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Special session on

"NONLINEAR EVOLUTION EQUATIONS"

Multispike solutions to nonlinear elliptic equations

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An abstract framework is outlined to study the existence and Morse index of spike-layer solutions to a class of singular perturbation problems. We then show how the abstract results apply in the following situation. Let $\epsilon > 0$ be a small positive parameter and Ω, a smooth bounded domain in $\mathbf{R}^n$ for $n \geq 2$. Consider

$$\begin{cases} \epsilon^2 \Delta v - av + f(v) = 0, & x \in \Omega, \\ \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $a > 0$, $f(v)$ is a smooth positive function, superlinear as $v \to \infty$, and such that $f(0) = f'(0) = 0$. We seek solutions u_ε which tend to zero as $\varepsilon \to 0$ uniformly on compact subsets of $\bar{\Omega}/P_N$ Here P_N is a collection of N points in $\bar{\Omega}$ which is characterized in terms of the geometry, and $u_\varepsilon(p)$ does not approach zero with ε for each $p \in P_N$.

To be communicated

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Global Existence of a Strongly-Coupled Quasilinear Parabolic System Arising from Electrochemistry

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We considered a strongly coupled quasilinear parabolic system

$$\frac{\partial v_i}{\partial t} = \sum_{j=1}^{m} \frac{\partial}{\partial x}\left(a^{ij}(\mathbf{v})\frac{\partial v_j}{\partial x}\right), \qquad i = 1, \cdots, m$$

modeling an electrochemical problem. Its coefficient a_{ij} may become discontinuous. First we proved the global existence of weak solutions of a strongly-coupled parabolic system with continuous coefficent. Using it to approximate the original problem, we established the global existence of the weak solution to the electrochemistry problem. Global stability of the steady-state solutions were also shown.

Operators of Thin-Film Type: Qualitative Properties and Open Problems

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This class of (nonlinear degenerate higher order) operators has recently attracted an enormous interest both for its relevance in a number of applications (from fluid dynamics to material sciences) and for its mathematical significance as relatively uncharted territory in the theory of PDE's. The mathematical investigation has revealed a great richness of structure, but still several challenging problems remain open.

Geometric Estimates for the Vanishing Behavior of Solutions to the Logarithmic Fast Diffusion Equation

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We study the vanishing behavior of maximal solutions to the logarithmic fast diffusion equation $u_t = \Delta \log u$ on the plane. This equation describes the evolution of a metric g, conformal to the standard metric under the Ricci flow. We derive upper and lower bounds on the geometric width of the solution and on the maximum curvature. Using these geometric estimates we deduce upper and lower pointwise bounds on the solution near its vanishing time. More precise bounds on the solution are obtained in the rotationally symmetric case, using comparison methods.

Repelling Spikes in the Dynamics of a System of Reaction-Diffusion Equations

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We consider an activator-inhibitor system of the form

$$\left\{\begin{array}{ll} u_t = \epsilon^2 \Delta u + F(u) + \sigma(u-v), & x \in \Omega, \\ \tau v_t = \epsilon^2 \Delta v + F(v) + \sigma(u-v), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega, \end{array}\right. \tag{1}$$

where: $F(s) = s^2 - s$, $\epsilon > 0$ is a small parameter, τ, σ positive constants and $\tau < 1, \sigma > \bar{\sigma}$ for some $\bar{\sigma}$.

We show that,given an integer $N \geq 1$, there is $\epsilon_N > 0$ such that for $\epsilon < \epsilon_N$ system (1) has a solution $t \to u^\epsilon(t)$ which exhibits N spikes at points $\xi_1^\epsilon(t), \ldots, \xi_N^\epsilon(t) \in \Omega$. We also derive a precise asymptotic formula for the speed of the N spikes. We show that, for small ϵ, the points $\xi_i^\epsilon(t)$ **repel** each other and are also **repelled** by the boundary $\partial\Omega$ of Ω. A consequence of this fact is that the solution $t \to u^\epsilon(t)$ exists globally in time keeping for ever its spike structure. The repelling character of the **spike interaction** has also interesting consequences on the structure of the set of stationary spike solutions. For instance one can deduce the existence of unstable spike stationary solutions in a $\epsilon^{\frac{1}{2}}-$neighborhood of stable ones.

Presented by G. Fusco

Intermediate Scaling Laws for Spreading Droplets: PDE Methods and Asymptotic Analysis

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The spreading of a thin droplet of viscous liquid on a plane surface driven by capillarity is modeled — in the standard lubrication approximation — by a fourth order nonlinear degenerate parabolic equation for the droplet height h. If the evolution is limited by the no–slip boundary condition at the liquid–solid interface, then the problem is ill–posed, since solutions in fact don't spread due to a singularity at the moving contact line (in the mythological words of Huh and Scriven, "... not even Herakles could sink a solid"). Ad-hoc relaxations of the no–slip boundary conditions, such as positive slippage models or shear-thinning rheologies, remove this paradox but introduce new physical, microscopic length scales b.

