Doubling properties of the harmonic measure associated with degenerate elliptic operators

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Let $X = \{X_1, \ldots, X_p\}$ be a family of smooth vector fields in $\mathbb{R}^n$, and let $\Omega \subset \mathbb{R}^n$ be a connected open subset with sufficiently regular boundary $\partial\Omega$ (for instance, take $\partial\Omega$ of class $C^{1,\alpha}$, $0 < \alpha \leq 1$). If X satisfies the so-called Hörmander rank condition, i.e. if the rank of the Lie algebra generated by X equals n at each point of a neighborhood of $\bar{\Omega}$, very few results have been proved so far concerning the behavior up to the boundary of positive solutions of boundary value problems associated with the degenerate elliptic operator $\mathcal{L} = \sum_{j=1}^{p} X_j^* X_j$. This is basically due to the presence of *characteristic points* of $\partial\Omega$, that behave – in a sense – as cusps do for the (usual elliptic) Laplace operator.

So far, the sharpest results in the literature concerning boundary behavior of solutions of $\mathcal{L}u = 0$, and that are the exact counterpart of the elliptic theory, have been proved for NTA-domains related to the Carnot-Carathédory metric associated with X. On the other hand, the characterization of NTA-domains is a very delicate point already for the simplest examples, since it is related to the geometry of metric balls.

Thus, we are interested in seeking what we can say when the structure of Carnot-Carathédory balls is too complicated to be handled explicitly. In fact we stress that, already in simple step 3 Carnot groups like the so-called Engel group, the geometry of the CC-balls and its interaction with that of the boundary becomes essentially more complicated. In other words, our aim is to describe as precisely as we can what happens when no structure assumptions are made on the vector fields besides Hörmander's condition.

More precisely, we prove a (in general not scale-invariant) doubling property for the "harmonic measure" associated with $\mathcal{L}$, and a (in general not scale-invariant) boundary Harnack principle for a large class of domains.

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