TOWARDS A THEORY OF AREA IN HOMOGENEOUS GROUPS

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Abstract. A general approach to compute the spherical measure of submanifolds in homogeneous groups is provided. We focus our attention on the *homogeneous tangent space*, that is a suitable weighted algebraic expansion of the submanifold. This space plays a central role for the existence of blow-ups. Main applications are area-type formulae for new classes of C^1 smooth submanifolds and the equality between spherical measure and Hausdorff measure on all horizontal submanifolds.

Contents

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1. Introduction

The notion of surface area is fundamental in several branches of mathematics, such as geometric analysis, differential geometry and geometric measure theory. Area formulae for rectifiable sets in Riemannian manifolds and general metric spaces are well known [32], [2]. When the metric space is not Riemannian, as a noncommutative homogeneous group (Section 2.1), even smooth sets need not be rectifiable in the standard metric sense [17]. Such unrectifiability occurs when the Hausdorff dimension is greater than the topological dimension. In Carnot-Carathéodory spaces all smooth submanifolds "generically" have this dimensional gap [28, Section 0.6.B], so several well known tools of geometric measure theory do not apply. The basic question of computing an area formula for the Hausdorff measure remains a difficult task, even for smooth submanifolds.

Hausdorff measure plays a fundamental role in geometric measure theory, as it is well witnessed by the following Federer's words [18]. "It took five decades, beginning with Carathéodory's fundamental paper on measure theory in 1914, to develop the intuitive conception of an m dimensional surface as a mass distribution into an efficient instrument of mathematical analysis, capable of significant applications in the calculus of variations. The first three decades were spent learning basic facts on how subsets of $\mathbb{R}^n$ behave with respect to m dimensional Hausdorff measure $\mathcal{H}^m$. During the next two decades this knowledge was fused with many techniques from analysis, geometry and algebraic topology, finally to produce new and sometimes surprising but classically acceptable solutions to old problems."

Federer's comments remain extremely appealing when applied to the Hausdorff measure in nilpotent groups, that have a more complicated geometric structure. The wider program of studying analysis and geometry in such groups and general Carnot-Carathéodory spaces already appeared in the seminal works by Hörmander [29], Folland [19], Stein [53], Gromov [28], Rothschild and Stein [50], Nagel, Stein and Wainger [48] and many others. An impressive number of papers prove the always expanding interest on understanding geometric measure theory in such non-Euclidean frameworks.

Among the many topics that have been studied, we mention projection theorems, unrectifiability [4], [5], [30], [16], sets of finite h-perimeter, intrinsic regular sets, intrinsic differentiability, rectifiability [1], [23] [22], [24], [25], [36], [3], [39], [21], [26], [41], [14], differentiation of measures and covering theorems, uniform measures, singular integrals [40], [34], [13], [12], [11] and minimal surfaces [6], [44], [47], [46], [10], [49], [8], [9], [15], [31], [27], [45], [52]. These works represent only a small part of a vaster and always growing literature.

Aim of the present work is to establish area formulas for the spherical measure of new classes of C^1 smooth submanifolds. One of the key tools is the intrinsic blow-up, performed by translations and dilations that are compatible with the metric structure of the group (Section 2). The blow-up is expected to exist on "metric regular points". Precisely, these are those points having maximum pointwise degree (2.10), that is a kind of "pointwise Hausdorff dimension". The pointwise degree was introduced by Gromov in [28, Section 0.6.B]. It was subsequently rediscovered in [43], through an algebraic definition that also provides the density of the spherical measure.

However, pointwise degree does not possess enough information to describe the local behavior of the submanifold. We show how a more precise local geometric description is available through the *homogeneous tangent space*, in short *h-tangent space*. It is not difficult to find submanifolds of the same topological dimension, having the same pointwise degree at a fixed point, but whose corresponding h-tangent spaces are algebraically different (Remark 2.12).

The construction of the h-tangent space is purely algebraic. It arises from a formal "weighted homogeneous expansion" of the standard tangent space (Definition 2.7). The h-tangent space appeared in [43] to represent the intrinsic blow-up at points of maximum degree of a $C^{1,1}$ smooth submanifold. In the same paper it was proved that the h-tangent space is a homogeneous subgroup (Definition 2.2). Indeed, the $C^{1,1}$ regularity allows to consider a.e. commutators of vector fields tangent to the submanifold, finally leading to the Lie group structure of the h-tangent space. This kind of "algebraic regularity" joined with $C^{1,1}$ smoothness was central to establish the blow-up.

The present work can be seen as a development of [43] for C^1 submanifolds. With this lower regularity, extracting more information on the structure of the h-tangent space becomes crucial. We focus our attention on *algebraically regular points*, i.e. those points whose h-tangent space is a homogeneous subgroup. Somehow, this algebraic regularity compensates the lack of $C^{1,1}$ smoothness.

We may consider those submanifolds that at least at points of maximum degree have the h-tangent space in a specific family of subgroups. In Section 4 and Section 5 we focus our attention on horizontal submanifolds and transversal submanifolds, that satisfy this condition. For these submanifolds we can compute their spherical measure. The same approach also allows us to improve some previous results.

Horizontal submanifolds are defined by having the h-tangent space everywhere isomorphic to a *horizontal subgroup* (Definition 4.1). The crucial relation is the inclusion

$$T_p\Sigma \subset H_p\mathbb{G}, \tag{1.1}$$

for the submanifold Σ at every point p, with horizontal fiber $H_p\mathbb{G}$ defined in (2.4). This condition is everywhere satisfied by all horizontal submanifolds (Remark 4.5). To more easily detect and construct horizontal submanifolds, it is important to verify whether the everywhere validity of (1.1) implies that Σ is a horizontal submanifold.

We notice that when (1.1) is satisfied at a single point p, this does not necessarily imply that p is horizontal (Example 2.9). If (1.1) holds on an open subset of a C^2 submanifold Σ, then the approach of the classical Frobenius theorem implies that Σ is horizontal (Proposition 4.6). However, for C^1 smooth submanifolds the question becomes more delicate, since commutators of vector fields are not defined. Surprisingly, with C^1 regularity the classical proof of Frobenius theorem can be replaced by a differentiability result. The horizontality condition (1.1) implies a suitable differentiability of the parametrization of Σ (Theorem 4.7), that is the well known as Pansu differentiability. As a result, the area formulas (1.7) holds for all C^1 smooth horizontal submanifolds satisfing the condition (1.1) at every point. These submanifolds include for instance horizontal curves and Legendrian submanifolds.

Transversal submanifolds are in some sense at the opposite side of horizontal submanifolds. They can be defined through *transversal points*, which are those points whose h-tangent space is a vertical subgroup (Definition 5.1). Due to this transversality, with arguments similar to those of [37, Section 4], one could see that generically every smooth submanifold is transversal. All C^1 smooth hypersurfaces are special instances of transversal submanifolds. Every transversal submanifold is characterized by having maximal Hausdorff dimension among all C^1 smooth submanifolds with the same topological dimension [42]. The same condition characterizes vertical subgroups with respect to homogeneous subgroups. Theorem 1.3 also includes the area formula for these submanifolds (1.7). The first step to obtain these area formulas is the blow-up of the submanifold, that is the main technical tool of this work.

Theorem 1.1 (Blow-up)**.** *Let $\Sigma \subset \mathbb{G}$ be a C^1 smooth submanifold of topological dimension* n *and degree* N. *Let $p \in \Sigma$ be an algebraically regular point of maximum degree* N *and let $A_p\Sigma$ be its homogeneous tangent space. We assume that one of the following assumptions holds:*

1. *p is a horizontal point,*
2. *$\mathbb{G}$ has step two,*
3. *Σ is a one dimensional submanifold,*
4. *p is a transversal point.*

For the translated submanifold

$$\Sigma_p = p^{-1}\Sigma,$$

we introduce the C^1 smooth homeomorphism $\eta : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{\mathrm{n}}$ by

$$\eta(t) = \left(\frac{|t_1|^{b_1}}{b_1} \mathrm{sgn}\,(t_1), \ldots, \frac{|t_p|^{b_{\mathrm{n}}}}{b_{\mathrm{n}}} \mathrm{sgn}\,(t_{\mathrm{n}}) \right), \tag{1.2}$$

where each b_i is defined in (3.6). *If ψ denotes the mapping of Theorem 3.1 applied to the translated submanifold Σ_p, we define the C^1 smooth mapping*

$$\Gamma = \psi \circ \eta \tag{1.3}$$

and we define the subset of indexes $I \subset \{1, \ldots, \mathrm{q}\}$ such that

$$A_0\Sigma_p = \mathrm{span}\,\{e_l : l \in I\} = \mathrm{span}\left\{e_1, \ldots, e_{\alpha_1}, e_{\mathrm{m}_1+1}, \ldots, e_{\mathrm{m}_1+\alpha_2}, \ldots, e_{\mathrm{m}_{\iota-1}+\alpha_\iota}\right\}, \tag{1.4}$$

then the following local expansion holds

$$\Gamma_s(t) = \begin{cases} \frac{|t_{s-\mathrm{m}_{d_s-1}+\mu_{d_s-1}}|^{d_s}}{d_s} \mathrm{sgn}\,(t_{s-\mathrm{m}_{d_s-1}+\mu_{d_s-1}}) & \text{if } s \in I \\ o(|t|^{d_s}) & \text{if } s \notin I \end{cases}. \tag{1.5}$$

This theorem establishes the existence of the blow-up at an algebraically regular point of a C^1 smooth submanifold, under different conditions. Its proof, besides including new cases, also simplifies the previous arguments.

The second step to establish the area formula is to turn the blow-up of Σ into a suitable differentiation of its intrinsic measure μ_Σ (Definition 7.3). This measure, first introduced in [43], takes into account the degree N of Σ and the graded structure of the group. Finding the relationship between μ_Σ and the spherical measure of Σ corresponds to establish an area formula, due to the explicit form of μ_Σ. We use a suitable differentiation of the intrinsic measure, that works in metric spaces [40]. In Section 7 we adapt the general differentiation to homogeneous groups. The point is to find an explicit formula for the Federer density $\theta^{\mathrm{N}}(\mu_\Sigma,\cdot)$ that works in any metric space and it appears in the measure theoretic area formula (7.5). The Federer density is defined in (7.6). The metric differentiation leads us to an "upper blow-up" of the intrinsic measure, that is our second result.

Theorem 1.2 (Upper blow-up)**.** *Let $\Sigma \subset \mathbb{G}$ be a C^1 smooth submanifold of topological dimension* n *and degree* N*. Let $p \in \Sigma$ be an algebraically regular point of maximum degree* N *and let $A_p\Sigma$ be the* n*-dimensional homogeneous tangent space. We assume that one of the following assumptions holds:*

(1) *p is a horizontal point,*
(2) *$\mathbb{G}$ has step two,*
(3) *Σ is a one dimensional submanifold,*
(4) *p is a transversal point.*

Then the Federer density satisfies the following formula

$$\theta^{\mathrm{N}}(\mu_\Sigma, p) = \beta_d\big(A_p\Sigma\big). \tag{1.6}$$

The degree of Σ is the maximum integer N among all pointwise degrees of Σ. The number $\beta_d(A_p\Sigma)$ is the *spherical factor* (Definition 7.6) associated to the h-tangent space $A_p\Sigma$ of Σ at p. Such a number amounts to the maximal area of the intersection of $A_p\Sigma$ with any metric unit ball whose center moves in the metric unit ball centered at the origin. We are then arrived at our third result.

Theorem 1.3 (Area formula)**.** *Let $\Sigma \subset \mathbb{G}$ be a C^1 smooth* n*-dimensional submanifold of degree* N*. Suppose that one of the following conditions hold:*

(1) *Σ is a horizontal submanifold,*
(2) *$\mathbb{G}$ has step 2, every point of maximum degree is algebraically regular and points of lower degree are $\mathcal{S}^{\mathrm{N}}$ negligible,*
(3) *Σ is a transversal submanifold,*
(4) *Σ is one dimensional.*

Then for any Borel set $B \subset \Sigma$ we have

$$\int_B \|\tau^{\tilde g}_{\Sigma,\mathrm{N}}(p)\|_g \, d\sigma_{\tilde g}(p) = \int_B \beta_d(A_p\Sigma)\, d\mathcal{S}^{\mathrm{N}}_0(p). \tag{1.7}$$

We refer the reader to Section 7 for the definitions of the projected $\tilde g$-unit tangent n-vector $\tau^{\tilde g}_{\Sigma,\mathrm{N}}$ and the "nonrenormalized" spherical measure $\mathcal{S}_0^{\mathrm{N}}$. Notice that this formula also includes the case of finite dimensional Banach spaces, where $\mathbb{G}$ is commutative and made by only the first layer. Indeed, taking the Euclidean distance in $\mathbb{R}^n$, we get $\beta_{d_E}(A_p\Sigma)\equiv\omega_{\mathrm{n}}$, that is the volume of the unit ball in $\mathbb{R}^n$.

Theorem 1.3 is the union of different results of Section 8.2. Precisely, the implication from (1) to (1.7) corresponds to Theorem 8.8, where $\mathrm{n}=\mathrm{N}=\deg\Sigma$. In particular, the area formula (1.7) holds for C^1 smooth submanifolds everywhere tangent to the horizontal subbundle, due to Theorem 4.7. In other words, we can compute the spherical measure of all C^1 smooth Legendrian submanifold in any Heisenberg group.

The other implications of Theorem 1.3 all need a negligibility result for the set of points of lower degree. If Σ has degree N greater than its topological dimension, we have to prove that the (generalized) characteristic set

$$\mathcal{C}_\Sigma=\{p\in\Sigma: d_\Sigma(p)<\mathrm{N}\} \tag{1.8}$$

is $\mathcal{S}^{\mathrm{N}}$ negligible. The implication from assumption (2) to (1.7) follows from Theorem 8.2. Let us point out that by results of [38], when Σ is $C^{1,1}$ smooth in a two step group, we have $\mathcal{S}^{\mathrm{N}}(\mathcal{S}_\Sigma)=0$ and every point of maximum degree is algebraically regular [43]. Thus, assumptions (2) are more general than the conditions required in [38]. The validity of (1.7) from hypothesis (3) is a consequence of Theorem 8.1, where the $\mathcal{H}^{\mathrm{N}}$ negligibility of $\mathcal{C}_\Sigma$ is a nontrivial fact [42]. The implication from (4) to (1.7) comes from Theorem 8.3, slightly extending the results of [33].

Let us point out that (1.7) cannot be obtained through $C^{1,1}$ smooth approximation of C^1 submanifolds, since continuity theorems for the spherical measure require strong topological constraints. Additional efforts may arise to preserve the degree of the approximating submanifolds. Furthermore, possible "isolated submanifolds" of specific degree (2.11) could also appear. Such difficulties justify why working with C^1 submanifolds is important and meets a number of difficulties.

Formula (1.7) provides an explicit relationship between the intrinsic measure and the spherical measure. The latter is constructed by a homogeneous distance, that may also arise from a sub-Riemannian metric on a Carnot group. This somehow justifies the terminology "sub-Riemannian measure" for the intrinsic measure on the left-hand side of (1.7).

Our last application provides the first explicit formula relating spherical measure and Hausdorff measure on the class of horizontal submanifolds in homogeneous groups. Such a result requires some symmetry conditions on the distance.

Theorem 1.4 (Hausdorff and spherical measures)**.** *Let d be a multiradial distance and let $\Sigma\subset\mathbb{G}$ be a horizontal submanifold. Then the following equality holds*

$$\mathcal{H}^{\mathrm{n}}_d \llcorner \Sigma=\mathcal{S}^{\mathrm{n}}_d \llcorner \Sigma, \tag{1.9}$$

where $\mathcal{S}^{\mathrm{n}}_d=\omega(\mathrm{n},\mathrm{n})\mathcal{S}^{\mathrm{n}}_0$, $\mathcal{H}^{\mathrm{n}}_d=\omega(\mathrm{n},\mathrm{n})\mathcal{H}^{\mathrm{n}}_0$ and $\omega(\mathrm{n},\mathrm{n})$ is defined in (8.40).

Multiradial distances are introduced in Definition 8.5. They include for instance the Cygan-Korányi distance for groups of Heisenberg type and can be found in any homogeneous group (Remark 8.6). Clearly formula (1.9) also includes the classical one in Euclidean spaces, whose proof relies on the classical isodiametric inequality. In general, the constant $\omega(\mathrm{n},\mathrm{n})$ is the area of the metric unit ball intersected with an n-dimensional space contained in the first layer of $\mathbb{G}$.

The results of this paper provide a strong evidence that a unified approach to the area formula in homogeneous groups can be achieved. However, several questions are still to be understood. Whether or not an "algebraic classification" of submanifolds is required certainly represents a first question, which may have an independent interest. Other issues may arise from the study of general negligibility results for points of low degree. These questions and many others are a matter for future investigations.

