A Maximum Principle for X–elliptic Operators and Applications

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Let $\{X_1, \cdots, X_m\}$ be a family of Lipschitz vector fields in $\mathbb{R}^N$, and the linear second order differential operator

$$Lu = \sum_{i,j=1}^{N} \partial_i \left(b_{ij}\, \partial_j u + d_i u \right) + \sum_{i=1}^{N} b_i\, \partial_i u + c\, u,$$

where $b_{ij}(x) = b_{ji}(x), d_i, b_i$ and c are measurable functions. Set $B = (b_{ij})$, $d = (d_1, \ldots, d_N)$ and $b = (b_1, \ldots, b_N)$. *The operator L is X–elliptic in an open subset Ω of $\mathbb{R}^N$* if it satisfies the following conditions: (a) there exist positive constants λ, Λ such that $\lambda \sum_{j=1}^{m} \langle X_j(x), \xi\rangle^2 \leq \langle B(x)\xi, \xi\rangle \leq \Lambda \sum_{j=1}^{m} \langle X_j(x), \xi\rangle^2$, $\forall\, \xi \in \mathbb{R}^N, x \in \Omega$,; and (b) there exists a function $\gamma(x) \geq 0$ such that $\langle d(x), \xi\rangle^2 + \langle b(x), \xi\rangle^2 \leq \gamma(x)^2 \sum_{j=1}^{m} \langle X_j(x), \xi\rangle^2$, $\forall\, \xi \in \mathbb{R}^N, x \in \Omega$. Here $\langle\, ,\rangle$ denotes the standard inner product in $\mathbb{R}^N$.

Assume that $\gamma \in L^{2p}(\Omega)$, and $c \in L^p(\Omega)$ for some $\frac{Q}{2} < p < \infty$, with $Q > 2$ the exponent in the Sobolev inequality for the vector fields. We establish a maximum principle for operators L of the form above, and use it to prove Harnack inequality for nonhomogeneous equations, uniform Harnack inequalities on rings, and as a consequence Liouville–type results.

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