Here we show that these microscopic length scales only enter logarithmically in the effective (that is, macroscopic) spreading behavior. For positive slippage models, we prove a scaling law in time for both the total energy and the diameter of the apparent (that is, macroscopic) support of the droplet. This is an intermediate scaling law: It takes an initial layer to "forget" the initial droplet shape — whereas after a long time, the droplet is so thin that its spreading is governed by the physics on the scale b. Our approach mimics a simple heuristic argument based on the gradient flow structure, and works by deriving suitable estimates for physically relevant integral quantities: the free energy, the length of the apparent support and their respective rates of change. For shear-thinning models we present formal arguments, based on the analysis of a class of qusi–selfsimilar solutions, which suggest analogous logarithmic corrections on time scales which in this case may not be intermediate. Results are based on joint works with Felix Otto and Lidia Ansini, respectively.

Stability in a Mathematical Model in Combustion Theory

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I will discuss the problem of the stability of the planar travelling wave solution to a well known free boundary parabolic system modelling the propagation of near-equidiffusional premixed flames in the whole plane or in a two-dimensional strip.

I will give stability and instability results, according to the value of the reduced Lewis number.

One-dimensional stability of relaxation shocks

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Consider a general *hyperbolic relaxation model* of form

$$\begin{cases} u_t + f(u,v)_x = 0 \\ v_t + g(u,v)_x = q(u,v), \end{cases} \tag{1}$$

with u, $f \in \mathbf{R}^n$, v, g, $q \in \mathbf{R}^r$, under the assumption $Re\,\sigma\big(q_v(u, v^*(u))\big) < 0$ along a smooth *equilibrium manifold* defined by $q(u, v^*(u)) \equiv 0$.

System (1) supports smooth traveling front solutions, known as *relaxation shocks*, i.e. solutions of form $(u,v)(x,t) = (\bar{u},\bar{v})(x-st)$ such that $\lim\limits_{z\to\pm\infty}(\bar{u},\bar{v}) = (u_\pm, v^*(u_\pm))$. Such relaxation shocks are the counterpart of shock solutions to the associated *equilibrium*, or "relaxed" system of conservation laws $u_t + f(u, v^*(u))_x = 0$, hence these solutions are expected to be stable.

Under the weak assumption of spectral stability, or stable point spectrum of the linearized operator about the wave, we establish sharp pointwise Green's function bounds and consequent linear and nonlinear orbital stability for relaxation shocks. A consequence is stability of small-amplitude profiles of Broadwell and Jin-Xin models for each of which spectral stability has been verified in other works. These are the first complete stability results for relaxation models with nonscalar equilibrium equations.

Presented by Corrado Mascia

Diffusion and Cross-Diffusion in Pattern Formation: from Single Equations to Systems

Wei-Ming Ni
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In this lecture I plan to explain how diffusions and cross-diffusions are used in modeling pattern formation, both from a modeling point of view and from a mathematical point of view. Examples will be used to illustrate various approaches.

Diffusive N-Waves and Metastability in Burgers Equation

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Athanasios E. Tzavaras

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We study the effect of viscosity on the large time behavior of the viscous Burgers equation by using a transformed version of Burgers (in self-similar variables) that captures efficiently the mechanism of transition to the asymptotic states, and allows to estimate the time of evolution from an N-wave to the final stage of a diffusion wave. Then, we construct certain special solutions of diffusive N-waves with unequal masses. Finally, using a set of similarity variables and a variant of the Cole-Hopf transformation, we obtain an integrated Fokker-Planck equation. The latter is solvable and provides an explicit solution of the viscous Burgers in a series of Hermite polynomials. This format captures the long time - small viscosity interplay, as the diffusion wave and the diffusive N-waves correspond respectively to the first two terms in the Hermite polynomial expansion.

Presented by Athanasios E. Tzavaras

Qualitative Behavior of Solutions of Chemotactic Diffusion Systems

Xuefeng Wang

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Chemotaxis is the oriented movement of cells in response to the concentration gradient of chemical substances in their environment. It is “anti-diffusion”. We are interested in the effects of chemotaxis and diffusion on the growth of cells. This kind of problems leads to new challenges in nonlinear analysis, even in special cases. We first consider the situation of a single bacterial population which responds chemotactically to a nutrient diffusing from an adjacent phase not accessible to the bacteria. The bacteria and the chemical are assumed to be diffusive. The concentration and density of the substrate and cells (resp.) satisfy a quasi-linear parabolic system , with nonlinear boundary condition. Our first set of results addresses the effects of two important biological parameters $\lambda > 0$ and $\chi > 0$ on the steady states, where λ measures the (random) motility of bacteria and χ the magnitude of chemotactic response (or sensitivity) to the chemical.

I shall also talk about the dynamics of the system: global boundedness of time-dependent solutions and stability issues of trivial and nontrivial steady states with small amplitudes.

Finally, I will present some results (joint work with Yaping Wu) for the situation when two species of cells compete for the same source of substrate.