2. Basic notions

2.1. **Graded nilpotent Lie groups and their metric structure.** A connected and simply connected *graded nilpotent Lie group* can be regarded as a graded linear space $\mathbb{G} = H^1 \oplus \cdots \oplus H^\iota$ equipped with a polynomial group operation such that its Lie algebra $\mathrm{Lie}(\mathbb{G})$ is *graded*. The subspaces H^j are called the *layers* of $\mathbb{G}$. This grading corresponds to the following conditions

$$\mathrm{Lie}(\mathbb{G}) = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_\iota, \qquad [\mathcal{V}_i, \mathcal{V}_j] \subset \mathcal{V}_{i+j} \tag{2.1}$$

for all integers $i, j \geq 0$ and $\mathcal{V}_j = \{0\}$ for all $j > \iota$, with $\mathcal{V}_\iota \neq \{0\}$. The integer $\iota \geq 1$ is the *step* of the group. The graded structure of $\mathbb{G}$ allows us to introduce intrinsic dilations $\delta_r : \mathbb{G} \to \mathbb{G}$ as linear mappings such that $\delta_r(p) = r^i p$ for each $p \in H^i$, $r > 0$ and $i = 1, \ldots, \iota$. The graded nilpotent Lie group $\mathbb{G}$ equipped with intrinsic dilations is called *homogeneous group*, [20]. With the stronger assumption that

$$[\mathcal{V}_1, \mathcal{V}_j] = \mathcal{V}_{j+1} \tag{2.2}$$

for each $j = 1, \ldots, \iota$ and $[\mathcal{V}_1, \mathcal{V}_\iota] = \{0\}$, we say that $\mathbb{G}$ is a *stratified group*. Identifying further $\mathbb{G}$ with the tangent space $T_0\mathbb{G}$ at the origin 0, we have a canonical isomorphism between H^j and $\mathcal{V}_j$, that associates to each $v \in H^j$ the unique left invariant vector field $X \in \mathcal{V}_j$ such that $X(0) = v$.

We may also assume that $\mathbb{G}$ is equipped with a Lie product that induces a Lie algebra structure, where its group operation is given through the Baker-Campbell-Hausdorff formula:

$$xy = \sum_{j=1}^{\iota} c_j(x, y) = x + y + \frac{[x, y]}{2} + \sum_{j=3}^{\iota} c_j(x, y) \tag{2.3}$$

with $x, y \in \mathbb{G}$. Here c_j denote homogeneous polynomials of degree j with respect to the nonassociative Lie product on $\mathbb{G}$. We will refer to (2.3) in short as BCH. It is always possible to have these additional conditions, since the exponential mapping

$$\exp : \mathrm{Lie}(\mathbb{G}) \to \mathbb{G}$$

of any connected and simply connected nilpotent Lie group $\mathbb{G}$ is a bianalytic diffeomorphism. In addition, the given Lie product and the Lie algebra associated to the induced group operation are compatible, according to the following standard fact.

Proposition 2.1. *Let G be a nilpotent, connected and simply connected Lie group and consider the new group operation given by* (2.3)*. Then the Lie algebra associated to this Lie group structure is isomorphic to the Lie algebra of G.*

We will denote by q the dimension of $\mathbb{G}$, seen as a linear space.

Definition 2.2. A linear subspace S of $\mathbb{G}$ that satisfies $\delta_r(S) \subset S$ for every $r > 0$ is a *homogeneous subspace* of $\mathbb{G}$. If in addition S is a Lie subgroup of $\mathbb{G}$ then we say that S is a *homogeneous subgroup* of $\mathbb{G}$.

Using dilations it is not difficult to check that $S \subset \mathbb{G}$ is a homogeneous subspace if and only if we have the direct decomposition

$$S = S_1 \oplus \cdots \oplus S_\iota,$$

where each S_j is a subspace of H^j.

A *homogeneous distance* d on a graded nilpotent Lie group $\mathbb{G}$ is a left invariant distance with $d(\delta_r x, \delta_r y) = r\, d(p,q)$ for all $p, q \in \mathbb{G}$ and $r > 0$. We define the open and closed balls

$$B(p,r) = \big\{q \in \mathbb{G} : d(q,p) < r\big\} \quad \text{and} \quad \mathbb{B}(p,r) = \big\{q \in \mathbb{G} : d(q,p) \le r\big\}\,.$$

The corresponding homogeneous norm is denoted by $\|x\| = d(x,0)$ for all $x \in \mathbb{G}$. When the graded nilpotent Lie group is equipped with the corresponding dilations, along with a homogeneous norm, is called *homogeneous group*.

In the special case $\mathbb{G}$ is a stratified group, the distribution of subspaces given by the so-called *horizontal fibers*

$$H_p\mathbb{G} = \{X(p) \in T_p\mathbb{G} : X \in \mathcal{V}_1\} \tag{2.4}$$

with $p \in \mathbb{G}$ satisfies the Lie bracket generating condition. In view of Chow's theorem, a left invariant sub-Riemannian metric, that is restricted to horizontal fibers, leads to the well known Carnot-Carathéodory distance. This is an important example of homogeneous distance. With this metric the Lie group $\mathbb{G}$ is also called *Carnot group*. We denote by $H\mathbb{G}$ the *horizontal subbundle* of $\mathbb{G}$, whose fibers are the ones of (2.4). A *graded basis* $(e_1, \ldots, e_{\mathrm{q}})$ of a homogeneous group $\mathbb{G}$ is a basis of vectors such that

$$(e_{\mathrm{m}_{j-1}+1}, e_{\mathrm{m}_{j-1}+2}, \ldots, e_{\mathrm{m}_j}) \tag{2.5}$$

is a basis of H^j for each $j = 1, \ldots, \iota$, where

$$\mathrm{m}_j = \sum_{i=1}^{j} \mathrm{h}_i \quad \text{and} \quad \mathrm{h}_j = \dim H^j, \tag{2.6}$$

we have set $\mathrm{m}_0 = 0$. We also set $\mathrm{m} = \mathrm{m}_1$ and observe that $\mathrm{m}_\iota = \mathrm{q}$. A graded basis provides the associated *graded coordinates* $x = (x_1, \ldots, x_{\mathrm{q}}) \in \mathbb{R}^{\mathrm{q}}$, then defining the unique element $p = \sum_{j=1}^{\mathrm{q}} x_j e_j \in \mathbb{G}$.

Remark 2.3. It is easy to realize that one can always equip a homogeneous subgroup with graded coordinates.

Throughout this work, a *graded left invariant Riemannian metric* g is fixed on the homogeneous group $\mathbb{G}$. This metric automatically induces a scalar product on $T_0\mathbb{G}$, therefore our identification of $\mathbb{G}$ with $T_0\mathbb{G}$ yields a fixed Euclidean structure in $\mathbb{G}$. The fact that our left invariant Riemannian metric g is "graded" means that the induced scalar product on $\mathbb{G}$ is *graded*, namely, all subspaces H^i with $i = 1, \ldots, \iota$ are orthogonal to each other. With a slight abuse of notation, the Euclidean norm on $\mathbb{G}$ and the norm arising from the Riemannian metric g on tangent spaces will be denoted by the same symbol $|\cdot|$.

For the sequel, it is also useful to recall that when a Riemannian metric $\tilde{g}$ is fixed on $\mathbb{G}$, then a scalar product on $\Lambda_k(T_p\mathbb{G})$ is automatically induced for every $p \in \mathbb{G}$. The corresponding norm on k-vectors is denoted by $\|\cdot\|_{\tilde{g}}$. A *$\tilde{g}$-unit k-vector* $v \in \Lambda_k(T_p\mathbb{G})$ satisfies $\|v\|_{\tilde{g}} = 1$.

Remark 2.4. One can easily check that when a graded scalar product is fixed, we can find a graded basis that is also orthonormal with respect to this scalar product.

2.2. **Degrees, multivectors and projections.** In this section we present a suitable notion of degree and of projection on k-vectors. Let us consider a graded basis $(e_1, \ldots, e_{\mathrm{q}})$ of $\mathbb{G}$ and the corresponding left invariant vector fields $X_j \in \mathrm{Lie}(\mathbb{G})$ such that $X_j(0) = e_j$ for each $j = 1, \ldots, \mathrm{q}$. We have obtained a basis $(X_1, \ldots, X_{\mathrm{q}})$ of the Lie algebra $\mathrm{Lie}(\mathbb{G})$. If the graded basis is orthonormal with respect to g, then our frame automatically becomes orthonormal. In the sequel, we will consider graded orthonormal frames.

If $(x_1, \ldots, x_{\mathrm{q}})$ are graded coordinates of graded basis $(e_1, \ldots, e_{\mathrm{q}})$, we assign *degree* j to each coordinate x_i such that $e_i \in H^j$. We analogously assign degree j to each left invariant vector field of $\mathcal{V}_j$. In different terms, for each $i \in \{1, \ldots, \mathrm{q}\}$ we consider the unique integer d_i on $\{1, \ldots, \iota\}$ such that

$$\mathrm{m}_{d_i-1} < i \leq \mathrm{m}_{d_i}.$$

It is easy to observe that d_i is the degree of both the coordinate x_i and the left invariant vector field X_i.

We denote by $\mathcal{I}_{k,\mathrm{q}}$ the family of all multi-index $I = (i_1, \ldots, i_k) \in \{1, \ldots, \mathrm{q}\}^k$ such that $1 \leq i_1 < \cdots < i_k \leq \mathrm{q}$. For each $I \in \mathcal{I}_{k,\mathrm{q}}$, we define the k-vector

$$X_I = X_{i_1} \wedge \cdots \wedge X_{i_k} \in \Lambda_k(\mathrm{Lie}(\mathbb{G}))\,, \tag{2.7}$$

whose degree is defined as follows

$$d(X_I) = d_{i_1} + \cdots + d_{i_k}.$$

Remark 2.5. The set $\{X_I : I \in \mathcal{I}_{k,\mathrm{q}}\}$ constitutes a basis of $\Lambda_k(\mathrm{Lie}(\mathbb{G}))$. We also notice that the degree of $X_1 \wedge \cdots \wedge X_{\mathrm{q}}$ is precisely

$$\mathrm{Q} = d_1 + \cdots + d_{\mathrm{q}},$$

where this number coincides with the Hausdorff dimension of $\mathbb{G}$ with respect to an arbitrary homogeneous distance.

The space $\Lambda_k(\mathrm{Lie}(\mathbb{G}))$ can be identified with the space of *left invariant k-vector fields*. The sections ξ of the vector bundle $\Lambda_k\mathbb{G} = \bigcup_{p\in\mathbb{G}} \Lambda_k(T_p\mathbb{G})$ are precisely the k-vector fields of $\mathbb{G}$. The left invariance of ξ is expressed by the equality

$$(\Lambda_k l_p)_*(\xi) = \xi$$

for every $p \in \mathbb{G}$, where $z \to l_p z = pz$ denotes the *left translation* by p. On a simple k-vector field $Z_1 \wedge \cdots \wedge Z_k$ made by the vector fields $Z_1, \ldots, Z_k$ of G, we have defined

$$(\Lambda_k l_p)_*(Z_1 \wedge \cdots \wedge Z_k) = (l_p)_* Z_1 \wedge \cdots \wedge (l_p)_* Z_k,$$

where $(l_p)_* Z_j$ is the push-forward of Z_j by l_p. In the sequel, we will automatically identify the space of k-vectors $\Lambda_k(\mathrm{Lie}G)$ with the space of left invariant k-vector fields. Indeed, whenever ξ is a left invariant k-vector field, the mapping that associates ξ to $\xi(0) \in \Lambda_k(T_0\mathbb{G})$ is an isomorphism and $\Lambda_k(T_0\mathbb{G})$ is isomorphic to $\Lambda_k(\mathrm{Lie}\mathbb{G})$.

Definition 2.6 (Projections on k-vectors)**.** Let $(X_1, \ldots, X_{\mathrm{q}})$ be a graded orthonormal frame, let $1 \le k \le \mathrm{q}$ and $1 \le M \le \mathrm{Q}$ be integers. For each left invariant k-vector field $\xi \in \Lambda_k(\mathrm{Lie}(\mathbb{G}))$, written as $\xi = \sum_{I\in\mathcal{I}_{k,\mathrm{q}}} c_I X_I$ for a suitable set of real numbers $\{c_I\}$, we define the *M-projection of ξ* as follows

$$\pi_M(\xi) = \sum_{\substack{I\in\mathcal{I}_{k,\mathrm{q}}\\ d(X_I)=M}} c_I X_I \in \Lambda_k(\mathrm{Lie}(\mathbb{G})).$$

This defines a mapping $\pi_M : \Lambda_k(\mathrm{Lie}(\mathbb{G})) \to \Lambda_k^M(\mathrm{Lie}(\mathbb{G}))$, where we have set

$$\Lambda_k^M(\mathrm{Lie}(\mathbb{G})) = \left\{ \sum_{I\in\mathcal{I}_{k,\mathrm{q}}} c_I X_I : d(X_I) = M,\ c_I \in \mathbb{R} \right\}.$$

For each $p \in \mathbb{G}$, we can also introduce the fibers

$$\Lambda_k^M\big(T_p\mathbb{G}\big) = \big\{\xi(p) \in \Lambda_k(T_p\mathbb{G}) : \xi \in \Lambda_k^M(\mathrm{Lie}(\mathbb{G}))\big\},$$

along with the following pointwise M-projection

$$\pi_{p,\mathrm{M}}(z) = \pi_{\mathrm{M}}(\xi)(p) \in \Lambda_k(T_p\mathbb{G}), \tag{2.8}$$

where $z \in \Lambda_k(T_p\mathbb{G})$ and there exists a unique $\xi \in \Lambda_k(\mathrm{Lie}(\mathbb{G}))$ such that $\xi(p) = z$. We clearly have $\pi_{p,\mathrm{M}} : \Lambda_k(T_p\mathbb{G}) \to \Lambda_k^M\big(T_p\mathbb{G}\big)$. By our identification of $\mathbb{G}$ with $T_0\mathbb{G}$, we introduce the translated projection of k-vectors at a point p to the origin:

$$\pi^0_{p,\mathrm{M}} : \Lambda_k(T_p\mathbb{G}) \to \Lambda_k\mathbb{G}. \tag{2.9}$$

For each $z \in \Lambda_k(T_p\mathbb{G})$, we consider the unique element $\xi \in \Lambda_k(\mathrm{Lie}(\mathbb{G}))$ such that

$$\xi(p) = z \quad \text{and} \quad \pi^0_{p,\mathrm{M}}(z) = \pi_{\mathrm{M}}(\xi)(0) \in \Lambda_k(T_0\mathbb{G}) \simeq \Lambda_k(\mathbb{G}).$$

2.3. **The homogeneous tangent space.** In this section and in the sequel Σ denotes an n-dimensional C^1 smooth submanifold embedded in a homogeneous group $\mathbb{G}$. A *tangent* n*-vector* of Σ at $p \in \Sigma$ is

$$\tau_\Sigma(p) = t_1 \wedge \cdots \wedge t_{\rm n} \in \Lambda_{\rm n}(T_p\Sigma),$$

where $(t_1, \ldots, t_{\rm n})$ is a basis of $T_p\Sigma$. This vector is not uniquely defined, but any other choice of the basis of $T_p\Sigma$ yields a proportional n-vector. We define the *pointwise degree* $d_\Sigma(p)$ of Σ at p as the integer

$$d_\Sigma(p) = \max\left\{M \in \mathbb{N} : \pi_{p,M}\left(\tau_\Sigma(p)\right) \neq 0\right\} \tag{2.10}$$

and the *degree* of Σ is the positive integer

$$d(\Sigma) = \max\{d_\Sigma(p) : p \in \Sigma\} \in \mathbb{N} \setminus \{0\}\,. \tag{2.11}$$

We say that $p \in \Sigma$ has *maximum degree* if $d_\Sigma(p) = d(\Sigma)$.

Definition 2.7 (Homogeneous tangent space)**.** Let $p \in \Sigma$ and set $d_\Sigma(p) = {\rm N}$. If $\tau_\Sigma(p)$ is a tangent n-vector to Σ at p and $\xi_{p,\Sigma} \in \Lambda_{\rm n}({\rm Lie}(\mathbb{G}))$ is the unique left invariant n-vector field such that $\xi_{p,\Sigma}(p) = \tau_\Sigma(p)$, then we define the *Lie homogeneous tangent space* of Σ at p, in short the *Lie h-tangent space* as follows

$$\mathcal{A}_p\Sigma = \left\{X \in {\rm Lie}(\mathbb{G}) : X \wedge \pi_{\rm N}(\xi_{p,\Sigma}) = 0\right\}.$$

We say that $p \in \Sigma$ is *algebraically regular* if $\mathcal{A}_p\Sigma$ is a subalgebra of ${\rm Lie}(\mathbb{G})$. In this case we call the corresponding subgroup

$$A_p\Sigma = \exp \mathcal{A}_p\Sigma$$

the *homogeneous tangent space* of Σ at p, or simply the *h-tangent space* of Σ at p.

Remark 2.8. It is very important that the h-tangent space can be defined at any point of a smooth submanifold of a graded group. In many cases, it precisely coincides with the blow-up of the submanifold, when it is performed by intrinsic dilations and the group operation.

Any point of a C^1 smooth curve of $\mathbb{G}$ is algebraically regular, since any one dimensional linear subspace of a layer H^j is automatically a homogeneous subalgebra. Points that are not algebraically regular may appear in submanifolds of dimension higher than one, according to the next example.

