Cases of equality in perimeter inequalities for Steiner symmetrization

Andrea Cianchi

Abstract. One of the several remarkable properties of Steiner symmetrization is that if E is any set of finite perimeter $P(E)$ in $\mathbb{R}^n$, $n \geq 2$, and H is any hyperplane, then also its Steiner symmetral E^s about H is of finite perimeter, and

$$P(E^s) \leq P(E) \; . \tag{0.1}$$

Recall that E^s can be defined as follows. Label the points $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ as $x = (x', y)$, where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $y = x_n$, assume, without loss of generality, that $H = \{(x', 0) : x' \in \mathbb{R}^{n-1}\}$, and set $E_{x'} = \{y \in \mathbb{R} : (x', y) \in E\}$, $\ell(x') = \mathcal{L}^1(E_{x'})$ for $x' \in \mathbb{R}^{n-1}$, and $\pi(E)^+ = \{x' \in \mathbb{R}^{n-1} : \ell(x') > 0\}$ where $\mathcal{L}^m$ denotes the outer Lebesgue measure in $\mathbb{R}^m$. Then

$$E^s = \{(x', y) \in \mathbb{R}^n : x' \in \pi(E)^+, \, |y| \leq \ell(x')/2\} \; . \tag{0.2}$$

The purpose of this joint work with Miroslav Chlebík and Nicola Fusco is to investigate on the cases of equality in (0.1), i.e. to characterize those sets of finite perimeter E which satisfy

$$P(E^s) = P(E) \; . \tag{0.3}$$

Partial results about this problem can be found in the literature. It is classical that if E is convex and fulfills (0.3), then it is equivalent to E^s (up to translations along the y-axis). On the other hand, as far as we know, the only available result concerning a general set of finite perimeter $E \subset \mathbb{R}^n$ satisfying (0.3) states that its section $E_{x'}$ is equivalent to a segment for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi(E)^+$. As a first step, we strengthen this result and show that, in fact, the generalized inner normals to E at the endpoints of $E_{x'}$ are symmetric about H for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi(E)^+$.

It might seem surprising that this information is not sufficient to conclude that E is equivalent to E^s (up to translations along the y-axis). However, simple counterexamples can be exhibited showing that such an equivalence cannot be inferred without any additional assumption on E^s. Our second and main result consists in finding out minimal conditions of geometric nature on E^s ensuring that if (0.3) is in force, then necessarily E is equivalent to E^s (up to translations along the y-axis).