Approaching a partial differential equation of mixed elliptic-hyperbolic type

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Suppose an isotropic, non-conducting, non-dissipative medium and a monochromatic electromagnetic field interact in absence of electric charges. Let n and ν denote the refractive index and the wave number, respectively. Here n is a scalar real-valued field, whose reciprocal is proportional to the relevant velocity of propagation through the medium, and ν is a large positive parameter, whose reciprocal is proportional to the length of waves involved. The following Helmholtz equation

$$\Delta U + \nu^2 n^2 U = 0 \tag{1}$$

is an archetype of those partial differential equations that ensue from Maxwell's system and model the subject matter mathematically. A distinctive feature of (1) is *stiffness* — the order of magnitude of ν is significantly greater than that of the other coefficients involved.

An expansion, which represents solutions to (1) asymptotically as $\nu \to +\infty$, originates from WKBJ method and reads thus

$$U \simeq \exp(i\nu S) \sum_{k=0}^{\infty} A_k \cdot (i\nu)^{-k}. \tag{2}$$

Here S and A_k are scalar fields, independent on ν. The former, named *eikonal,* is real-valued and governed by

$$|\nabla S|^2 = n^2;$$

the latter is complex-valued and governed by the so-called *transport equations.*

Inference built upon expansion (2) amounts to geometrical optics. Though successful in describing both the propagation of light and the concurrence of caustics via the mechanism of rays, geometrical optics is inherently unable to account for those phenomena, such as the development of evanescent waves, that take place beyond a caustic.

A more powerful asymptotic expansion, which is apt to represent solutions to (1) on *both sides of a caustic,* simultaneously in the region covered by geometric optical rays and in the opposite region where geometrical optics breaks down, is provided by a theory of Kravtsov and Ludwig. In case the caustic involved is smooth and convex, such an expansion reads

$$U \simeq e^{i\nu v}\Big\{\mathrm{Ai}(\nu^{2/3}u)\sum_{k=0}^{\infty} A_k\cdot(i\nu)^{-k} + i\nu^{-1/3}\mathrm{Ai}'(\nu^{2/3}u)\sum_{k=0}^{\infty} B_k\cdot(i\nu)^{-k}\Big\}. \tag{3}$$

Here u, v, A_k, B_k are scalar fields, independent on ν; u and v are real-valued, A_k and B_k are complex-valued; Ai denotes the *Airy function.*

Properties of Ai inform us that the right-hand side of (3) oscillates rapidly where u is negative, approaches smoothly a limit if u approaches 0, quenches fast where u is positive. Therefore (3) matches geometrical optics in the region where u is negative and predicts the occurrence of damped waves in the region where u is positive; a caustic take place on the level surface where $u = 0$.

Assembling (1) and (3) results in

$$\begin{array}{l} u\ |\nabla u|^2 - |\nabla v|^2 + n^2 = 0 \\ \nabla u \cdot \nabla v = 0 \end{array} \tag{4}$$

— a fully nonlinear, first-order partial differential system governing u and v.

In the present paper we sketch some lineaments of (4) in the case where the space dimension equals 2, i.e. we let x and y denote rectangular coordinates in the Euclidean plane and investigate the following system

$$\begin{array}{lcl} u\ (u_x^2 + u_y^2) - v_x^2 - v_y^2 + n^2(x,y) & = & 0 \\ u_x v_x + u_y v_y & = & 0. \end{array} \tag{5}$$