Example 2.9. Let the first Heisenberg group $\mathbb{H}$ be identifed with $\mathbb{R}^3$ through the coordinates (x_1, x_2, x_3) such that the group operation reads as follows

$$(x_1, x_2, x_3)(x_1', x_2', x_3') = (x_1 + x_1', x_2 + x_2', x_3 + x_3' + x_1x_2' - x_2x_1').$$

Let $\Sigma = \{(x_1, x_2, x_3) \in \mathbb{H} : x_3 = x_1^2 + x_2^2\}$ be a 2-dimensional submanifold. Let us show that the origin $p = (0, 0, 0) \in \Sigma$ is not algebraically regular. It is easy to observe that $d_\Sigma(p) = 2$. We have $T_p\Sigma = {\rm span}\,\{e_1, e_2\}$, where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$, therefore $\tau_\Sigma(p) = e_1 \wedge e_2$. Introducing the left invariant vector fields

$$X_1(x) = e_1 - x_2e_3 \quad \text{and} \quad X_2 = e_2 + x_1e_3,$$

have may define $\xi = X_1 \wedge X_2$, of degree two, such that $\xi(0) = e_1 \wedge e_2$. This implies that $\pi_{p,2}(e_1 \wedge e_2) \neq 0$ and $\pi_{p,j}(e_1 \wedge e_2) = 0$ for all $j \geq 3$. The Lie h-tangent space is defined as follows

$$\mathcal{A}_p\Sigma = \{X \in \mathrm{Lie}(\mathbb{G}) : X \wedge X_1 \wedge X_2 = 0\} = \mathrm{span}\,\{X_1, X_2\}\,,$$

that is not a Lie subalgebra of Lie($\mathbb{H}$). The homogeneous tangent space

$$A_p\Sigma = \exp \mathcal{A}_p\Sigma = \{(x, y, 0) \in \mathbb{H} : x, y \in \mathbb{R}\}$$

is a subspace of $\mathbb{H}$, but it is not a subgroup.

Remark 2.10 (Characteristic points)**.** We observe that in the previous example the origin p is also a *characteristic point* of Σ. The general definition states that a characteristic point q of a C^1 smooth hypersurface $\Sigma \subset \mathbb{G}$ satisfies $H_q\mathbb{G} \subset T_q\Sigma$. This kind of point behaves as a singular point with respect to the metric strucure of $\mathbb{G}$.

The notion of algebraic regularity fits with this picture in that characteristic points are not algebraically regular, as it can be seen arguing as in Example 2.9 and taking into account the invariance of the pointwise degree under left translations, as shown in Proposition 3.6. On the other hand, for all C^1 smooth hypersurfaces, characteristic points are negligible with respect to the $(Q-1)$-dimensional Hausdorff measure [36].

Example 2.11. Let p be a point of a 2-dimensional Legendrian submanifold Σ, see Section 4, that is embedded in the second Heisenberg group $\mathbb{H}^2$. Then $d_\Sigma(p) = 2$ and p is an algebraically regular point whose homogeneous tangent space is a commutative horizontal subgroup of $\mathbb{H}^2$. Here we consider $\mathbb{H}^2$ as $\mathbb{R}^5$ equipped with the horizontal left invariant vector fields

$$X_1(x) = e_1 - x_3 e_5, \quad X_2 = e_2 - x_4 e_5, \quad X_3(x) = e_3 + x_1 e_5, \quad X_4 = e_4 + x_2 e_5,$$

spanning the first layer of the stratified Lie algebra Lie($\mathbb{H}^2$).

Remark 2.12. Examples 2.9 and 2.11 show that one can find different submanifolds of the same dimension with points of the same degree, where only one of these points is algebraically regular. This shows somehow that algebraic regularity encodes the "behavior" of the submanifold around the point. The pointwise degree clearly provides less information.

3. Special coordinates around points of submanifolds

Throughout this section, the symbol $\Sigma \subset \mathbb{G}$ will denote a C^1 smooth submanifold embedded in a homogeneous group $\mathbb{G}$, if not otherwise stated. To perform the blow-up of Σ at a fixed point, finding special coordinates is of capital importance. They are also useful to determine degree and homogeneous tangent space of a fixed point.

From the proof of [43, Lemma 3.1], it is not difficult to see that special coordinates can be found around any point of a submanifold, that need not have maximum degree. This is the content of the following theorem.

Theorem 3.1. *Let $\Sigma \subset \mathbb{G}$ be a C^1 smooth submanifold of topological dimension* n *and let $0 \in \Sigma$. There exist $\alpha_1, \ldots, \alpha_\iota \in \mathbb{N}$ with $\alpha_j \leq \mathrm{h}_j$ for all $j = 1, \ldots, \iota$, an orthonormal graded basis $(e_1, \ldots, e_{\mathrm{q}})$ with respect to the fixed graded scalar product on $\mathbb{G}$, a bounded open neighborhood $U \subset \mathbb{R}^{\mathrm{n}}$ of the origin and a C^1 smooth embedding $\Psi : U \to \Sigma$ with the following properties. There holds $\Psi(0) = 0 \in \mathbb{G}$, for all $y \in U$*

$$\Psi(y) = \sum_{j=1}^{\mathrm{q}} \psi_j(y) e_j, \quad \psi(y) = (\psi_1(y), \ldots, \psi_{\mathrm{q}}(y))$$

and the Jacobian matrix of ψ at the origin is

$$D\psi(0) = \left(\begin{array}{c|c|c|c|c|c} I_{\alpha_1} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & * & \cdots & \cdots & \cdots & * \\ \hline 0 & I_{\alpha_2} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & * & \cdots & \cdots & * \\ \hline 0 & 0 & I_{\alpha_3} & 0 & \cdots & 0 \\ 0 & 0 & 0 & * & \cdots & * \\ \hline \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & \cdots & \cdots & I_{\alpha_\iota} \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \end{array}\right). \tag{3.1}$$

The blocks containing the identity matrix I_{α_j} have h_j rows, for every $j = 1, \ldots, \iota$. The blocks $*$ are $(\mathrm{h}_j - \alpha_j) \times \alpha_i$ matrices, for all $j = 1, \ldots, \iota - 1$ and $i = j+1, \ldots, \iota$. The mapping ψ can be assumed to have the special graph form given by the conditions

$$\psi_s(y) = y_{s - \mathrm{m}_{j-1} + \mu_{j-1}} \tag{3.2}$$

for every $s = \mathrm{m}_{j-1} + 1, \ldots, \mathrm{m}_{j-1} + \alpha_j$ and $j = 1, \ldots, \iota$, where we have defined

$$\mu_0 = 0 \quad \textit{and} \quad \mu_j = \sum_{i=1}^{j} \alpha_i \quad \textit{for } j = 1, \ldots, \iota. \tag{3.3}$$

Remark 3.2. The numbers α_j provided by Theorem 3.1 are uniquely defined and do not depend on the choice of the special coordinates ψ_j. One may also observe that

$$\mu_\iota = \mathrm{n} \quad \text{and} \quad d_\Sigma(0) = \sum_{i=1}^{\iota} i\, \alpha_i, \tag{3.4}$$

where n is the topological dimension of Σ and $d_\Sigma(0)$ is the degree of Σ at the origin.

Proposition 3.3. *Under the assumptions of Theorem 3.1, the homogeneous tangent space of Σ at the origin can be represented as follows*

$$A_0\Sigma = \operatorname{span}\left\{e_1, \ldots, e_{\alpha_1}, e_{\mathrm{m}_1+1}, \ldots, e_{\mathrm{m}_1+\alpha_2}, \ldots, e_{\mathrm{m}_{\iota-1}+1}, \ldots, e_{\mathrm{m}_{\iota-1}+\alpha_\iota}\right\}. \tag{3.5}$$

Proof. From the form of the Jacobian matrix (3.1) and the definition of homogeneous tangent space, there holds

$$\pi^0_{0,\mathrm{N}}\left(\partial_1\psi(0) \wedge \partial_2\psi(0) \wedge \cdots \wedge \partial_{\mathrm{n}}\psi(0)\right) = e_1 \wedge \cdots \wedge e_{\alpha_1} \cdots \wedge e_{\mathrm{m}_{\iota-1}+1} \wedge \cdots \wedge e_{\mathrm{m}_{\iota-1}+\alpha_1}.$$

The unique left invariant n-vector field $\xi \in \Lambda_{\mathrm{n}}(\mathrm{Lie}(\mathbb{G}))$ such that

$$\xi(0) = \partial_1\psi(0) \wedge \partial_2\psi(0) \wedge \cdots \wedge \partial_{\mathrm{n}}\psi(0)$$

then satisfies

$$\pi_{\mathrm{N}}(\xi) = X_1 \wedge \cdots \wedge X_{\alpha_1} \wedge \cdots \wedge X_{\mathrm{m}_{\iota-1}+1} \wedge \cdots \wedge X_{\mathrm{m}_{\iota-1}+\alpha_1}.$$

As a result, in view of Definition 2.7 our claim is established. □

The special coordinates of Theorem 3.1 allow us to introduce an "induced degree" on Σ, as in the next definition.

Definition 3.4. In the notation of Theorem 3.1, we define

$$b_i = j \quad \text{if and only if} \quad \mu_{j-1} < i \le \mu_j \tag{3.6}$$

for every $i = 1, \dots, \mathrm{n}$. The integer b_i is the *induced degree of* y_i, with respect to the coordinates $y = (y_1, \dots, y_{\mathrm{n}})$ of Σ around the origin, in Theorem 3.1. We define accordingly the *induced dilations* $\sigma_r : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{\mathrm{n}}$ as follows

$$\sigma_r(t_1, \dots, t_{\mathrm{n}}) = (r^{b_1} t_1, \dots, r^{b_{\mathrm{n}}} t_{\mathrm{n}}) \quad \text{where } r > 0. \tag{3.7}$$

The coordinates y of Theorem 3.1 allow us to act on the homogeneous tangent space $A_p\Sigma$ of Σ through the induced dilations σ_r. This is an important fact, that will be used in the sequel.

Corollary 3.5. *Under the assumptions of Theorem 3.1, we consider the frame of left invariant vector fields*

$$X_1, \dots, X_{\mathrm{q}}$$

adapted to the coordinates of the theorem, namely we impose the condition $X_j(0) = e_j$ *for each* $j = 1, \dots, \mathrm{q}$. *Then there exist unique continuous coefficients* C_i^s *such that*

$$\partial_i \psi = \sum_{s=1}^{\mathrm{q}} C_i^s(\psi)\, X_s(\psi) \quad \textit{for all } i = 1, \dots, \mathrm{n}. \tag{3.8}$$

If $0 \in \Sigma$ *has maximum degree, then the* $\mathrm{q} \times \mathrm{n}$ *matrix-valued function* C *of coefficients* C_i^s *satisfies the following formula*

$$C = \left(\begin{array}{c|c|c|c|c|c}
I_{\alpha_1} + o(1) & o(1) & \cdots & \cdots & \cdots & o(1) \\
o(1) & * & \cdots & \cdots & \cdots & * \\
\hline
o(1) & I_{\alpha_2} + o(1) & o(1) & \cdots & \cdots & o(1) \\
0 & o(1) & * & \cdots & \cdots & * \\
\hline
o(1) & o(1) & I_{\alpha_3} + o(1) & o(1) & \cdots & o(1) \\
0 & 0 & o(1) & * & \cdots & * \\
\hline
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\hline
o(1) & o(1) & \cdots & \cdots & \cdots & I_{\alpha_\iota} + o(1) \\
0 & 0 & \cdots & \cdots & \cdots & o(1)
\end{array}\right). \tag{3.9}$$

The symbols $o(1)$ denote a continuous submatrix that vanishes at 0. The constantly null submatrices in (3.9) are denoted by 0. In the case $0 \in \Sigma$ is not of maximum degree these submatrices are replaced by other matrices $o(1)$ vanishing at 0.

Proof. The form (3.9) of C follows from (3.1) joined with the assumption that $0 \in \Sigma$ has maximum degree. The assumption on the maximum degree of the origin is needed only to obtain the constantly vanishing submatrices of (3.9). □

Theorem 3.1 and Corollary 3.5 provide special coordinates around any point of a smooth submanifold. The next proposition shows that translations preserve the "algebraic structure of points".

Proposition 3.6. *If Σ is a C^1 smooth submanifold, $p \in \Sigma$ and we define the translated submanifold $\Sigma_p = p^{-1}\Sigma$, then*

$$d_\Sigma(p) = d_{\Sigma_p}(0), \quad \mathcal{A}_p\Sigma = \mathcal{A}_0\Sigma_p \quad \text{and} \quad A_p\Sigma = A_0\Sigma_p.$$

Proof. We consider the tangent n-vector

$$\tau_\Sigma(p) = \sum_{I \in \mathcal{I}_{k,\mathrm{q}}} c_I X_I(p),$$

where X_I are defined in (2.7) and $c_I \in \mathbb{R}$ and the translated one

$$\tau_{\Sigma_p}(0) = dl_{p^{-1}} \left(\sum_{I \in \mathcal{I}_{k,\mathrm{q}}} c_I X_I(p) \right) = \sum_{I \in \mathcal{I}_{k,\mathrm{q}}} c_I X_I(0).$$

We have used the left invariance of the basis $(X_1, \ldots, X_{\mathrm{q}})$, that defines the k-vectors X_I. This invariance of the coefficients c_I joint with the definition of degree and of homogeneous tangent space immediately lead us to our claim. □

As we have previously seen, the continuous matrix (3.9) is related to the algebraic structure of the homogeneous tangent space $A_0\Sigma$ and it plays an important role in the proof of the blow-up of Theorem 1.1. This result considers four distinct cases that correspond to different "shapes" of the submanifold around the blow-up point. It is then imporant to make the form of the continuous matrix (3.9) explicit in each of the four cases.

If $\mathbb{G}$ is of step two, the continuous matrix C of (3.9) takes the form

$$C = \left(\begin{array}{c|c} I_{\alpha_1} + o(1) & o(1) \\ o(1) & * \\ \hline o(1) & I_{\alpha_2} + o(1) \\ 0 & o(1) \end{array} \right). \tag{3.10}$$

In the case Σ is curve embedded in $\mathbb{G}$, namely $\mathrm{n}=1$, $\alpha_{\mathrm{N}}=1$, we have

$$C=\left(\begin{array}{c}\vdots\\ \hline *\\ \hline I_{\alpha_{\mathrm{N}}}+o(1)\\ o(1)\\ \hline 0\\ \hline \vdots\\ \hline 0\end{array}\right), \tag{3.11}$$

where $I_{\alpha_{\mathrm{N}}}$ in this case denotes the 1×1 matrix equal to one. The remaining two cases, related to the special structure of the homogeneous tangent space, need to be treated in more detail. They are indeed related to specific classes of submanifolds.

4. Horizontal points and horizontal submanifolds

Horizontal points are a specific class of algebraically regular points, associated to a class of subgroups. The interesting fact is that they have a corresponding class of submanifolds, where all points are horizontal.

Definition 4.1 (Horizontal subgroup)**.** We say that $H\subset\mathbb{G}$ is a *horizontal subgroup* if it is a homogeneous subgroup contained in the first layer H^1 of $\mathbb{G}$.

Clearly horizontal subgroups are automatically commutative.

Definition 4.2 (Horizontal points and horizontal submanifolds)**.** A *horizontal point* p of a C^1 smooth submanifold Σ embedded in a homogeneous group $\mathbb{G}$ is an algebraically regular one whose homogeneous tangent space is a horizontal subgroup. The submanifold Σ is *horizontal* if all of its points are horizontal.

Horizontal points determine a special form of the matrix C in Corollary 3.5, as shown in the next proposition.

Proposition 4.3. *In the assumptions of Corollary 3.5, if the origin $0\in\Sigma$ is a horizontal point, then $\alpha_1=\mathrm{n}$, $\alpha_j=0$ for each $j=2,\ldots,\iota$ and the continuous matrix* (3.9) *takes the following form*

$$C=\left(\begin{array}{c}I_{\alpha_1}+o(1)\\ o(1)\\ \hline 0\\ \hline \vdots\\ \hline 0\end{array}\right). \tag{4.1}$$

Proof. From Proposition 3.3, we have $A_0\Sigma=\mathrm{span}\,\{e_1,\ldots,e_{\alpha_1}\}$. This immediately shows the form of C given in (4.1). $\square$

Remark 4.4. Joining the previous proposition with Remark 3.2 and taking into account the left invariance pointed out in Proposition 3.6, one immediately observes that all points of an n-dimensional horizontal submanifold Σ have degree n. Therefore the degree of Σ coincides with its topological dimension.

Remark 4.5. Proposition 4.3 shows in particular that a horizontal point p of a C^1 smooth submanifold Σ must satisfy the condition

$$T_p\Sigma \subset H_p\mathbb{G}. \tag{4.2}$$

Then any C^1 smooth horizontal submanifold is tangent to the horizontal subbundle $H\mathbb{G}$. In different terms, Σ is an integral submanifold of the distribution made by the fibers $H_p\mathbb{G}$.

The inclusion (4.2) alone does not imply that p is horizontal, see Example 2.9.

