3D Protter problems for mixed type equations

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At a conference of AMS in New York in 1952 M.H.Protter formulated and studied some boundary value problems for the wave equation in a $3D$ domain Ω_0, bounded by two characteristic cones and a plane domain, which are three-dimensional analogues of Darboux-problems (or Cauchy-Goursat problems) on the plane. In 1954 he initiated the study of such 3-dimensional problems for the mixed type equation

$$Lu := K(t)\left(\;u_{x_1x_1} + u_{x_2x_2}\right) \;- u_{tt} = f, \qquad tK(t) > 0, \;\; t \neq 0 \tag{1}$$

in domain Ω bounded by two characteristic cones in the hyperbolic halfspace $\{t > 0\}$ and an elliptic part in $\{t < 0\}$. These problems have been formulated by M. Protter as three-dimensional analogues of a plane problem given by C. Morawetz and examined by C. Morawetz, P. Lax and R. Phillips.

What is the situation around both these problems now, 50 years later?

1. Many authors studied these problems using different methods, like: Wiener-Hopf method, special Legendre functions, a priori estimates, nonlocal regularization and others. In the case of the wave equation it is shown that for any $n \in N$ there exists a $C^n(\bar{\Omega}_0)$ - function, for which the corresponding unique *generalized solution* belongs to $C^n(\bar{\Omega}_0 \backslash O)$, but it has a strong power-type singularity $\left(x_1^2 + x_2^2 + t^2\right)^{-n/2}$ at the point O. This singularity is isolated only at the vertex O of the characteristic cone and does not propagate along the cone. We will present here some final results for the exact behavior of the singular solutions at the point O. Also, we will give some necessary and sufficient conditions for the function f, under which only classical solution exists. Finally, some weight a priori estimates are stated.

2. According to the mixed type equation (1) in Ω, the situation is still away from the final results. Some uniqueness results have been obtained by many authors. But up to now there are no general existence results for thes problem in Ω. This situation can be interpreted in terms of improperly posed (or ill-posed) problems. Using Friedrichs' theory of symmetric positive operators, we find and investigate a non-local problems which are regularisers, in some sence, of these ill-posed problems.