(31/8) > \frac{24}{13}\frac{23}{8} - 3\frac{56}{75} - \left(\frac{56}{75}\right)^2 = \frac{183557}{73125} > \frac{5}{2}. \tag{5.9}$$

By Cauchy–Schwarz,

$$\frac{5}{2}V \le -\int G_c u\,ds \le \sqrt{Y_c V}, \qquad \text{so} \qquad Y_c \ge \frac{25}{4}V.$$

Finally, under a dilation of factor $2\pi/L$, $G_c$ scales as length$^{-3}$ and $Y_c$ as length$^{-5}$. Since $V = d/(2\pi)$ in the normalized coordinates, this becomes (5.4) at arbitrary length. □

**Proposition 5.3.** *Under the hypothesis of Theorem 1.1 for the length-preserving flow, set $A_* = A(8/3)$, with $A$ defined in (4.1). Then $A_* < 21/50$ and, for all $t_0 < t_1$,*

$$\frac{|c(t_1) - c(t_0)|}{L} \le \frac{A_*}{10\pi^2}(U(t_0) - U(t_1)) < \frac{1}{8\pi^2}. \tag{5.10}$$

*Moreover, for every unit vector $e$ and every $t \in [0, +\infty)$,*

$$\rho_e^{\pm} \ge \frac{3}{4}\frac{L}{2\pi}. \tag{5.11}$$

*Proof.* The energy is nonincreasing by (5.2); hence $d(t) \le 8/3$, and Lemma 2.2 ensures turning number $\pm 1$. The $L^2$ estimate for $H = k(\gamma - c) + \nu$ in (4.5) is geometric and applies unchanged. By (5.3) and Cauchy–Schwarz,

$$|c'| \le A(d)\sqrt{\frac{d}{L}}\sqrt{Y_c}.$$

For $d > 0$, Proposition 5.2 and $-U' = LY_c$ imply

$$\frac{|c'|}{L} \le \frac{A(d)}{10\pi^2}(-U') \le \frac{A_*}{10\pi^2}(-U'). \tag{5.12}$$

At $d = 0$ the curve is a stationary circle, so the same inequality holds. The elementary bounds in the proof of Lemma 4.4 give $a(8/3) < 53/100$ and therefore $A_* < 1/4 + 53/(100\pi) < 21/50$. Integrating (5.12) on $[t_0, t_1]$ and using $0 \le U(t_0) - U(t_1) \le 8/3$

proves (5.10), since $\frac{21}{50}\cdot\frac{8}{3}\cdot\frac{1}{10}=\frac{14}{125}<\frac{1}{8}$. The geometric margin estimate (4.7), together with Lemma 4.4, gives (5.11). □

## 6. Proof of the main result

*Proof of Theorem 1.1.* Lemma 2.2 shows that the initial turning number is $\pm1$, which is preserved throughout the smooth evolution. We reverse orientation if necessary and assume it is $+1$.

For the free flow, Proposition 3.1 gives (1.6); the lower bound $U\geq 4\pi^2$ is (1.4) with $n=\pm1$. The estimates of Lemmas 4.2, 4.3 and 4.4, together with $L/(2\pi)\geq R_E$, give (1.10). If (1.7) held, then

$$(c(t_0)-c(t_1))\cdot e\geq \rho_e^-(t_0)+\rho_e^+(t_1)\geq \frac{3}{4}[R_E(t_0)+R_E(t_1)].$$

On the other hand, integrating the bound $|c'|\leq \frac{3}{4}R_E'$ of (1.10) and using $R_E'>0$, we get

$$|c(t_1)-c(t_0)|\leq \frac{3}{4}[R_E(t_1)-R_E(t_0)],$$

Since $|(c(t_0)-c(t_1))\cdot e|\leq |c(t_1)-c(t_0)|$, the two inequalities give $R_E(t_0)\leq 0$, which is impossible.

For the length-preserving flow, monotonicity (1.6) follows directly from (5.2). If the two half-plane conditions held, (5.11) would give

$$(c(t_0)-c(t_1))\cdot e\geq \rho_e^-(t_0)+\rho_e^+(t_1)\geq \frac{3L}{4\pi}.$$

This contradicts (5.10), as $L/(8\pi^2)<3L/(4\pi)$. In both cases the persistent bound $d(t)\leq 8/3$ and the second assertion of Lemma 2.2 also give embeddedness at every time. This proves the theorem. □

**Corollary 6.1.** *Under the free-flow hypotheses of Theorem 1.1, for every $e\in S^1$ both*

$$\frac{3}{4}R_E(t)+c(t)\cdot e,\qquad \frac{3}{4}R_E(t)-c(t)\cdot e$$

*are nondecreasing. Moreover $\rho_e^{\pm}\geq (3/4)R_E$.*

*Proof.* The derivatives of these two functions are $\frac{3}{4}R_E'\pm c'\cdot e\geq 0$ by (1.10). □

*Remark* 6.2. Our result concerns the free and length-preserving equations in the stated energy sublevel. Huisken's original question is about the distinct length-penalized equation [8, 7].

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