Proposition 4.6. *If Σ is a C^2 smooth submanifold such that $T_p\Sigma \subset H_p\mathbb{G}$ for every $p \in \Sigma$, then Σ is a horizontal submanifold.*

Proof. Fix $p \in \Sigma$ and consider two arbitrary C^1 smooth sections X and Y of the tangent bundle $T\Sigma$, which are defined on a neighborhood U of p. There exist a_j, b_l C^1 smooth coefficients on U such that

$$X = \sum_{j=1}^{\mathrm{m}} a_j X_j \quad \text{and} \quad Y = \sum_{j=1}^{\mathrm{m}} b_j X_j$$

where $(X_1, \ldots, X_{\mathrm{m}})$ is a frame of horizontal left invariant vector fields, namely a basis of the first layer $\mathcal{V}_1 \subset \mathrm{Lie}(\mathbb{G})$. It follows that

$$\begin{aligned}[X,Y](p) = &\sum_{j,l=1}^{\mathrm{m}} a_j(p) b_l(p)[X_j, X_l](p) + \sum_{j,l=1}^{\mathrm{m}} a_j(p) X_j b_l(p) X_l(p) \\ &- \sum_{l,l=1}^{\mathrm{m}} b_l(p) X_l a_j(p) X_j(p) \in H_p\mathbb{G} \cap T_p\Sigma.\end{aligned} \tag{4.3}$$

Due to (4.3), we have proved that

$$\left[\sum_{j=1}^{\mathrm{m}} a_j(p) X_j, \sum_{l=1}^{\mathrm{m}} b_l(b) X_l\right](p) = \sum_{i,j=1}^{\mathrm{m}} a_j(p) b_l(p) [X_j, X_l](p) \in H_p\mathbb{G}.$$

The coefficients a_j, b_l are arbitrarily chosen to get any possible couple of sections of $T\Sigma$ around p. In particular, we can choose any couple of vectors in $T_p\Sigma \subset H_p\mathbb{G}$, consider their associated left invariant vector fields and observe that their Lie bracket evaluated at the origin is in $dl_{p^{-1}}(T_p\Sigma) \subset H_0\mathbb{G}$, namely their Lie bracket is in $\mathcal{V}_1$. We have proved that $\mathcal{A}_p\Sigma$ is a commutative subalgebra of $\mathcal{V}_1$, hence p is a regular point and its homogeneous tangent space $A_p\Sigma = \exp \mathcal{A}_p\Sigma$ is a horizontal subgroup. □

Theorem 4.7. *If Σ is a C^1 smooth submanifold such that $T_p\Sigma \subset H_p\mathbb{G}$ for every $p \in \Sigma$, then Σ is a horizontal submanifold.*

Proof. Let us consider a C^1 smooth local chart $\Psi : \Omega \to U$ of the C^1 smooth horizontal submanifold $\Sigma \subset \mathbb{G}$. Here $\Omega \subset \mathbb{R}^k$ is an open set and U is an open subset of Σ. The fact that Σ is horizontal precisely means that

$$d\Psi(x)(\mathbb{R}^k) \subset H_{\Psi(x)}\mathbb{G}$$

for a every $x \in \Omega$. These conditions coincides with the validity of contact equations, according to [39]. However, they do not ensure a priori that the subspace of $\mathcal{V}_1$ associated to the subspace $d\Psi(x)(\mathbb{R}^k)$ is a commutative subalgebra. To obtain this information we use [39, Theorem 1.1], according to which Ψ is also differentiable with respect to dilations and the group operation. In particular, this gives the existence of the following limit

$$\lim_{t\to 0^+} \delta_{1/t}\left(\Psi(x)^{-1}\Psi(x+tv)\right) = L_x(v) \tag{4.4}$$

where $v \in \mathbb{R}^k$ and $L_x : \mathbb{R}^k \to \mathbb{G}$ is a Lie group homomorphism. We fix now a point $p = \Psi(x_0) \in \Sigma$, observing that

$$H_0 = L_{x_0}(\mathbb{R}^k)$$

is a horizontal subgroup of $\mathbb{G}$. We fix a graded basis $(e_1, \ldots, e_{\mathrm{q}})$ of $\mathbb{G}$, hence we set

$$\Psi(x) = \sum_{j=1}^{\mathrm{q}} \psi_j(x)e_j \quad \text{and} \quad L_{x_0}(v) = \sum_{j=1}^{\mathrm{m}} (L_{x_0})_j(v)e_j.$$

The Baker-Campbell-Hausdorff formula joined with the limit (4.4) yields

$$d\psi_j(x_0)(v) = (L_{x_0})_j(v) \quad \text{for all } j = 1, \ldots, \mathrm{m}. \tag{4.5}$$

The same formula shows that the left invariant vector fields $X_1, \ldots, X_{\mathrm{q}}$ have a special polynomial form. Indeed assuming that $X_j(0) = e_j$, with the identification of $\mathbb{G}$ with $T_0\mathbb{G}$, being $\mathbb{G}$ a linear space, we have

$$X_j(x) = e_j + \sum_{l=\mathrm{m}+1}^{\mathrm{q}} a_{jl}(x)e_l,$$

where $a_{jl} : \mathbb{G} \to \mathbb{R}$ a polynomials. We have

$$\begin{aligned}
\frac{\partial\Psi}{\partial x_k}(x) &= \sum_{j=1}^{\mathrm{q}} \frac{\partial\psi_j}{\partial x_k}(x)e_j = \sum_{j=1}^{\mathrm{m}} \frac{\partial\psi_j}{\partial x_k}(x)e_j + \sum_{j=\mathrm{m}+1}^{\mathrm{q}} \frac{\partial\psi_j}{\partial x_k}(x)e_j \\
&= \sum_{j=1}^{\mathrm{m}} \frac{\partial\psi_j}{\partial x_k}(x)\, X_j(\Psi(x)) - \sum_{l=\mathrm{m}+1}^{\mathrm{q}} \sum_{j=1}^{\mathrm{m}} \frac{\partial\psi_j}{\partial x_k}(x)a_{jl}(\Psi(x))e_l + \sum_{j=\mathrm{m}+1}^{\mathrm{q}} \frac{\partial\psi_j}{\partial x_k}(x)e_j \\
&= \sum_{j=1}^{\mathrm{m}} \frac{\partial\psi_j}{\partial x_k}(x)\, X_j(\Psi(x)),
\end{aligned}$$

where in the last equality we have used the fact that any $\partial_{x_k}\Psi(x)$ must be horizontal, namely $\partial_{x_k}\Psi(x) \in H_{\Psi(x)}\mathbb{G}$ for all $x \in \Omega$. Applying the definition of algebraically regular point, we consider the left invariant vector fields

$$Y_k = \sum_{j=1}^{\mathrm{m}} \frac{\partial \psi_j}{\partial x_k}(x_0)\, X_j \in \mathcal{V}_1 \quad \text{for } k = 1, \ldots, \mathrm{m}.$$

Setting $(E_1, \ldots, E_k)$ as the canonical basis of $\mathbb{R}^k$, by (4.5) we define

$$v_k = L_{x_0}(E_k) = \sum_{j=1}^{\mathrm{m}} (L_{x_0})_j(E_k) e_j = \sum_{j=1}^{\mathrm{m}} \frac{\partial \psi_j}{\partial x_k}(x_0) e_j \in H_0.$$

Being H_0 a horizontal subgroup, it is in particular commutative, therefore

$$[v_k, v_s] = \sum_{j,l=1}^{\mathrm{m}} \frac{\partial \psi_j}{\partial x_k}(x_0) \frac{\partial \psi_l}{\partial x_s}(x_0) [e_j, e_l] = 0.$$

This proves that

$$[Y_k, Y_s] = \sum_{j,l=1}^{\mathrm{m}} \frac{\partial \psi_j}{\partial x_k}(x_0) \frac{\partial \psi_l}{\partial x_s}(x_0) [X_j, X_l] = 0,$$

due to the isomorphism between the Lie product on $\mathbb{G}$ and Lie($\mathbb{G}$), see Proposition 2.1. We have shown that

$$\mathcal{A}_p\Sigma = \operatorname{span}\{Y_1, \ldots, Y_k\}$$

is commutative, hence $\Psi(x_0)$ is an algebraically regular point and the homogeneous tangent space $A_p\Sigma = \exp \mathcal{A}_p\Sigma$ is a horizontal subgroup. □

Remark 4.8. As a consequence of the previous theorem, all C^1 smooth Legendrian submanifolds in the Heisenberg group are horizontal submanifolds.

5. Transversal points and transversal submanifolds

This section is devoted to a class of submanifolds containing a specific type of algebraically regular point. We start with the following definition.

Definition 5.1 (Vertical subgroup)**.** We say that a homogeneous subgroup $N \subset \mathbb{G}$ is a *vertical subgroup* if

$$N = N_\ell \oplus H^{\ell+1} \oplus \cdots \oplus H^\iota \tag{5.1}$$

for some $\ell \in \{1, \ldots, \iota\}$ and a linear subspace $N_\ell \subset H^\ell$.

One may easily observe that any vertical subgroup is also a normal subgroup of $\mathbb{G}$.

Definition 5.2 (Transversal points and transversal submanifolds)**.** Let $\Sigma \subset \mathbb{G}$ be a C^1 smooth submanifold. A *transversal point* p of Σ is an algebraically regular point, whose homogeneous tangent space is a vertical subgroup. The submanifold Σ is *transversal* if it contains at least one *transversal point*.

Transversal points can be characterized by their degree. To see this, we introduce the following integer valued functions $\ell_\cdot, r_\cdot : \{1, \ldots, \mathrm{q}\} \to \mathbb{N}$. For every $\mathrm{n} = 1, \ldots, \mathrm{q}$, the inequalities

$$\begin{cases} \ell_{\mathrm{n}} = \iota & \text{if } 1 \le \mathrm{n} \le \mathrm{h}_\iota \\ \displaystyle\sum_{j=\ell_{\mathrm{n}}+1}^{\iota} \mathrm{h}_j < \mathrm{n} \le \sum_{j=\ell_{\mathrm{n}}}^{\iota} \mathrm{h}_j & \text{if } \mathrm{h}_\iota < \mathrm{n} \le \mathrm{q} \end{cases} \tag{5.2}$$

uniquely define the integer $\ell_{\mathrm{n}} \in \{1, \ldots, \iota\}$. Thus, we also define

$$\mathrm{r}_{\mathrm{n}} := \begin{cases} \mathrm{n} & \text{if } 1 \le \mathrm{n} \le \mathrm{h}_\iota \\ \mathrm{n} - \displaystyle\sum_{j=\ell_{\mathrm{n}}+1}^{\iota} \mathrm{h}_j & \text{if } \mathrm{h}_\iota < \mathrm{n} \le \mathrm{q} \end{cases} \tag{5.3}$$

for every $\mathrm{n} = 1, \ldots, \mathrm{q}$, where $\mathrm{r}_{\mathrm{n}} \ge 1$. We finally set

$$Q_{\mathrm{n}} = \ell_{\mathrm{n}}\, \mathrm{r}_{\mathrm{n}} + \sum_{j=\ell_{\mathrm{n}}+1}^{\iota} j\, \mathrm{h}_j\,, \tag{5.4}$$

where the sum is understood to be zero only in the case $1 \le \mathrm{n} \le \mathrm{h}_\iota$, that is $\ell_{\mathrm{n}} = \iota$.

If $N \subset \mathbb{G}$ is an n-dimensional vertical subgroup of the form (5.1), it is not difficult to observe that the degree at every point of N equals Q_{n} given in (5.4) with

$$\dim N_\ell = \mathrm{r}_{\mathrm{n}} \quad \text{and} \quad \ell = \ell_{\mathrm{n}}.$$

From formula (3.4), taking into account Proposition 3.6, it is not difficult to realize that

$$Q_{\mathrm{n}} = \max_{\Sigma \in \mathcal{S}_{\mathrm{n}}(\mathbb{G})} d(\Sigma). \tag{5.5}$$

The set $\mathcal{S}_{\mathrm{n}}(\mathbb{G})$ denotes the family of n-dimensional submanifolds of class C^1 that are contained in $\mathbb{G}$. The integer $d(\Sigma)$ is the degree of Σ introduced in (2.11).

We are now in the position to prove the following characterization.

Proposition 5.3. *A point p of an* n*-dimensional C^1 smooth submanifold $\Sigma \subset \mathbb{G}$ is transversal if and only if $d_\Sigma(p) = Q_{\mathrm{n}}$.*

Proof. If p is transversal, using left translations we may assume that it coincides with the origin. Using the coordinates of Theorem 3.1 and applying formula (3.5), the fact that $A_0\Sigma$ is a transversal subgroup gives

$$A_0\Sigma = \operatorname{span}\left\{e_{\mathrm{m}_{\ell-1}+1}, \ldots, e_{\mathrm{m}_{\ell-1}+\mathrm{r}}, e_{\mathrm{m}_\ell+1}, e_{\mathrm{m}_\ell+2}, \ldots, e_{\mathrm{q}}\right\}. \tag{5.6}$$

We have assumed that $A_0\Sigma$ has the form of (5.1) and $\dim N_\ell = \mathrm{r}$. From (3.4) we immediately get

$$d_\Sigma(0) = \mathrm{r}\ell + \sum_{j=\ell+1}^{\iota} j\, \mathrm{h}_j\,,$$

where it must be $\mathrm{r} = \mathrm{r_n}$ and $\ell = \ell_\mathrm{n}$, from (5.2) and (5.3). We have proved that $d_\Sigma(0) = Q_\mathrm{n}$. It is not restrictive to assume $p = 0$ also for the converse implication. In this case we only know that $d_\Sigma(0) = Q_\mathrm{n}$. Again, referring to the special coordinates of Theorem 3.1 and the corresponding formula (3.4), the previous equality implies that

$$\begin{cases} \alpha_j = 0 & \text{if } j < \ell_\mathrm{n} \\ \alpha_j = \mathrm{r_n} & \text{if } j = \ell_\mathrm{n} \\ \alpha_j = \mathrm{h}_j & \text{if } j > \ell_\mathrm{n} \end{cases} . \tag{5.7}$$

Applying formula (3.5), we have shown that $A_0\Sigma$ must be a vertical subgroup. □

Remark 5.4. The previous proposition and formula (5.5) show that any transveral point has maximum degree.

We finally observe that with the assumptions of Corollary 3.5, when $0 \in \Sigma$ is transversal, the matrix C of (3.9) becomes

$$C = \left(\begin{array}{c|c|c|c|c} \vdots & \vdots & \cdots & \cdots & * \\ \hline * & * & \cdots & \cdots & * \\ \hline I_{\mathrm{r_n}} + o(1) & o(1) & \cdots & \cdots & o(1) \\ o(1) & * & \cdots & \cdots & * \\ \hline o(1) & I_{\mathrm{h}_{\ell_\mathrm{n}+1}} + o(1) & o(1) & \cdots & o(1) \\ \hline \vdots & o(1) & \ddots & \cdots & \vdots \\ \hline \vdots & \vdots & \cdots & \ddots & o(1) \\ \hline \vdots & \vdots & \cdots & o(1) & Id_{\mathrm{h}_\iota} + o(1) \end{array}\right), \tag{5.8}$$

where $\mathrm{r_n}$ and ℓ_n are defined in (5.2) and (5.3), respectively. Indeed Proposition 5.3 shows that $d_\Sigma(0) = Q_\mathrm{n}$ holds and this implies the validity of the conditions (5.7).

6. Proof of the blow-up theorem

The general structure of (3.9) is important for the proof of the blow-up theorem.

Proof of Theorem 1.1. Taking into account Proposition 3.6, the translated manifold Σ_p has the same degree of Σ, therefore

$$d_{\Sigma_p}(0) = d_\Sigma(p) = \mathrm{N}.$$

Thus, the origin $0 \in \Sigma_p$ is a point of maximum degree for Σ_p. By Theorem 3.1, following its notation, there exists a special graded basis $(e_1, \dots, e_\mathrm{q})$, along with a C^1 smooth embedding $\Psi : U \to \Sigma_p$ with $\Psi(0) = 0 \in \mathbb{G}$ and

$$\Psi(y) = \sum_{j=1}^{\mathrm{q}} \psi_j(y) e_j, \tag{6.1}$$

that satisfies both conditions (3.1) and (3.2). For our purposes, it is not restrictive to assume that Ψ is a C^1 diffeomorphism. We also introduce the basis $(X_1, \dots, X_\mathrm{q})$

of Lie($\mathbb{G}$) such that $X_i(0) = e_i$ for all $i = 1, \dots, \mathrm{q}$ and consider graded coordinates (x_i) of a point p, such that $p = \sum_{i=1}^{\mathrm{q}} x_i e_i \in \mathbb{G}$. With respect to these coordinates, the vector fields

$$X_i = \sum_{l=1}^{n} a_i^l \, \partial_{x_l} \tag{6.2}$$

satisfy the following conditions

$$a_i^l = \begin{cases} \delta_i^l & d_l \le d_i \\ \text{polynomial of homogeneous degree } d_l - d_i & d_l > d_i \end{cases} . \tag{6.3}$$

The homogeneity here refers to intrinsic dilations of the group, namely

$$a_i^l(\delta_r x) = r^{d_l - d_i} a_i^l(x) \tag{6.4}$$

for all $r > 0$ and $x \in \mathbb{G}$, see e.g. [54]. We can further assume that there exists $c_1 > 0$ sufficiently small such that the domain U of the above diffeomorphism Ψ is defined on $(-c_1, c_1)^{\mathrm{n}}$. The continuous functions C_i^s in (3.8) can be assumed to be defined on a common interval $(-c_1, c_1)$, where $C_i^s(0)$ is the (s,i) entry of the matrix (3.1). For the sequel, it is convenient to recall formula (3.8) here

$$(\partial_i \psi)(y) = \sum_{s=1}^{\mathrm{q}} C_i^s(\psi(y)) X_s(\psi(y)) \quad \text{for all} \quad i = 1, \dots, \mathrm{n} \tag{6.5}$$

for all $y \in (-c_1, c_1)^{\mathrm{n}}$. Thus, from (1.2) and (1.3) we have the partial derivatives

$$\partial_{t_i} \Gamma(t) = |t_i|^{b_i - 1} \, (\partial_i \psi)(\eta(t)) = |t_i|^{b_i - 1} \sum_{l,s=1}^{\mathrm{q}} C_i^l(\Gamma(t)) \, a_l^s(\Gamma(t)) \, \partial_{x_s} \tag{6.6}$$

for all $i = 1, \dots, \mathrm{n}$, where we have used both (6.2) and (6.5).

The main point is to prove by induction the validity of the following statement. For each $j = 1, \dots, \iota$, if $0 \le \alpha_j < \mathrm{h}_j$ there holds

$$\Gamma_s(t) = o(|t|^j) \quad \text{for} \quad \mathrm{m}_{j-1} + \alpha_j < s \le \mathrm{m}_j. \tag{6.7}$$

Notice that in the case $\alpha_j = \mathrm{h}_j = \mathrm{m}_j - \mathrm{m}_{j-1}$ there is nothing to prove and the statement is automatically satisfied.

Let us first establish the case $j = 1$. If $\alpha_1 = 0$, in all of the four assumptions where this condition applies, we have $b_i \ge 2$ for each $i = 1, \dots, \mathrm{n}$, therefore (6.6) gives

$$\nabla \Gamma_s(0) = 0 \quad \text{for every } s = 1, \dots, \mathrm{q}.$$

If $0 < \alpha_1 < \mathrm{m}_1$ and $\alpha_1 < s \le \mathrm{m}_1$, again in all four assumptions, due to (3.1), we get

$$\partial_{x_i} \Gamma_s(0) = 0 \quad \text{for all } i = 1, \dots, \alpha_1.$$

In view of (6.6), the previous equalities extend to all $i = \alpha_1 + 1, \dots, \mathrm{n}$, being $b_i \ge 2$. In both cases, the vanishing of $\Gamma_s(0)$ and $\nabla \Gamma_s(0)$ for any $s = \alpha_1 + 1, \dots, \mathrm{m}_1$ and in all of our four assumptions proves our inductvie assumption (6.7) for $j = 1$.

Now, we assume by induction the validity of (6.7) for all $j=1,\dots,k-1$, where $2\le k\le \iota$. We wish to prove this formula for $j=k$, in the nontrivial case $0\le \alpha_k<\mathrm{h}_k$. Let us write the general formula (6.6) for partial derivatives

$$\partial_{t_i}\Gamma_s(t)=|t_i|^{b_i-1}\bigg(C_i^s(\Gamma(t))+\sum_{l:d_l<d_s}C_i^l(\Gamma(t))\,a_l^s(\Gamma(t))\bigg), \tag{6.8}$$

where $s=\mathrm{m}_{k-1}+1,\dots,\mathrm{m}_k$. We consider the following possibilities:

$$b_i<k,\quad b_i=k\quad\text{ and }\quad b_i>k.$$

Let us begin with the case $b_i<k$. If $\alpha_k>0$ and consider $\mathrm{m}_{k-1}+\alpha_k<s\le \mathrm{m}_k$, then the structure of (3.9) and the fact that $b_i<k$ yield

$$C_i^s\equiv 0. \tag{6.9}$$

If $\alpha_k=0$ and the fourth assumption holds, then the special structure of C, see (5.8), implies that $\alpha_j=0$ for all $j=1,\dots,k-1$. This gives $b_i\ge k+1$ for all $i=1,\dots,\mathrm{n}$. Taking into account the form (1.2) of η and the composition (1.3) we clearly have

$$\Gamma_s(t)=O(|t|^{k+1})=o(|t|^k)$$

for all $s=1,\dots,\mathrm{q}$ and in particular (6.7) is established. If $\alpha_k=0$ and the first assumption holds, then the form (4.1) always gives

$$C_i^s\equiv 0\quad\text{for}\quad \mathrm{m}_1\le \mathrm{m}_{k-1}<s\le \mathrm{q}\quad\text{and}\quad i=1,\dots,\mathrm{n}. \tag{6.10}$$

If $\alpha_k=0$ and the second assumption holds, then $\iota=2$ and we only have the case $k=2$, namely $\alpha_2=0$. From the form (3.10), then

$$C_i^s\equiv 0\quad\text{for}\quad \mathrm{m}_1<s\le \mathrm{q}\quad\text{and}\quad i=1,\dots,\mathrm{n}. \tag{6.11}$$

If $\alpha_k=0$ and the third assumption holds, then $\mathrm{n}=1$ and the condition $b_i<k$ gives

$$b_1=\mathrm{N}<k=d_s\quad\text{for all } s=\mathrm{m}_{k-1}+1,\dots,\mathrm{m}_k, \tag{6.12}$$

so that the form (3.11) yields

$$C_1^s\equiv 0\quad\text{for}\quad \mathrm{m}_{k-1}<s\le \mathrm{m}_k. \tag{6.13}$$

We are interested in the case $s=\mathrm{m}_{k-1}+1,\dots,\mathrm{m}_k$ and $i=1,\dots,\mu_{k-1}$, therefore the vanishing of C_i^s joined with (6.8) gives

$$\begin{aligned}\partial_{t_i}\Gamma_s(t)&=|t_i|^{b_i-1}\sum_{l:d_l<k}C_i^l(\Gamma(t))\,a_l^s(\Gamma(t))\\&=|t_i|^{b_i-1}\sum_{l:d_l<b_i<k}C_i^l(\Gamma(t))\,a_l^s(\Gamma(t))+|t_i|^{b_i-1}\sum_{l:d_l=b_i<k}C_i^l(\Gamma(t))\,a_l^s(\Gamma(t))\\&+|t_i|^{b_i-1}\sum_{l:b_i<d_l<k}C_i^l(\Gamma(t))\,a_l^s(\Gamma(t))=T_1+T_2+T_3.\end{aligned} \tag{6.14}$$

We have denoted by T_1, T_2 and T_3 the first, second and third addend, respectively. To study T_1, we use the graph form of ψ given by (3.2). In fact, whenever $\alpha_j > 0$ we have the identity

$$b_{s-\mathrm{m}_{j-1}+\mu_{j-1}} = j \tag{6.15}$$

for $\mathrm{m}_{j-1} < s \le \mathrm{m}_{j-1} + \alpha_j$ and $j = 1, \dots, \iota$, hence (1.2) and (1.3) yield

$$\Gamma_s(t) = \frac{|t_{s-\mathrm{m}_{j-1}+\mu_{j-1}}|^j}{j} \operatorname{sgn}(t_{s-\mathrm{m}_{j-1}+\mu_{j-1}}) = \frac{|t_{s-\mathrm{m}_{j-1}+\mu_{j-1}}|^{d_s}}{d_s} \operatorname{sgn}(t_{s-\mathrm{m}_{j-1}+\mu_{j-1}}). \tag{6.16}$$

Each polynomial a_l^s in the sum of T_1 has homogeneous degree $k - d_l$, hence it does not depend on the variables x_i, with $i > \mathrm{m}_{k-1}$. As a consequence of (6.16), for all $s = \mathrm{m}_{k-1} + 1, \dots, \mathrm{m}_k$, the homogeneity (6.4) of a_l^s, when joined with our inductive assumption also implies that

$$a_l^s(\Gamma(t)) = a_l^s(\Gamma_1(t), \dots, \Gamma_{\mathrm{m}_{k-1}}(t)) = O(|t|^{k-d_l}).$$

This immediately shows that $T_1(t) = O(|t|^k) = o(|t|^{k-1})$. We now consider the second addend

$$T_2(t) = |t_i|^{b_i-1} \sum_{l:d_l=b_i<k} C_i^l(\Gamma(t))\, a_l^s(\Gamma(t))$$

and set $j = b_i$. The conditions $d_l = b_i < k$ give

$$\mu_{j-1} < i \le \mu_j \quad \text{and} \quad \mathrm{m}_{j-1} < l \le \mathrm{m}_j. \tag{6.17}$$

We consider the general case where $0 \le \alpha_k < \mathrm{h}_k$. Since $b_i = j$ we have $\alpha_j > 0$, therefore taking into account (3.9), for $\mathrm{m}_{j-1} < l \le \mathrm{m}_{j-1} + \alpha_j$ it follows that

$$C_i^l = \delta_{i-\mu_{j-1}}^{l-\mathrm{m}_{j-1}} + o_i^l(1) \tag{6.18}$$

where $o_i^l(1)$ vanish at the origin. When $\mathrm{m}_{j-1} + \alpha_j < l \le \mathrm{m}_j$, we have

$$C_i^l = o_i^l(1)$$

and $o_i^l(1)$ vanish at zero. In view of (6.18), for i and l in the ranges (6.17), we set

$$\mathrm{m}_{j-1} < l_{ij} := i - \mu_{j-1} + \mathrm{m}_{j-1} \le \mathrm{m}_{j-1} + \alpha_j,$$

therefore we obtain the expression

$$T_2(t) = |t_i|^{b_i-1} \Bigg(\sum_{\substack{l:d_l=b_i<k \\ l \ne l_{ij}}} o_i^l(1)\, a_l^s(\Gamma(t)) + a_{l_{ij}}^s(\Gamma(t)) \Bigg). \tag{6.19}$$

Arguing as before, formulae (6.16) and the inductive assumption imply that

$$|t_i|^{b_i-1} a_l^s(\Gamma(t)) = |t_i|^{j-1} O(|t|^{k-d_l}) = |t_i|^{j-1} O(|t|^{k-j}) = O(|t|^{k-1}).$$

It follows that

$$T_2(t) = o(|t|^{k-1}) + |t_i|^{b_i-1} a_{l_{ij}}^s(\Gamma(t)). \tag{6.20}$$

The behavior of the second addend in the previous equality requires a special study, that precisely relies on the group structure that is assumed on $A_0\Sigma_p$. Taking into account the definition of the set of indexes I defined through (1.4), in view of [43, Lemma 2.5], if the group operation is given by the polynomial formula

$$xy = x + y + Q(x,y)$$

with respect to our fixed graded coordinates, then the polynomial Q_s, with $s \notin I$, is given by the formula

$$Q_s(x,y) = \sum_{v:d_v<k,v\notin I} x_v R_{sv}(x,y) + y_v U_{sv}(x,y).$$

Both polynomials R_{sv} and U_{sv} have homogeneous of degree $k - d_v$. Since we have $\mathrm{m}_{j-1} < l_{ij} \le \mathrm{m}_{j-1} + \alpha_j$, the condition $l_{ij} \in I$ gives

$$\frac{\partial Q_s}{\partial y_{l_{ij}}}(x,0) = a^s_{l_{ij}}(x) = \sum_{v:d_v\le k-j,v\notin I} x_v \frac{\partial R_{sv}}{\partial y_{l_{ij}}}(x,0),$$

where we have used the relationship between left invariant vector fields and group operation, along with the fact that $v \neq l_{ij}$ for all $v \notin I$. As we have already observed, $a^s_{l_{ij}}$ only depends on $(x_1, \dots, x_{\mathrm{m}_{k-1}})$ and by our inductive assumption (6.7)

$$\Gamma_v(t) = o_v(|t|^{d_v}) \quad \text{ whenever } \quad d_v < k \text{ and } v \notin I.$$

Precisely, for all of these $v's$, we have $o_v(|t|^{d_v})/|t|^{d_v} \to 0$ as $t \to 0$ and there holds

$$a^s_{l_{ij}}(\Gamma(t)) = \sum_{v:d_v\le k-j,v\notin I} o_v(|t|^{d_v})\, \frac{\partial R_{sv}}{\partial y_{l_{ij}}}(\Gamma(t),0),$$

Again, the inductive assumption gives $\partial_{y_{l_{ij}}} R_{sv}(\Gamma(t),0) = O(|t|^{k-d_v-j})$, that is

$$o_v(|t|^{d_v})\, \frac{\partial R_{sv}}{\partial y_{l_{ij}}}(\Gamma(t),0) = o(|t|^{k-j}),$$

therefore $a^s_{l_{ij}}(\Gamma(t)) = o(|t|^{k-j})$. We have finally proved that

$$T_2(t) = o(|t|^{k-1}).$$

The treatment of the addend

$$T_3 = |t_i|^{b_i-1} \sum_{l:b_i<d_l<k} C^l_i(\Gamma(t))\, a^s_l(\Gamma(t))$$

in (6.14) strongly relies on our special four assumptions. Without these assumptions, it is not clear whether for instance the factors $C^l_i(\Gamma(t))$ for $b_i < d_l < k$ behave like $o(|t|^{d_l-b_i})$, since C^l_i are only continuous.

If the first assumption holds, then the special form (4.1) of C immediately proves that there cannot exist nonvanishing coefficients C^l_i whenever $b_i < d_l$, hence $T_3 \equiv 0$. If the second assumption holds, then $1 \le b_i < d_l < k$ implies $k \ge 3$, that conflicts with the 2-step assumption on $\mathbb{G}$, therefore $T_3 \equiv 0$. If the third assumption holds, then $\mathrm{n} = 1 = i$ and (6.12) gives

$$b_1 = \mathrm{N} < d_l$$

that joined with the special form (3.11) gives $C_1^l \equiv 0$, therefore $T_3 \equiv 0$ also in this case. In the fourth assumption, where p is a transveral point, we consider the integer ℓ_n defined in (5.2). By definition (5.3), according to (5.8), we have

$$\alpha_{\ell_\mathrm{n}} = \mathrm{r}_n \geq 1 \quad \text{and} \quad b_i \geq \ell_\mathrm{n},$$

therefore $k > \ell_\mathrm{n}$. This implies that $\alpha_k = \mathrm{h}_k$, hence the inductive assumption is automatically satisfied. Collecting all of the previous cases, we conclude that in any of the four assumptions for $b_i < k$, we have that either the inductive assumption (6.7) is satisfied or we have

$$\partial_{t_i} \Gamma_s(t) = o(|t|^{k-1}).$$

In the case $b_i = k$, then $\alpha_k > 0$ and the condition $\mathrm{m}_{k-1} + \alpha_k < s \leq \mathrm{m}_k$ joined with the form of (3.9) yields

$$C_i^s(\Gamma(t)) = o(1),$$

therefore (6.8) gives

$$\partial_{t_i} \Gamma_s(t) = |t_i|^{k-1} \bigg(o(1) + \sum_{l : d_l < k} C_i^l(\Gamma(t))\, a_l^s(\Gamma(t)) \bigg).$$

In the previous sum the condition $d_s = k > d_l$ yields $a_l^s(0) = 0$, therefore also in the case $b_i = k$ we have

$$\partial_{t_i} \Gamma_s(t) = o(|t|^{k-1}).$$

When $b_i > k$, there obviously holds

$$\begin{aligned} \partial_{t_i} \Gamma_s(t) &= |t_i|^{b_i - 1} \bigg(C_i^s(\Gamma(t)) + \sum_{l : d_l < d_s} C_i^l(\Gamma(t))\, a_l^s(\Gamma(t)) \bigg) \\ &= |t_i|^{b_i - 1} O(1) = o(|t|^{k-1}). \end{aligned}$$

Joining all the previous results, it follows that $\nabla \Gamma_s = o(|t|^{k-1})$, hence

$$\Gamma_s(t) = o(|t|^k),$$

proving the induction step. This proves our claim (1.5). □

7. Measure theoretic area formula in homogeneous groups

We introduce some preliminary results and notions that will be needed in the next sections. The symbol $\mathbb{G}$ always denotes a homogeneous group equipped with a homogeneous distance d.

7.1. **Differentiation of measures in homogeneous groups.** We denote by $\mathcal{F}_b$ the family of closed balls in $\mathbb{G}$ having positive radius. The properties of the homogeneous distance give $\operatorname{diam}(B(x,r)) = 2r$ for all $x \in \mathbb{G}$ and $r > 0$. Indeed, $\operatorname{diam}(B(x,r)) \le 2r$ is trivial and the opposite inequality follows considering a horizontal segment passing through x. As a consequence, if $\mu : \mathcal{P}(X) \to [0,+\infty]$ is a measure that is finite on bounded sets, then one easily realizes that

$$\mathcal{S}_{\mu,\zeta_{b,\alpha}} = \mathcal{F}_b \setminus \{S \in \mathcal{F}_b : \zeta_{b,\alpha}(S) = \mu(S) = 0 \text{ or } \zeta_{b,\alpha}(S) = \mu(S) = +\infty\} = \mathcal{F}_b, \tag{7.1}$$

where we have defined

$$\zeta_{b,\alpha} : \mathcal{F}_b \to [0,+\infty), \quad \zeta_{b,\alpha}(S) = \frac{\operatorname{diam}(S)^\alpha}{2^\alpha}.$$

Definition 7.1 (Carathéodory construction)**.** Let $\mathcal{F} \subset \mathcal{P}(\mathbb{G})$ denote a nonempty family of closed subsets and fix $\alpha > 0$. If $\delta > 0$ and $E \subset \mathbb{G}$, we define

$$\phi_\delta^\alpha(E) = \inf\left\{ \sum_{j=0}^{\infty} \frac{\operatorname{diam}(B_j)^\alpha}{2^\alpha} : E \subset \bigcup_{j\in\mathbb{N}} B_j,\ \operatorname{diam}(B_j) \le \delta,\ B_j \in \mathcal{F} \right\}, \tag{7.2}$$

where the diameter $\operatorname{diam} B_j$ is computed with respect to the distance d on $\mathbb{G}$. If $\mathcal{F}$ coincides with the family of closed balls $\mathcal{F}_b$, then we set

$$\mathcal{S}_0^\alpha(E) = \sup_{\delta>0} \phi_\delta^\alpha(E) \tag{7.3}$$

to be the *α-dimensional spherical measure* of E. In the case $\mathcal{F}$ is the family of all closed sets and $k \in \{1,2,\ldots,\mathrm{q}-1\}$, we define the Hausdorff measure

$$\mathcal{H}^k_{|\cdot|} = \mathcal{L}^k(\{x \in \mathbb{G} : |x| \le 1\}) \sup_{\delta>0} \phi_\delta^k(E) \tag{7.4}$$

where $\mathcal{L}^k$ denotes the Lebesgue measure and $|\cdot|$ is the norm arising from the fixed graded scalar product on $\mathbb{G}$.

Observing that $\mathcal{F}_b$ covers any subset finely, according to the terminology in [17, 2.8.1] and that condition (7.1) holds, we can apply Theorem 11 in [40] to the metric space $(\mathbb{G}, d)$, establishing the following result.

Theorem 7.2. *Let $\alpha > 0$ and let μ be a Borel regular measure over $\mathbb{G}$ such that there exists a countable open covering of $\mathbb{G}$, whose elements have μ finite measure. If $B \subset A \subset \mathbb{G}$ are Borel sets, then $\theta^\alpha(\mu,\cdot)$ is Borel on A. In addition, if $\mathcal{S}_0^\alpha(A) < +\infty$ and $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{S}_0^\alpha \llcorner A$, then we have*

$$\mu(B) = \int_B \theta^\alpha(\mu,x)\, d\mathcal{S}_0^\alpha(x)\,. \tag{7.5}$$

The *spherical Federer density* $\theta^\alpha(\mu,\cdot)$ in (7.5) was introduced in [40]. We will use its explicit representation

$$\theta^\alpha(\mu,x) = \inf_{\varepsilon>0}\ \sup\left\{ \frac{2^\alpha \mu(\mathbb{B})}{\operatorname{diam}(\mathbb{B})^\alpha} : x \in \mathbb{B} \in \mathcal{F}_b,\ \operatorname{diam}\mathbb{B} < \varepsilon \right\}. \tag{7.6}$$

7.2. **Intrinsic measure and spherical factor.** The next definition introduces the intrinsic measure associated to a submanifold in a homogeneous group, see [43]. For hypersurfaces in Carnot groups this measure is precisely the h-perimeter measure with respect to the sub-Riemannian structure of the group.

Definition 7.3 (Intrinsic measure)**.** Let $\Sigma \subset \mathbb{G}$ be an n-dimensional submanifold of class C^1 and degree N. We consider our fixed graded left invariant Riemannian metric g on $\mathbb{G}$. To present a coordinate free version of this measure, we fix an auxiliary Riemannian metric $\tilde g$ on $\mathbb{G}$. Let τ_Σ be a $\tilde g$-unit tangent n-vector field on Σ, namely,

$$\|\tau_\Sigma(p)\|_{\tilde g} = 1 \quad \text{for each} \quad p \in \Sigma.$$

We consider its corresponding N-tangent n-vector field, defined as follows

$$\tau^{\tilde g}_{\Sigma,\mathrm{N}}(p) := \pi_{p,\mathrm{N}}(\tau_\Sigma(p)) \quad \text{for each} \quad p \in \Sigma. \tag{7.7}$$

Then we define the *intrinsic measure* of Σ in $\mathbb{G}$ as follows

$$\mu_\Sigma = \|\tau^{\tilde g}_{\Sigma,\mathrm{N}}\|_g \, \sigma_{\tilde g}, \tag{7.8}$$

where $\sigma_{\tilde g}$ is the n-dimensional Riemannian measure induced by $\tilde g$ on Σ. This can be also seen as the n-dimensional Hausdorff measure with respect to the Riemannian distance induced by $\tilde g$ and restricted to Σ.

Remark 7.4. By definition of pointwise degree (2.10), we realize that under the assumptions of Definition 7.3 a point $p \in \Sigma$ has maximum degree N if and only if

$$\tau^{\tilde g}_{\Sigma,\mathrm{N}}(p) = \pi_{p,\mathrm{N}}(\tau_\Sigma(p)) \neq 0,$$

as it follows from the definition of pointwise N-projection, see (2.8).

Proposition 7.5. *If $H \subset \mathbb{R}^{\mathrm{n}}$ is an open subset and $\Phi : H \to \mathbb{G}$ is a C^1 smooth local chart for an* n*-dimensional C^1 smooth submanifold Σ of degree* N*, then*

$$\mu_\Sigma\big(\Phi(H)\big) = \int_H \|\pi_{\Phi(y),\mathrm{N}}\big(\partial_{y_1}\Phi(y) \wedge \cdots \wedge \partial_{y_\mathrm{n}}\Phi(y)\big)\|_g \, dy. \tag{7.9}$$

Proof. By our local chart, using (7.7) we can write

$$\tau^{\tilde g}_{\Sigma,\mathrm{N}}(\Phi(y)) := \frac{\pi_{\Phi(y),\mathrm{N}}\big(\partial_{y_1}\Phi(y) \wedge \cdots \wedge \partial_{y_\mathrm{n}}\Phi(y)\big)}{\|\partial_{y_1}\Phi(y) \wedge \cdots \wedge \partial_{y_\mathrm{n}}\Phi(y)\|_{\tilde g}},$$

therefore the integral

$$\int_{\Phi(H)} \|\tau^{\tilde g}_{\Sigma,\mathrm{N}}(p)\|_g \, d\sigma_{\tilde g}(p),$$

after the standard change of variables $p = \Phi(y)$, becomes equal to

$$\int_H \left\| \frac{\pi_{\Phi(y),\mathrm{N}}\big(\partial_{y_1}\Phi(y) \wedge \cdots \wedge \partial_{y_\mathrm{n}}\Phi(y)\big)}{\|\partial_{y_1}\Phi(y) \wedge \cdots \wedge \partial_{y_\mathrm{n}}\Phi(y)\|_{\tilde g}} \right\|_g \|\partial_{y_1}\Phi(y) \wedge \cdots \wedge \partial_{y_\mathrm{n}}\Phi(y)\|_{\tilde g} \, dy,$$

therefore concluding the proof of (7.9). □

The relationship between intrinsic meausure and spherical measure requires some geometric constants that can be associated to the homogeneous distance that defines the spherical measure. These constants may change, depending on the sections of the metric unit ball.

Definition 7.6 (Spherical factor). Let $S \subset \mathbb{G}$ a linear subspace and consider a fixed homogeneous distance d on $\mathbb{G}$. If $|\cdot|$ denotes our fixed graded scalar product on $\mathbb{G}$, then the *spherical factor* of d, with respect to S, is the number

$$\beta_d(S) = \max_{d(u,0)\le 1} \mathcal{H}^{\mathrm{n}}_{|\cdot|}\big(\mathbb{B}(u,1) \cap S\big),$$

where $\mathbb{B}(u,1) = \{v \in \mathbb{G} : d(v,u) \le 1\}$.

8. The upper blow-up and some applications

This section is divided into two parts. We give a proof of the upper blow-up theorem and we establish a number of applications, that are summerized in Theorem 1.3.

8.1. **Proof of the upper blow-up theorem.** The upper blow-up theorem is the second main result of this paper.

Proof of Theorem 1.2. We consider the special coordinates obtained in Theorem 3.1 for the translated manifold $\Sigma_p = p^{-1}\Sigma$. This assumption is possible by Proposition 3.6, since algebraic regularity along with the first and the fourth assumptions are automatically transferred to the origin of Σ_p. We follow notations of Theorem 1.1. In some parts of the proof the identification of $\mathbb{G}$ with $\mathbb{R}^{\mathrm{n}}$ with respect to the above mentioned coordinates will be understood. For instance, the algebraic tangent space $A_0\Sigma_p$ defined in (3.5) equals $A_p\Sigma$ by Proposition 3.6 and it can be also identified with $\mathbb{R}^{\mathrm{n}}$.

Let Ψ be defined as in the proof of Theorem 1.1 and define the translated mapping $\Phi : (-c_1, c_1)^p \to \Sigma$ as follows

$$\Phi(y) = p\Psi(y). \tag{8.1}$$

We are going to use the local expansion (1.5) in order to compute the Federer's density, that is defined as follows

$$\theta^{\mathrm{N}}(\mu_\Sigma, p) = \inf_{r>0} \sup_{\substack{z\in\mathbb{B}(p,\tilde r)\\ 0<\tilde r<r}} \frac{\mu_\Sigma(\mathbb{B}(z,\tilde r))}{\tilde r^{\mathrm{N}}}. \tag{8.2}$$

Taking $r > 0$ sufficiently small and $z \in B(p, \tilde r)$, in view of (7.9), we have

$$\frac{\mu_\Sigma(\mathbb{B}(z,\tilde r))}{\tilde r^{\mathrm{N}}} = \tilde r^{-\mathrm{N}} \int_{\Phi^{-1}(\mathbb{B}(z,\tilde r))} \|\pi_{\Phi(y),\mathrm{N}}\big(\partial_{y_1}\Phi(y) \wedge \cdots \wedge \partial_{y_{\mathrm{n}}}\Phi(y)\big)\|_g \, dy. \tag{8.3}$$

Taking into account the relations

$$\mathrm{N} = \sum_{i=1}^{\mathrm{n}} b_i = \sum_{j=1}^{\iota} j\,\alpha_j,$$

and the "induced dilations" σ_r introduced in (3.7), the change of variable $y=\sigma_r(t)$ implies that

$$\frac{\mu_\Sigma(\mathbb{B}(z,\tilde r))}{\tilde r^{\mathrm{N}}}=\int_{\sigma_{1/\tilde r}(\Phi^{-1}(\mathbb{B}(z,\tilde r)))}\|\pi_{\Phi(y),\mathrm{N}}\big(\partial_{y_1}\Phi(\sigma_{\tilde r}y)\wedge\cdots\wedge\partial_{y_{\mathrm{n}}}\Phi(\sigma_{\tilde r}y)\big)\|_g\,dy\,. \tag{8.4}$$

Our first claim is the uniform boundedness of the following rescaled sets

$$\sigma_{1/\tilde r}\big(\Phi^{-1}(\mathbb{B}(z,\tilde r))\big)=\sigma_{1/\tilde r}\big(\Psi^{-1}(\mathbb{B}(p^{-1}z,\tilde r))\big)$$

as $\tilde r<r$ and $d(p,z)\le\tilde r$ with $\tilde r$ sufficiently small. There holds

$$\sigma_{1/\tilde r}\big(\Phi^{-1}(\mathbb{B}(z,\tilde r))\big)=\big\{y\in\mathbb{R}^{\mathrm{n}}:\delta_{1/\tilde r}(z^{-1}p)\delta_{1/\tilde r}(\Psi(\sigma_{\tilde r}y))\in\mathbb{B}(0,1)\big\}\,. \tag{8.5}$$

We first observe that

$$\zeta(\tau)=\left(\mathrm{sgn}\,(\tau_1)\,\sqrt[b_1]{b_1|\tau_1|},\ldots,\mathrm{sgn}\,(\tau_p)\,\sqrt[b_p]{b_p|\tau_p|}\right)$$

is the inverse of η, hence in view of (1.3) and (6.1) we have

$$\psi(\sigma_{\tilde r}y)=\Gamma(\zeta(\sigma_{\tilde r}y))=\Gamma(\tilde r\,\zeta(y))\,. \tag{8.6}$$

In view of (6.15), we can write (1.5) as follows

$$\Gamma_s(t)=\begin{cases}\eta_{s-\mathrm{m}_{d_s-1}+\mu_{d_s-1}}(t) & \text{if } s\in I\\ o(|t|^{d_s}) & \text{if } s\notin I\end{cases}, \tag{8.7}$$

therefore whenever $s\in I$ we get

$$\begin{aligned}\Gamma_s(\zeta(\sigma_{\tilde r}y))&=(\eta\circ\zeta)_{s-\mathrm{m}_{d_s-1}+\mu_{d_s-1}}(\sigma_{\tilde r}y)\\&=(\sigma_{\tilde r}y)_{s-\mathrm{m}_{d_s-1}+\mu_{d_s-1}}\\&=(\tilde r)^{b_{s-\mathrm{m}_{d_s-1}+\mu_{d_s-1}}}y_{s-\mathrm{m}_{d_s-1}+\mu_{d_s-1}}\\&=(\tilde r)^{d_s}y_{s-\mathrm{m}_{d_s-1}+\mu_{d_s-1}}.\end{aligned} \tag{8.8}$$

As a result, taking into account that $d\left(\delta_{1/\tilde r}(z^{-1}p),0\right)\le1$, an element $y\in\mathbb{R}^{\mathrm{n}}$ of (8.5) satisfies the condition

$$\begin{aligned}&y_1e_1+\cdots+y_{\alpha_1}e_{\alpha_1}+\frac{\Gamma_{\alpha_1+1}(r\zeta(y))}{\tilde r}e_{\alpha_1+1}+\cdots+\frac{\Gamma_{\mathrm{m}_1}(\tilde r\zeta(y))}{\tilde r}e_{\mathrm{m}_1}\\&+y_{\alpha_1+1}e_{\mathrm{m}_1+1}+\cdots+y_{\mu_2}e_{\mathrm{m}_1+\alpha_2}+\frac{\Gamma_{\mathrm{m}_1+\alpha_2+1}(\tilde r\zeta(y))}{(\tilde r)^2}e_{\mathrm{m}_1+\alpha_2+1}+\cdots+\frac{\Gamma_{\mathrm{m}_2}(\tilde r\zeta(y))}{(\tilde r)^2}e_{\mathrm{m}_2}\\&\vdots\qquad\vdots\qquad\vdots\qquad\vdots\qquad\vdots\qquad\vdots\qquad\vdots\\&+y_{\mu_{\iota-1}+1}e_{\mathrm{m}_{\iota-1}+1}+\cdots+y_{\mathrm{n}}e_{\mathrm{m}_{\iota-1}+\alpha_\iota}+\frac{\Gamma_{\mathrm{m}_{\iota-1}+\alpha_{\iota-1}+1}(\tilde r\zeta(y))}{(\tilde r)^\iota}e_{\mathrm{m}_{\iota-1}+\alpha_\iota+1}+\cdots\\&\cdots+\frac{\Gamma_{\mathrm{m}_\iota}(\tilde r\zeta(y))}{(\tilde r)^\iota}e_{\mathrm{m}_\iota}\in\mathbb{B}(0,2).\end{aligned}$$

Since $\mathbb{B}(0,2)$ is also bounded with respect to the fixed Euclidean norm on $\mathbb{G}$ and $(e_1,\dots,e_{\mathrm{q}})$ is an orthonormal basis the previous expression implies the existence of a bounded set $V\subset A_p\Sigma$ such that

$$\sigma_{1/\tilde r}\big(\Phi^{-1}(\mathbb{B}(z,\tilde r))\big)\subset V \tag{8.9}$$

for $r>0$ sufficiently small, $0<\tilde r<r$ and $d(z,p)\le\tilde r$. We notice that the previous sums can be also written as follows

$$\sum_{l=1}^{\mathrm{n}} y_l e_{\mathrm{m}_{b_l-1}+l-\mu_{b_l-1}}+\sum_{l\notin I}\frac{\Gamma_l(\tilde r\zeta(y))}{(\tilde r)^{d_l}}e_l\in\mathbb{B}(0,2).$$

The uniform boundedness (8.9) joined with (8.4) implies that $\theta^{\mathrm{N}}(\tilde\mu\llcorner\Sigma,p)<+\infty$, hence there exist a sequence $\{r_k\}\subset(0,+\infty)$ converging to zero and a sequence of elements $z_k\in\mathbb{B}(p,r_k)$ such that

$$\theta^{\mathrm{N}}(\mu_\Sigma,p)=\lim_{k\to\infty}\int_{\sigma_{1/r_k}(\Phi^{-1}(\mathbb{B}(z_k,r_k)))}\|\pi_{\Phi(y),\mathrm{N}}\big(\partial_{y_1}\Phi(\sigma_{r_k}y)\wedge\cdots\wedge\partial_{y_{\mathrm{n}}}\Phi(\sigma_{r_k}y)\big)\|_g\,dy.$$

Possibly extracting a subsequence, there exists $u_0\in\mathbb{B}(0,1)$ such that

$$\delta_{1/r_k}(z_k^{-1}p)\to u_0^{-1}\in\mathbb{B}(0,1). \tag{8.10}$$

We define the following subsets of the algebraic tangent space

$$F_k=\sigma_{1/r_k}\big(\Phi^{-1}(\mathbb{B}(z_k,r_k))\big)\quad\text{and}\quad F(u_0)=\mathbb{B}(u_0,1)\cap A_p\Sigma.$$

Our second claim is the validity of the following limit

$$\lim_{k\to\infty}\mathbf{1}_{F_k}(w)=0 \tag{8.11}$$

for each $w\in A_p\Sigma\setminus F(u_0)$. Arguing by contradiction, if there exists a sequence of positive integers j_k such that

$$\mathbf{1}_{F_{j_k}}(w)=1$$

for every $k\in\mathbb{N}$, then (8.5) gives

$$\sum_{l=1}^{\mathrm{n}} w_l e_{\mathrm{m}_{b_l-1}+l-\mu_{b_l-1}}+\sum_{l\notin I}\frac{\Gamma_l(r_{j_k}\zeta(w))}{(r_{j_k})^{d_l}}e_l\in\delta_{1/r_{j_k}}(p^{-1}z_{j_k})\mathbb{B}(0,1), \tag{8.12}$$

since the previous element precisely coincides with $\delta_{1/r_{j_k}}(\Psi(\sigma_{r_{j_k}}w))$. The estimate (8.7) joined with the limit (8.10), as $k\to\infty$ give

$$\sum_{l=1}^{\mathrm{n}} w_l e_{\mathrm{m}_{b_l-1}+l-\mu_{b_l-1}}\in\mathbb{B}(u_0,1)\cap A_p\Sigma=F(u_0),$$

that is a contradiction. We now define

$$I_k=\int_{F_k}\|\pi_{\Phi(y),\mathrm{N}}\big(\partial_{y_1}\Phi(\sigma_{r_k}y)\wedge\cdots\wedge\partial_{y_{\mathrm{n}}}\Phi(\sigma_{r_k}y)\big)\|_g\,dy, \tag{8.13}$$

along with

$$
\begin{aligned}
J_{1,k} &= \int_{F_k\cap F(u_0)} \|\pi_{\Phi(y),\mathrm{N}}\big(\partial_{y_1}\Phi(\sigma_{r_k}y)\wedge\cdots\wedge\partial_{y_{\mathrm{n}}}\Phi(\sigma_{r_k}y)\big)\|_g\,dy,\\
J_{2,k} &= \int_{F_k\setminus F(u_0)} \|\pi_{\Phi(y),\mathrm{N}}\big(\partial_{y_1}\Phi(\sigma_{r_k}y)\wedge\cdots\wedge\partial_{y_{\mathrm{n}}}\Phi(\sigma_{r_k}y)\big)\|_g\,dy,
\end{aligned}
\tag{8.14}
$$

so that $I_k = J_{1,k} + J_{2,k}$ for each $k \geq 0$. Taking the limit of the following inequality

$$
J_{1,k} \leq \int_{F(u_0)} \|\pi_{\Phi(y),\mathrm{N}}\big(\partial_{y_1}\Phi(\sigma_{r_k}y)\wedge\cdots\wedge\partial_{y_{\mathrm{n}}}\Phi(\sigma_{r_k}y)\big)\|_g\,dy,
\tag{8.15}
$$

we obtain

$$
\limsup_{k\to\infty} J_{1,k} \leq \mathcal{H}^{\mathrm{n}}_{|\cdot|}(F(u_0))\,\|\pi_{\Phi(y),\mathrm{N}}\big(\partial_{y_1}\Phi(0)\wedge\cdots\wedge\partial_{y_{\mathrm{n}}}\Phi(0)\big)\|_g.
\tag{8.16}
$$

Joining (8.14) with (8.9), we also get

$$
J_{2,k} \leq \int_{V\setminus F(u_0)} \mathbf{1}_{F_k}(y)\|\pi_{\Phi(y),\mathrm{N}}\big(\partial_{y_1}\Phi(\sigma_{r_k}y)\wedge\cdots\wedge\partial_{y_{\mathrm{n}}}\Phi(\sigma_{r_k}y)\big)\|_g\,dy.
\tag{8.17}
$$

The boundedness of V and (8.11) joined with the classical Lebesgue's convergence theorem imply that

$$
\lim_{k\to\infty} J_{2,k} = 0.
\tag{8.18}
$$

In view of (8.16) and (8.18), we have proved that

$$
\theta^{\mathrm{N}}(\mu_\Sigma, p) \leq \mathcal{H}^{\mathrm{n}}_{|\cdot|}\big(\mathbb{B}(u_0,1)\cap A_p\Sigma\big)\,\|\pi_{\Phi(y),\mathrm{N}}\big(\partial_{y_1}\Phi(0)\wedge\cdots\wedge\partial_{y_{\mathrm{n}}}\Phi(0)\big)\|_g,
\tag{8.19}
$$

where $u_0 \in \mathbb{B}(0,1)$, therefore the definition of spherical factor yields

$$
\theta^{\mathrm{N}}(\mu_\Sigma, p) \leq \beta_d(A_p\Sigma)\,\|\pi_{\Phi(y),\mathrm{N}}\big(\partial_{y_1}\Phi(0)\wedge\cdots\wedge\partial_{y_{\mathrm{n}}}\Phi(0)\big)\|_g.
\tag{8.20}
$$

Our third claim is the validity of the equality in (8.20). Let $v_0 \in \mathbb{B}(0,1)$ be such that

$$
\beta_d(A_p\Sigma) = \mathcal{H}^{\mathrm{n}}_{|\cdot|}\big(\mathbb{B}(v_0,1)\cap A_p\Sigma\big),
\tag{8.21}
$$

define $v_{\tilde r} = p\delta_{\tilde r}v_0 \in \mathbb{B}(p,\tilde r)$ for $\tilde r > 0$ and fix $\lambda > 1$. We observe that

$$
\sup_{0<\tilde r<r} \frac{\mu_\Sigma\big(\mathbb{B}(v_{\tilde r},\lambda\tilde r)\big)}{(\lambda\tilde r)^{\mathrm{N}}} \leq \sup_{\substack{u\in\mathbb{B}(p,r')\\ 0<r'<\lambda r}} \frac{\mu_\Sigma\big(\mathbb{B}(u,r')\big)}{(r')^{\mathrm{N}}}
$$

for each $r > 0$ sufficiently small. From the definition of spherical Federer density (8.2), it follows that

$$
\limsup_{\tilde r\to 0^+} \frac{\mu_\Sigma\big(\mathbb{B}(v_{\tilde r},\lambda\tilde r)\big)}{(\lambda\tilde r)^{\mathrm{N}}} \leq \theta^{\mathrm{N}}(\mu_\Sigma, p)\,.
\tag{8.22}
$$

We wish to write a formula for $\mu_\Sigma\big(\mathbb{B}(v_{\tilde r},\lambda\tilde r)\big)$, therefore we consider (8.4) and apply (8.5), replacing $\tilde r$ with $\lambda\tilde r$ and z with $v_{\tilde r}$. It follows that the set

$$E_{\tilde r}=\delta_{1/(\lambda\tilde r)}\big(\Phi^{-1}(\mathbb{B}(v_{\tilde r},\lambda\tilde r))\big)=\Big\{y\in\mathbb{R}^{\rm n}:\delta_{1/\tilde r}(\Psi(\sigma_{\lambda\tilde r}y))\in\mathbb{B}(v_0,\lambda)\Big\}\tag{8.23}$$

gives the equality

$$\frac{\mu_\Sigma\big(\mathbb{B}(v_{\tilde r},\lambda\tilde r)\big)}{(\lambda\tilde r)^{\rm N}}=\int_{E_{\tilde r}}\|\pi_{\Phi(y),{\rm N}}\big(\partial_{y_1}\Phi(\sigma_{\lambda\tilde r}y)\wedge\cdots\wedge\partial_{y_{\rm n}}\Phi(\sigma_{\lambda\tilde r}y)\big)\|_g\,dy\,.\tag{8.24}$$

Setting $\tilde E_{\tilde r}=\sigma_\lambda(E_{\tilde r})$ and performing the change of variables $y=\sigma_{1/\lambda}\tilde y$, we get

$$\frac{\mu_\Sigma\big(\mathbb{B}(v_{\tilde r},\lambda\tilde r)\big)}{(\lambda\tilde r)^{\rm N}}=\frac{1}{\lambda^{\rm N}}\int_{\tilde E_{\tilde r}}\|\pi_{\Phi(y),{\rm N}}\big(\partial_{y_1}\Phi(\sigma_{\tilde r}\tilde y)\wedge\cdots\wedge\partial_{y_{\rm n}}\Phi(\sigma_{\tilde r}\tilde y)\big)\|_g\,d\tilde y\,,\tag{8.25}$$

where we have defined

$$\tilde E_{\tilde r}=\Big\{y\in\mathbb{R}^{\rm n}:\delta_{1/\tilde r}(\Psi(\sigma_{\tilde r}y))\in\mathbb{B}(v_0,\lambda)\Big\}.$$

Now, we fix $1<\tilde\lambda<\lambda$, the subset

$$H_{\tilde r}=\Big\{y\in\mathbb{R}^{\rm n}:\delta_{1/\tilde r}(\Psi(\sigma_{\tilde r}y))\in B(v_0,\tilde\lambda)\Big\}\tag{8.26}$$

and observe that (8.6), (8.7) and (8.8), in view of $\Psi(y)=\sum_{j=1}^{\rm q}\psi_j(y)e_j$, show that

$$\delta_{1/\tilde r}(\Psi(\sigma_{\tilde r}y))=\sum_{l=1}^{\rm n}y_l e_{{\rm m}_{b_l-1}+l-\mu_{b_l-1}}+\sum_{l\notin I}\frac{\Gamma_l(\tilde r\zeta(y))}{(\tilde r)^{d_l}}e_l\tag{8.27}$$

for each $y\in\mathbb{R}^{\rm n}$ converges to

$$\sum_{l=1}^{\rm n}y_l e_{{\rm m}_{b_l-1}+l-\mu_{b_l-1}}\in A_p\Sigma\quad\text{as}\quad\tilde r\to0^+.\tag{8.28}$$

As a result, for any $y\in B(v_0,\tilde\lambda)\cap A_p\Sigma$ there holds

$$\lim_{\tilde r\to0^+}\mathbf{1}_{H_{\tilde r}\cap B(v_0,\tilde\lambda)}(y)=1.$$

Thus, taking into account that (8.22), (8.25), (8.26), the following limit superior

$$\limsup_{\tilde r\to0^+}\frac{1}{\lambda^{\rm N}}\int_{B(v_0,\tilde\lambda)\cap A_p\Sigma}\mathbf{1}_{H_{\tilde r}\cap B(v_0,\tilde\lambda)}(\tilde y)\|\pi_{\Phi(y),{\rm N}}\big(\partial_{y_1}\Phi(\sigma_{\tilde r}\tilde y)\wedge\cdots\wedge\partial_{y_{\rm n}}\Phi(\sigma_{\tilde r}\tilde y)\big)\|_g\,d\tilde y$$

is not greater than $\theta^{\rm N}(\mu_\Sigma,p)$. Then Lebesgue's convergence theorem gives

$$\frac{1}{\lambda^{\rm N}}\mathcal{H}^{\rm n}_{|\cdot|}(B(v_0,\tilde\lambda)\cap A_p\Sigma)\,\|\pi_{\Phi(y),{\rm N}}\big(\partial_{y_1}\Phi(0)\wedge\cdots\wedge\partial_{y_{\rm n}}\Phi(0)\big)\|_g\le\theta^{\rm N}(\mu_\Sigma,p).$$

Letting first $\tilde\lambda\to1^+$ and then $\lambda\to1^+$, due to (8.21), we get

$$\beta_d(A_p\Sigma)\,\|\pi_{\Phi(y),{\rm N}}\big(\partial_{y_1}\Phi(0)\wedge\cdots\wedge\partial_{y_{\rm n}}\Phi(0)\big)\|_g\le\theta^{\rm N}(\mu_\Sigma,p).\tag{8.29}$$

Joining this inequality with (8.20) we get a formula for the Federer density

$$\theta^{\mathrm{N}}(\mu_\Sigma,p)=\beta_d(A_p\Sigma)\,\|\pi_{\Phi(y),\mathrm{N}}\big(\partial_{y_1}\Phi(0)\wedge\cdots\wedge\partial_{y_\mathrm{n}}\Phi(0)\big)\|_g. \tag{8.30}$$

Finally, by (3.1) and (8.1) we observe that

$$\begin{aligned}\pi_{\Phi(y),\mathrm{N}}\big(\partial_{y_1}\Phi(0)\wedge\cdots\wedge\partial_{y_\mathrm{n}}\Phi(0)\big)=&X_1\wedge\cdots\wedge X_{\alpha_1}\wedge X_{\mathrm{m}_1+1}\wedge\cdots\wedge X_{\mathrm{m}_1+\alpha_2}\wedge\cdots\\ &\cdots\wedge X_{\mathrm{m}_{\iota-1}+1}\wedge\cdots\wedge X_{\mathrm{m}_{\iota-1}+\alpha_\iota},\end{aligned}$$

which has unit norm with respect to $\|\cdot\|_g$. This completes our proof. □

8.2. **Area formulae for the spherical measure.** In this section we show how the upper blow-up theorem automatically leads to general area formulae for the spherical measure. By a special symmetry condition on the distance, we establish the relationship between Hausdorff measure and spherical measure on horizontal submanifolds.

Throughout this section $\mathbb{G}$ denotes an arbitrary homogeneous group and $\Sigma\subset\mathbb{G}$ is an n-dimensional C^1 smooth submanifold of degree N. Its *characteristic set* and its *subset of maximum degree* are defined by

$$\mathcal{C}_\Sigma=\{p\in\Sigma: d_\Sigma(p)<\mathrm{N}\}\quad\text{and}\quad \mathcal{M}_\Sigma=\{p\in\Sigma: d_\Sigma(p)=\mathrm{N}\}\,, \tag{8.31}$$

respectively. We also fix the intrinsic measure μ_Σ, along with the Riemannian metrics $\tilde g$ and g, as in Definition 7.3. We use the spherical measure $\mathcal{S}_0^{\mathrm{N}}$ of (7.3), that does not contain any geometric constant.

Theorem 8.1 (Transversal submanifolds)**.** *If* $\Sigma\subset\mathbb{G}$ *is an* n*-dimensional transversal submanifold of degree* N*, then for every Borel set* $B\subset\Sigma$ *we have*

$$\mu_\Sigma(B)=\int_B\|\tau^{\tilde g}_{\Sigma,\mathrm{N}}(p)\|_g\,d\sigma_{\tilde g}(p)=\int_B\beta_d(A_p\Sigma)\,d\mathcal{S}_0^{\mathrm{N}}(p), \tag{8.32}$$

where $\tilde g$ *is any fixed Riemannian metric.*

Proof. From the definitions of (8.31), by Theorem 1.2 of [42] we get $\mathcal{S}_0^{\mathrm{N}}(\mathcal{C}_\Sigma)=0$. The definition of intrinsic measure (7.8) joined with Remark 7.4 yield $\mu_\Sigma(\mathcal{C}_\Sigma)=0$. This allows us to restrict our attention to points of $\mathcal{M}_\Sigma$. For every Borel set $E\subset\mathcal{M}_\Sigma$, each point $p\in E$ has maxium degree, therefore Proposition 5.3 implies that it is a transversal point. Then we are in the position to apply part (4) of Theorem 1.2 to each $p\in E$, getting formula (1.6). The everywhere finiteness of the spherical Federer density $\theta^{\mathrm{N}}(\mu_\Sigma,\cdot)$ shows that $\mu_\Sigma\llcorner E$ is absolutely continuous with respect $\mathcal{S}_0^{\mathrm{N}}\llcorner E$ and the measure theoretic area formula (7.5) applied to μ_Σ yields

$$\mu_\Sigma(E)=\int_E\beta_d(A_p\Sigma)\,d\mathcal{S}_0^{\mathrm{N}}(p).$$

This formula joined with the negligibility of $\mathcal{C}_\Sigma$ immediately leads us to (8.32). □

The next results are the first important consequences of the upper blow-up theorem.

Theorem 8.2 (Submanifolds in two step groups). *If $\mathbb{G}$ has step two, $\mathcal{S}_0^{\mathrm{N}}(\mathcal{C}_\Sigma)=0$ and $p\in\Sigma$ is algebraically regular for all $p\in\mathcal{M}_\Sigma$, then for any Borel set $B\subset\Sigma$ we have*

$$\mu_\Sigma(B)=\int_B \|\tau_{\Sigma,\mathrm{N}}^{\tilde g}(p)\|_g\,d\sigma_{\tilde g}(p)=\int_B \beta_d(A_p\Sigma)\,d\mathcal{S}_0^{\mathrm{N}}(p), \tag{8.33}$$

where $\tilde g$ is any fixed Riemannian metric.

Proof. Taking into account definitions (8.31), by our assumptions joined with (7.8) and Remark 7.4, we get

$$\mu_\Sigma(\mathcal{C}_\Sigma)=\mathcal{S}_0^{\mathrm{N}}(\mathcal{C}_\Sigma)=0. \tag{8.34}$$

If $E\subset\mathcal{M}_\Sigma$ is any Borel set, we may apply part (2) of Theorem 1.2 to each $p\in E$, since all of these points are algebraically regular. This allows us to establish (1.6). In particular, the spherical Federer density $\theta^{\mathrm{N}}(\mu_\Sigma,\cdot)$ is everywhere finite on E, hence $\mu_\Sigma \llcorner E$ is absolutely continuous with respect $\mathcal{S}_0^{\mathrm{N}}\llcorner E$. We are in the conditions to apply the measure theoretic area formula (7.5), obtaining that

$$\mu_\Sigma(E)=\int_E \beta_d(A_p\Sigma)\,d\mathcal{S}_0^{\mathrm{N}}(p)\,,$$

therefore (8.33) holds. The previous equality joined with (8.34) gives (8.33). □

Theorem 8.3 (Curves in homogeneous groups). *Let $\Sigma\subset\mathbb{G}$ be a C^1 smooth embedded curve of degree N, let $\tilde g$ be a fixed Riemannian metric and consider a Borel set $B\subset\Sigma$. The following formula holds*

$$\int_B \|\tau_{\Sigma,\mathrm{N}}^{\tilde g}(p)\|_g\,d\sigma_{\tilde g}(p)=\int_B \beta_d(A_p\Sigma)\,d\mathcal{S}_0^{\mathrm{N}}(p). \tag{8.35}$$

Proof. Taking into account the definitions (8.31), Theorem 1.1 of [33] gives

$$\mathcal{S}_0^{\mathrm{N}}(\mathcal{C}_\Sigma)=0. \tag{8.36}$$

From the definition of intrinsic measure and taking into account Remark 7.4, one also notices that $\mu_\Sigma(\mathcal{C}_\Sigma)=0$. At any point p of $\mathcal{M}_\Sigma$, the N-projection $\pi_{p,\mathrm{N}}(\tau_\Sigma(p))$ is obviously a vector, hence the homogeneous tangent space $A_p\Sigma$ is automatically a one dimensional subgroup of $\mathbb{G}$. This shows that any point of $\mathcal{M}_\Sigma$ is algebraically regular. As a consequence, considering any Borel set $E\subset\mathcal{M}_\Sigma$, we apply part (3) of Theorem 1.2 at each point $p\in E$, getting

$$\theta^{\mathrm{N}}(\mu_\Sigma,x)=\beta_d(A_p\Sigma). \tag{8.37}$$

In particular, the finiteness of the spherical Federer density $\theta^{\mathrm{N}}(\mu_\Sigma,\cdot)$ on E yields the absolute continuity of $\mu_\Sigma\llcorner E$ with respect to $\mathcal{S}_0^{\mathrm{N}}\llcorner E$. Joining the measure theoretic area formula (7.5) with the negligibility condition (8.36) our claim (8.35) follows. □

Remark 8.4. In any Heisenberg group $\mathbb{H}^n=H^1\oplus H^2$ equipped with a homogeneous distance d the blow-up at nonhorizontal points of C^1 smooth curves is the one dimensional vertical subgroup H^2. Thus, from the Definition 7.6 of spherical factor and Theorem 8.3, we have the following area formula

$$\mathcal{S}_d^2 \llcorner \Sigma(B) = \int_B \|\tau_{\Sigma,\mathrm{N}}^{\tilde g}(p)\|_g \, d\sigma_{\tilde g}(p) \tag{8.38}$$

for any C^1 smooth nonhorizontal curve $\Sigma \subset \mathbb{H}^n$ and any $B \subset \Sigma$ Borel set. In this case have defined

$$\omega_d(1,2) = \beta_d(H^2) \quad \text{and} \quad \mathcal{S}_d^2 = \omega_d(1,2)\mathcal{S}_0^2.$$

The spherical factor β_d in all the previous theorems strongly depends on the choice of the homogeneous distance d. It is then worth to consider special classes of distances that make β_d a fixed geometric constant. This is in analogy to what occurs in Euclidean space $\mathbb{R}^n$ for the Hausdorff measure $\mathcal{H}^k_{|\cdot|}$, that in its definition includes the geometric constant ω_k. Indeed, such a constant corresponds to the volume of maximal k-dimensional sections of the unit ball in $\mathbb{R}^n$.

Definition 8.5 (Multiradial distance)**.** Let $d : \mathbb{G}\times\mathbb{G} \to \mathbb{R}$ be a homogeneous distance and let $\varphi : [0,+\infty)^\iota \to [0,+\infty)$ be continuous and monotone nondecreasing on each single variable, such that

$$d(x,0) = \varphi(|x_1|, \ldots, |x_\iota|), \tag{8.39}$$

$x_j = P_{H^j}(x)$ and $P_{H^j} : \mathbb{G} \to H^j$ is the canonical projection with respect to the direct sum decomposition of $\mathbb{G}$ into subspaces H^j. The function φ is also assumed to be *coercive* in the sense that

$$\varphi(x) \to +\infty \quad \text{as} \quad |x| \to +\infty.$$

Let us stress that the symbol $|\cdot|$ indicates the Euclidean norm arising from the fixed graded scalar product, see Section 2.

Remark 8.6. In any homogeneous group one can find a multiradial distance. Setting $\varepsilon_1 = 1$ and suitably small $\varepsilon_i > 0$, one can always construct a nonsmooth homogeneous distance defining

$$\|x\|_\infty = \max\{\varepsilon_i |x_i|^{1/i} : 1 \le i \le \mathrm{q}\}$$

and then $d(x,y) = \|x^{-1}y\|_\infty$ for $x,y \in \mathbb{G}$, see for instance [51]. One can easily realize that d is multiradial.

We use multiradial distances to study the relationship between Hausdorff measure and spherical measure on horizontal submanifolds of class C^1.

Proposition 8.7. *If d is a multiradial distance on $\mathbb{G}$, then there exists a geometric constant $\omega_d(\mathrm{n},\mathrm{n})$ such that*

$$\beta_d(V) = \mathcal{H}^{\mathrm{n}}_{|\cdot|}(\mathbb{B} \cap V) = \omega_d(\mathrm{n},\mathrm{n}) \tag{8.40}$$

for any n-dimensional horizontal subgroup $V \subset H^1$, where $\mathbb{B} = \{x \in \mathbb{G} : d(x,0) \le 1\}$.

Proof. Consider an n-dimensional horizontal subgroup $V \subset H^1$ and the intersection

$$V \cap \mathbb{B}(z,1) = \left\{ v \in V : z^{-1} v \in \mathbb{B} \right\}.$$

Since d is multiradial, we have

$$V \cap \mathbb{B}(z,1) = \left\{ v \in V : \varphi(|P_{H^1}(z^{-1}v)|, \ldots, |P_{H^\iota}(z^{-1}v|) \le 1 \right\}$$

and the monotonicity properties of φ give

$$V \cap \mathbb{B}(z,1) \subset \{ v \in V : \varphi(|v - P_{H^1}(z)|, 0, \ldots, 0) \le 1 \} = \zeta_1 + C,$$

where $\zeta_1 = P_{H^1}(z)$ and $C = \{ v \in V : \varphi(|v|, 0, \ldots, 0) \le 1 \}$. It follows that

$$\mathcal{H}^{\mathrm{n}}_{|\cdot|}\left(V \cap \mathbb{B}(z,1)\right) \le \mathcal{H}^{\mathrm{n}}_{|\cdot|}\left(\zeta_1 + C\right) = \mathcal{H}^{\mathrm{n}}_{|\cdot|}(C) = \mathcal{H}^{\mathrm{n}}_{|\cdot|}\left(\mathbb{B} \cap V\right).$$

This proves (8.40), along with the fact that $\beta_d(V)$ does not depend on the choice of the n-dimensional horizontal subgroup V. □

Theorem 8.8 (Horizontal submanifolds)**.** *If $\Sigma \subset \mathbb{G}$ is an* n*-dimensional horizontal submanifold, then for every Borel set $B \subset \Sigma$ we have*

$$\mu_\Sigma(B) = \int_B \|\tau^{\tilde g}_{\Sigma,\mathrm{N}}(p)\|_g \, d\sigma_{\tilde g}(p) = \int_B \beta_d(A_p\Sigma) \, d\mathcal{S}^{\mathrm{n}}_0(p), \tag{8.41}$$

where $\tilde g$ is any fixed Riemannian metric. If in addition d is multiradial, then for any homogeneous tangent space V of Σ, the spherical factor $\beta_d(V)$ equals a geometric constant $\omega_d(\mathrm{n},\mathrm{n})$ and defining $\mathcal{S}^{\mathrm{n}}_d = \omega_d(\mathrm{n},\mathrm{n})\, \mathcal{S}^{\mathrm{n}}_0$, there holds

$$\mathcal{S}^{\mathrm{n}}_d \llcorner \Sigma(B) = \int_B \|\tau^{\tilde g}_{\Sigma,\mathrm{N}}(p)\|_g \, d\sigma_{\tilde g}(p). \tag{8.42}$$

Proof. From the definition of horizontal submanifold, all points of Σ are algebraically regular. In view of Remark 4.4, we also observe that all points of Σ have degree n. As a result, $\Sigma = \mathcal{M}_\Sigma$ and choosing any Borel set $B \subset \Sigma$, we apply part (1) of Theorem 1.2 to each $p \in B$, getting formula (1.6). The everywhere finiteness of the spherical Federer density on B joined with the measure theoretic area formula (7.5) lead us to the integration formula (8.41). In the case d is multiradial, Proposition 8.7 allows us to define the geometric constant $\omega_d(\mathrm{n},\mathrm{n}) = \beta_d(V)$, independent of the choice of the homogeneous tangent space V at any point of Σ. Then (8.41) immediately gives (8.42), concluding the proof. □

As a consequence of the previous theorem, joined with the area formula of [35], we will find the formula relating spherical measure and Hausdorff measure on horizontal submanifolds.

A multiradial distance d is fixed from now on. We consider the set function ϕ^k_δ of (7.2) with respect to d, where $\mathcal{F}$ is the family of all closed sets $\alpha = \mathrm{n}$. In the sequel, we consider the Hausdorff measure

$$\mathcal{H}^{\mathrm{n}}_0(E) = \sup_{\delta > 0} \phi^{\mathrm{n}}_\delta(E) \tag{8.43}$$

for every $E \subset \mathbb{G}$. We will also use the "normalized Hausdorff measure"

$$\mathcal{H}^{\mathrm{n}}_d = \omega_d(\mathrm{n},\mathrm{n})\mathcal{H}^{\mathrm{n}}_0, \tag{8.44}$$

where $\omega_d(\mathrm{n},\mathrm{n})$ is defined in (8.40).

Lemma 8.9. *Let d be a multiradial distance on $\mathbb{G}$ and let $V, W \subset H^1$ be horizontal subgroups of dimension* n*. It follows that*

$$\mathcal{H}^{\mathrm{n}}_0(\mathbb{B}\cap V) = \mathcal{H}^{\mathrm{n}}_0(\mathbb{B}\cap W) = 1, \tag{8.45}$$

with $\mathbb{B} = \{x \in \mathbb{G} : d(x,0) \le 1\}$.

Proof. Let us consider the Euclidean isometry $\widetilde{T} : V \to W$ with respect to the fixed graded Euclidean norm $|\cdot|$ on $\mathbb{G}$. The same scalar product defines the multiradial distance, see (8.39). For each $x, y \in V$ there holds

$$d(\widetilde{T}x, \widetilde{T}y) = d((\widetilde{T}x)^{-1}\widetilde{T}(y), 0) = d(\widetilde{T}y - \widetilde{T}x, 0) = d(\widetilde{T}(y-x), 0)$$

where the last equality follows by the fact that W is commutative. By definition of d, we have

$$d(\widetilde{T}x, \widetilde{T}y) = \varphi(|\widetilde{T}y - \widetilde{T}x|, 0, \dots, 0) = \varphi(|y-x|, 0, \dots, 0) = d(x,y),$$

where the last equality holds, due to the commutativity of V. Choosing proper orthonormal bases, we extend $\widetilde{T}$ to an isometry $T : H^1 \to H^1$ with respect to $|\cdot|$ such that $T|_V = \widetilde{T}$. Since d is multiradial it is easy to observe that

$$T(\mathbb{B}\cap H^1) = \mathbb{B}\cap H^1.$$

By definition of $\widetilde{T}$, it follows that

$$T(\mathbb{B}\cap V) = T(\mathbb{B}\cap H^1 \cap V) = \mathbb{B}\cap H^1 \cap \widetilde{T}(V) = \mathbb{B}\cap W. \tag{8.46}$$

We now consider the Hausdorff measures

$$\widetilde{\mathcal{H}}^{\mathrm{n}}_V : \mathcal{P}(V) \to [0,+\infty] \quad \text{and} \quad \widetilde{\mathcal{H}}^{\mathrm{n}}_W : \mathcal{P}(W) \to [0,+\infty]$$

defined in (8.43), but where the metric space $\mathbb{G}$ is replaced by the horizontal subgroups V and W, respectively. Since $\widetilde{T}$ is an isometry also with respect to d, taking into account (8.46) and the standard property of Lipschitz functions with respect to the Hausdorff measure, we get

$$\widetilde{\mathcal{H}}^{\mathrm{n}}_W(\mathbb{B}\cap W) = \widetilde{\mathcal{H}}^{\mathrm{n}}_W(\widetilde{T}(\mathbb{B}\cap V)) = \widetilde{\mathcal{H}}^{\mathrm{n}}_V(\mathbb{B}\cap V).$$

Exploiting the special property of the Hausdorff meausure about restrictions

$$\widetilde{\mathcal{H}}^{\mathrm{n}}_V = \mathcal{H}^{\mathrm{n}}_0|_{\mathcal{P}(V)} \quad \text{and} \quad \widetilde{\mathcal{H}}^{\mathrm{n}}_W = \mathcal{H}^{\mathrm{n}}_0|_{\mathcal{P}(W)}$$

the first equality of (8.45) follows. Finally, we apply the isodiametric inequality in finite dimensional Banach spaces, see for instance [7, Theorem 11.2.1], and observe that the restriction $\|x\|_d = d(x,0)$ for $x \in V$ yields a Banach norm, due to the commutativity of V. By standard arguments, the isodiametric inequality in the Banach space $(V, \|\cdot\|_d)$ gives $\widetilde{\mathcal{H}}^{\mathrm{n}}_V(\mathbb{B}\cap V) = 1$, therefore concluding the proof. □

Proof of Theorem 1.4. Since our argument is local, it is not restrictive to consider an open set $\Omega\subset\mathbb{R}^{\mathrm{n}}$ and assume that there exists a C^1 smooth embedding $\Psi:\Omega\to\mathbb{G}$ such that $\Sigma=\Psi(\Omega)$. Joining Proposition 7.5 and Theorem 8.8, for every open subset $H\subset\Omega$ there holds

$$\mathcal{S}_d^{\mathrm{n}}\llcorner\Sigma\big(\Psi(H)\big)=\int_H\|\pi_{\Psi(y),\mathrm{n}}\big(\partial_{y_1}\Psi(y)\wedge\cdots\wedge\partial_{y_{\mathrm{n}}}\Psi(y)\big)\|_g\,dy. \tag{8.47}$$

From the area formula of [35]:

$$\mathcal{H}_d^{\mathrm{n}}(\Psi(H))=\int_H\frac{\mathcal{H}_d^{\mathrm{n}}(D\Psi(x)(B_E))}{\mathcal{L}^{\mathrm{n}}(B_E)}dx,$$

where $D\Psi(x):\mathbb{R}^{\mathrm{n}}\to\mathbb{G}$ is the Lie group homomorphism defining the differential, see [35] for more information. For each $x\in\Omega$ both $\mathcal{H}_d^{\mathrm{n}}$ and $\mathcal{H}_{|\cdot|}^{n}$ are Haar measures on the horizontal subgroup $V_x=D\Psi(x)(\mathbb{R}^{\mathrm{n}})\subset H^1$, therefore

$$\mathcal{H}_d^{\mathrm{n}}\llcorner V_x=\frac{\mathcal{H}_d^{\mathrm{n}}(V_x\cap\mathbb{B})}{\mathcal{H}_{|\cdot|}^{\mathrm{n}}(V_x\cap\mathbb{B})}\mathcal{H}_{|\cdot|}^{\mathrm{n}}\llcorner V_x.$$

The Haar property of these measures follows from the commutativity of V_x, hence the BCH yields $yA=y+A$, whenever $y\in V_x$ and $A\subset V_x$. By Proposition 8.7 and definition (8.44) we have

$$\mathcal{H}_d^{\mathrm{n}}\llcorner V_x=\mathcal{H}_0^{\mathrm{n}}(V_x\cap\mathbb{B})\,\mathcal{H}_{|\cdot|}^{\mathrm{n}}\llcorner V_x=\mathcal{H}_{|\cdot|}^{\mathrm{n}}\llcorner V_x, \tag{8.48}$$

in view of Lemma 8.9. We have proved that

$$\mathcal{H}_d^{\mathrm{n}}(\Psi(H))=\int_H\frac{\mathcal{H}_{|\cdot|}^{\mathrm{n}}(D\Psi(x)(B_E))}{\mathcal{L}^{\mathrm{n}}(B_E)}dx. \tag{8.49}$$

The Lie group homomorphism $D\Psi(x):\mathbb{R}^k\to\mathbb{G}$ is defined as the limit

$$D\Psi(x)(v)=\lim_{t\to0^+}\delta_{1/t}\left(\Psi(x)^{-1}\Psi(x+tv)\right), \tag{8.50}$$

that exists for all $x\in\Omega$, in view of [39, Theorem 1.1]. Exploiting the BCH formula in the limit (8.50) and the fact that the image of $D\Psi(x)$ must be in a horizontal subgroup, we have

$$D\Psi(x)(h)=d\widetilde{\Psi}(x)(h)\in V_x\subset\mathbb{G}, \tag{8.51}$$

where $\widetilde{\Psi}=P_{H^1}\circ\Psi$ and h is any vector of $\mathbb{R}^{\mathrm{n}}$. Finally, the expression

$$\pi_{\Psi(y),\mathrm{n}}\big(\partial_{y_1}\Psi(y)\wedge\cdots\wedge\partial_{y_{\mathrm{n}}}\Psi(y)\big)$$

can be more explicitly written as

$$\sum_{j_1,\dots,j_{\mathrm{n}}=1}^{\mathrm{m}}\pi_{\Psi(y),\mathrm{n}}\big((\partial_{y_1}\Psi(x))^{j_1}X_{j_1}(\Psi(x))\wedge\cdots\wedge\partial_{y_{\mathrm{n}}}\Psi(x))^{j_1}X_{j_1}(\Psi(x))\big)$$

where the special polynomial form of the vector fields X_j implies that the component $(\partial_{y_i}\Psi(x))^j$ of $\partial_{y_i}\Psi(x)$ with respect to the basis $(X_1(\Phi(x)),\ldots,X_{\mathrm{m}}(\Phi(x))$ coincides with $\partial_{y_1}\Psi^j(x)$. This allows us to conclude that

$$\|\pi_{\Psi(x),\mathrm{n}}\big(\partial_{y_1}\Psi(x)\wedge\cdots\wedge\partial_{y_\mathrm{n}}\Psi(x)\big)\|_g = J\tilde\Psi(x) = \frac{\mathcal{H}^{\mathrm{n}}_{|\cdot|}(d\tilde\Psi(x)(B_E))}{\mathcal{L}^{\mathrm{n}}(B_E)}.$$

As a consequence, by (8.51), (8.49) and (8.47), our claim follows. □

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